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REAL OPTIONS AND AMERICAN DERIVATIVES: THE DOUBLE CONTINUATION REGION

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Abstract

We thoroughly study the non-standard optimal exercise policy associated with relevant capital investment options and with the prepayment option of widespread collateralized-borrowing contracts like the gold loan. Option exercise is optimally postponed not only when moneyness is insufficient but also when it is excessive. We contribute an important extension of the classical optimal exercise properties for American options. Early exercise of an American call with a negative underlying payout rate can occur if the option is moderately in the money. We fully characterize the existence, the monotonicity, the continuity, the limits and the asymptotic behavior at maturity of the double free boundary that separates the exercise region from the double continuation region. We find that the finite-maturity non-standard policy conspicuously differs from the infinite-maturity one.

1 Introduction

A number of significant decision-making problems in finance can be reformulated as American option problems with an endogenous negative interest rate. Two chief examples are the prepayment option in collateralized borrowing like the recently popular gold loans and a notable class of capital investment options. Gold loans are familiar among Indian financial intermediaries.[1] An endogenous negative interest rate for the American derivatives embedded into loans collateralized by tradable assets appears whenever the loan rate is above the riskfree rate. An endogenous negative interest rate in waiting-to-invest real options appears whenever the risk-adjusted expected growth rate of the project value is above the rate used by the firm to discount it.

We show that such decision-making problems can imply a non-standard *double* continuation region: exercise is optimally postponed not only when the option is not enough in the money (the standard part of the continuation region) but also *when the option is too deep in the money* (the non-standard part of the continuation region). For finite-maturity and perpetual American puts and calls with a negative interest rate in a diffusive setting, we contribute by providing a detailed analysis of the conditions that enable the double continuation region and a comprehensive characterization of the double free boundary entailed by such a continuation region[2].

Importantly, we contribute to a thorougher understanding of the optimal exercise properties for American options. Given a positive riskfree rate r, it is well known that it is always suboptimal to exercise prior to maturity an American call on a tradable asset with payout rate π equal to zero (Merton (1973)) and, more generally, an American contingent claim for which the net benefit of exercising immediately is non-positive at all times (Detemple (2006)). For example, consider the optimal exercise date t^* of the prepayment option

[1] Muthoot Finance is one of the largest gold loan companies in India. J.P. Morgan Chase started accepting gold as loan collateral from institutional players since February 2011, amid a climate of soaring gold prices.

[2] Our single-underlying result of multiple continuation regions mirrors upside down the literature documenting multiple exercise regions in models with a single underlying asset, e.g. Chiarella and Ziogas (2005) and Detemple and Emmerling (2009).

embedded into a 5-year loan collateralized by gold. To maximize intuition, assume the absence of risk. The loan amount is q and the current gold price is G so that the optimal exercise date boils down to

$$t^* = \underset{0 \leq t \leq 5}{\arg\max} e^{-rt} \left(G e^{(r-\pi)t} - q e^{\gamma t} \right)^+ ,$$

where γ is the borrowing rate commanded by the loan contract. Focus on the in-the-money case ($G > q$). If γ had been zero, the standard Merton result of $t^* = 5$ would have applied as holding gold is burdened with the storage cost $-G\pi$ (the payout rate π is negative). A positive γ that dominates the risk-free rate ($\gamma > r$) introduces a prepayment incentive for the borrower. Such an incentive is overpowered by $-G\pi$ ($t^* > 0$) when gold is markedly dear, that is when the degree of in-the-moneyness is huge. However, the storage cost is not overwhelming and immediate prepayment does occur ($t^* = 0$) when the loan rate γ is sufficiently high and the degree of in-the-moneyness is moderate. Fix $r = 1\%$, $\pi = -1\%$, $\gamma = 7\%$ and $q = 1$. If $G = 7$ the prepayment option exercise is optimally delayed for three years ($t^* = 3.083$), whereas if $G = 2$ the borrower exercises immediately ($t^* = 0$). The deterministic decision-making example admits a neat restatement as an American option problem with a constant strike price q and an endogenous interest rate $\rho = r - \gamma$,

$$t^* = \underset{0 \leq t \leq 5}{\arg\max} e^{-\rho t} \left(G e^{\mu t} - q \right)^+ ,$$

where $\mu = r - \pi - \gamma$ is the gold price's adjusted drift rate. The restatement streamlines the optimal exercise analysis. If $\rho = -6\%$, $\mu = -5\%$ and $q = 1$, the spur to postpone exercise caused by a negative interest rate wins over the aversion to delay induced by the drift towards the out-of-the-money region ($t^* = 3.083$) for $G = 7$, whereas the spur is insufficient ($t^* = 0$) for $G = 2$.

Our results add to the vast literature on American options under diffusive risk, see for instance Broadie and Detemple (1996), (2004), Detemple and Tian (2002), Detemple (2006), and more recently Medvedev and Scaillet (2010). We conduct an in-depth study of the existence, the monotonicity, the continuity, the limits and the asymptotic behavior at maturity of both the upper and the lower free boundary. We start from the American put problem and prove the conditions for the existence of a double continuation region in the case of a negative interest rate via convexity, monotonicity and value-dominance arguments. We use the variational inequality approach to prove the continuity of the double free boundary. We then carefully characterize the double free boundary near to maturity (for asymptotic results on the (single) free boundary with non-negative interest rate see Medvedev and Scaillet (2010) and the references therein). Finally, we translate the results obtained for the American put problem into double-free-boundary statements for the American call problem via the American put-call symmetry (e.g. Carr and Chesney (1996) and Detemple (2001)).

In a gold loan the precious metal is the collateral, which the borrower has the right to redeem at any time before or on the loan maturity. We show that, since gold is a tradable investment asset with storage (and insurance) costs and without earnings, a double continuation region can ensue: the exercise of a deep in-the-money redemption option may be optimally postponed by the borrower. This is an interesting and distinct addition to the existing literature on the optimal redeeming strategy of tradable securities used as loan collateral: Xia and Zhou (2007) focus on perpetual stock loans; Ekström and Wanntorp (2008) deal with margin call stock loans; Zhang and Zhou (2009) look into stock loans in the presence of regime switching; Liu and Xu (2010) consider capped stock loans, whose subtle variational-inequality issues are studied by Liang and Zu (2012); Dai and Xu (2011) examine the impact of the dividend-distribution criterion on the stock loan. The stock loan problem comes with a standard (unique) free boundary as the risk-neutral percentage drift of the underlying stock price equals the riskfree rate minus a non-negative dividend yield.

By investigating the general American option problem with a negative interest rate with possibly finite maturities, our work thoroughly extends the specific perpetual-real-option analysis developed in Battauz, De Donno and Sbuelz (2012). We examine capital investment options akin to, for instance, the option of entering the lucrative but challenging business of nuclear energy. Projects may have values with conspicuous growth rates even after risk adjustment (say rates above the discount rate used by the firm), but the overall cost of entering them is likely to increase even more conspicuously in the future (uranium is a scarce resource and demand for safety is definitely increasing). Such a hierarchy in the risk-adjusted growth/discount rates for the real option is conducive to the non-standard optimal continuation policy. Our work focuses on mapping in detail the finite-maturity non-standard optimal exercise policy (see Sections 2 and 3) and clearly shows that the perpetual early-exercise region constitutes a rather poor proxy for the finite-maturity one (see the examples in Sections 4 and 5).

The rest of the paper is organized as follows. Sections 2 and 3 deal with the double continuation region for American puts and calls, respectively. Sections 4 and 5 discuss the double continuation region for the redemption option embedded in a gold loan and for an interesting class of real options. Section 6 concludes and an Appendix contains all the proofs.

2 The American put

We consider an American put option written on the log-normal asset X, whose drift under the valuation measure is positive and denoted with μ. We denote the volatility with σ, the strike with K, and the interest rate with ρ. The put value at time t is given by

$$\operatorname*{ess\,sup}_{t\le\tau\le T} \mathbb{E}\left[e^{-\rho(\tau-t)}\left(K - X(\tau)\right)^+ \middle| \mathcal{F}_t\right] = v(t, X(t))$$

where

$$v(t,x) = \sup_{0\le\Theta\le T-t} \mathbb{E}\left[e^{-\rho\Theta}\left(K - x\cdot\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma\, B(\Theta)\right)\right)^+\right] \tag{2.1}$$

and B is a standard Brownian motion under the valuation measure. In Sections 2 and 3, expectations and distributions of stochastic processes refer all to the valuation measure and, for the sake of simplicity, we will omit their dependence on the probability measure. If the option is perpetual, its value is

$$v_\infty(x) = \sup_{0\le\Theta} \mathbb{E}\left[e^{-\rho\Theta}\left(K - x\cdot\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma\, B(\Theta)\right)\right)^+\right].$$

Regardless of the sign of ρ, the function v in (2.1) dominates the payoff function, is convex and decreasing with respect to x, decreasing with respect to t, and dominated by the perpetual put v_∞, that is

$$(K-x)^+ \leq v(t,x) \leq v(t,0) \leq v_\infty(x) \text{ for all } t \in [0;T] \text{ and } x \geq 0. \tag{2.2}$$

(see for instance Karatzas and Shreve (1998), and Broadie and Detemple (1997)).

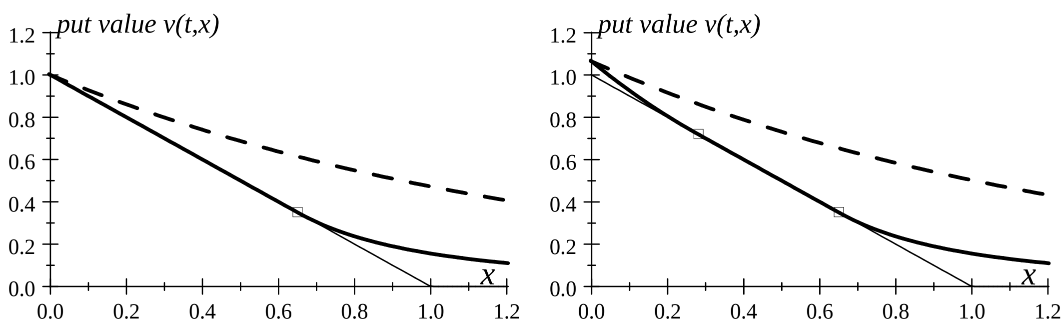
These properties interact with the sign of ρ to determine the shape of the free boundary, and the "geometry structure" of the exercise region. More precisely, if $\rho \geq 0$, for any $t < T$ we have that $v(t,0) = \sup_{0\leq\Theta\leq T-t} \mathbb{E}\left[e^{-\rho\Theta}(K-0)^+\right] = (K-0)^+$. Since $v(t,x)$ coincides for $x=0$ with the immediate exercise payoff, convexity and (2.2) imply that either $v(t,x) > (K-x)^+$ for all $x>0$ (see the thick dashed line in the left-hand panel of Figure 1) or $v(t,x) = (K-x)^+$ for any x belonging to the interval whose extremes are 0 and

$$x^*(t) = \sup\{x \geq 0 : v(t,x) = K-x\} \leq K$$

(see the thick solid line in the left-hand panel of Figure 1). The value $x^*(t)$ is the unique *put critical price at t* with nonnegative interest rates. Detemple and Tian (2002) and Detemple (2005) show that this is true for a large class of diffusion processes with nonnegative stochastic interest rates.

Figure 1: The value of the American put option $v(t,\cdot)$ (thick lines),

and the immediate exercise put payoff (thin line). $K=1$.



Finite-maturity put value with $\rho \geq 0$ Finite-maturity put value with $\rho < 0$

On the contrary, if $\rho < 0$, then $v(t,0) = \sup_{0\leq\Theta\leq T-t} \mathbb{E}\left[e^{-\rho\Theta}(K-0)^+\right] = e^{-\rho\cdot(T-t)} \cdot K > K$. The value of the American option for $x=0$ now strictly dominates the immediate exercise payoff: $v(t,0) = e^{-\rho(T-t)} \cdot K > (K-0)^+$. Then either early exercise is never optimal at date t, i.e. $v(t,x) > (K-x)^+$ for all $x>0$ (see the thick dashed line in the right-hand panel of Figure 1), or early exercise is optimal at time t for some $x' \in (0;K)$, i.e. $(K-x')^+ = v(t,x')$ (see the thick solid line in the right-hand panel of Figure 1). If x' is unique, then the exercise region collapses into a single point (the free boundary at time t). If x' is not unique, then by convexity and (2.2) the exercise region at time t is constituted by a connected segment defined by the extremes

$l(t) \leq u(t) \in [0; K]$ where[3]

$$l(t) = \inf\left\{x \geq 0 : v(t,x) = (K-x)^+\right\} \tag{2.3}$$

$$u(t) = \sup\left\{x \geq 0 : v(t,x) = (K-x)^+\right\} \wedge K \tag{2.4}$$

such that $v(t,x) = (K-x)^+$ for $l(t) \leq x \leq u(t)$ and $v(t,x) > (K-x)^+$ for $x < l(t)$ and $x > u(t)$. This implies that the continuation region at time t is splitted in two segments. Exercise is optimally postponed not only when the option is insufficiently in the money ($x > u(t)$) but also (surprisingly, at first sight) when the option is excessively in the money ($x < l(t)$). In the excessively in the money region ($x < l(t)$), moreover, the value of the American put decreases with steeper slope than the immediate put payoff, i.e. $\frac{\partial v}{\partial x}(t,x) < -1$, as it is readily seen in the right-hand panel of Figure 1. On the contrary, if $\rho \geq 0$, the derivative $\frac{\partial v}{\partial x}(t,x) \geq -1$ for all x. Thus, if the exercise region at date t is non-empty, it is the negativity of the interest rate that modifies its usual "geometry structure" (see Detemple and Tian (2002) and Detemple (2005)). Assumptions (2.6) and (2.7) in Proposition 2.2 are sufficient conditions for the non-emptiness of the exercise region in the perpetual case, and, consequently, in the finite-maturity case at *any* date t (see Theorem 2.3). In particular, Assumption (2.6) implies that the *'dividend yield'* $\delta = \rho - \mu$ is *negative.* Therefore, the negativity of both ρ and δ is crucial to determine the presence of the double continuation region. Clearly, the continuation region cannot be constituted by more than two non-connected segments, because the *convex* function $v(t,\cdot)$ must lie above the payoff function $(K-\cdot)^+$.
Let us denote with $\mathcal{ER} = \left\{(t,x) \in [0;T] \times [0;+\infty[\, : v(t,x) = (K-x)^+\right\}$, the early exercise region, and with $\mathcal{CR} = \left\{(t,x) \in [0;T] \times [0;+\infty[\, : v(t,x) > (K-x)^+\right\}$, the continuation region.

Given a finite maturity and a negative interest rate, Theorems 2.3 and 2.4 provide an accurate description of the *double continuation region*, which is separated from the (single) early exercise region by a *double free boundary.* Our findings contribute to the extant literature on multiple free boundaries that separate the (single) continuation region from the multiple exercise region for certain American options with multiple underlying assets, e.g. Broadie and Detemple (1997).

The function v in (2.1) can be expressed as the solution of the system of variational inequalities (see for instance Bensoussan and Lions (1982), Jaillet, Lamberton and Lapeyre (1990), Feng, Kovalov, Linetsky, Marcozzi (2007), and Kovalov, Linetsky, and Marcozzi (2007) for the related numerical solution):

$$\begin{cases} v(T,\cdot) = \pi(\cdot), \qquad v(t,\cdot) \geq \pi(\cdot) \quad \text{for any } t \in [0;T] \\ \frac{\partial}{\partial t} v + \mathcal{L}v - \rho v \leq 0 \text{ on } (0;T) \times \Re^+ \\ \frac{\partial}{\partial t} v + \mathcal{L}v - \rho v = 0 \text{ on } \{(t,x) \in (0;T) \times \Re^+ : v(t,x) > \pi(x)\} \end{cases} \tag{2.5}$$

where $\pi(x) = (K-x)^+$ and $(\mathcal{L}v)(t,x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} v(t,x) + \mu x \frac{\partial}{\partial x} v(t,x)$. When interest rates are non-negative, it is well known that (2.5) admits a smooth solution (see Jaillet, Lamberton and Lapeyre (1990)). The same conclusion can be achieved even if the interest rate is negative, as shown in the next proposition.

[3] Whenever $t < T$, we have $\sup\left\{x \geq 0 : v(t,x) = (K-x)^+\right\} \leq K$, because $(K-x)^+ = 0$ and $v(t,x) > 0$ for $x \geq K$. On the contrary, for $t = T$ the $\sup\left\{x \geq 0 : v(T,x) = (K-x)^+\right\} = +\infty$. Hence the cap at K in the definition of u is binding at T only.

Proposition 2.1 ***(Smoothness of the put value v, negative interest rate)*** *The solution of* (2.5) *admits partial derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$, $\frac{\partial^2 v}{\partial x^2}$ that are locally bounded on $[0;T) \times \Re^+$. Moreover, v enjoys the smooth-fit property, i.e. $\frac{\partial v}{\partial x}$ is continuous on $[0;T) \times \Re^+$.*

In the infinite-maturity case, the constant *double* free boundary can be explicitly computed by solving the differential equation implied by (2.5) in the continuation region and by applying the important *smooth-pasting* principle at the free boundary (see Battauz, De Donno and Sbuelz (2012). For the standard case of non-negative interest rates in models based on Lévy processes see e.g. Boyarchenko and Levendorskiĭ (2002a), Boyarchenko and Levendorskiĭ (2002b), Alili and Kyprianou (2005), and Jiang and Pistorius (2008). The result requires an *ad-hoc* direct verification, because v_∞ violates the usual boundedness requirements. Indeed, when $\rho < 0$ and $x = 0$ the optimal exercise time is $\Theta^* = +\infty$, and the value of the American option is $v_\infty(0) = \mathbb{E}\left[e^{-\rho\Theta^*}(K-0)^+\right] = +\infty$. Battauz et al. (2012) work out a closed-form solution for the special case of a perpetual real-option problem. The following proposition adapts their statement to our current framework (for convenience of the reader, the main steps of the proof are summoned in the Appendix).

Proposition 2.2 ***(Perpetual put, negative interest rate)*** *If $T = +\infty$,*

$$\rho < 0, \quad \mu - \frac{\sigma^2}{2} > 0 \tag{2.6}$$

and

$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0, \tag{2.7}$$

then the perpetual American put option value is

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0; l_\infty) \\ K - x & \text{for } x \in [l_\infty; u_\infty] \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty; +\infty) \end{cases} \tag{2.8}$$

where $\xi_u < \xi_l$ are the negative solutions of the equation

$$\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{\sigma^2}{2}\right)\xi - \rho = 0, \tag{2.9}$$

that is

$$\xi_l = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2} \text{ and } \xi_u = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2}.$$

The critical asset prices are

$$l_\infty, u_\infty = K\frac{\xi_i}{\xi_i - 1} \quad \text{for } i = l, u \tag{2.10}$$

and the constant A_l and A_u are given by

$$A_l = -\frac{(l_\infty)^{1-\xi_l}}{\xi_l} \text{ and } A_u = -\frac{(u_\infty)^{1-\xi_u}}{\xi_u} \tag{2.11}$$

Given a negative interest rate $\rho < 0$, the positive-drift condition (2.6) and the positive-discriminant condition (2.7) guarantee the existence of (negative) solutions of the equation (2.9) and rule out the potential explosive effect of a negative interest rate on the put value function. If the interest rate is negative, the holder of the option may obtain an infinite expected gain by deferring indefinitely the exercise of the option. Such an incentive to indefinite deferment can be counteracted by a significant chance that the option goes out of the money as time goes by. This is the case if the growth rate of the underlying asset X is high enough compared to the absolute value of the negative interest rate, as stated by the condition (2.7): $|\rho| < \frac{\left(\mu - \frac{\sigma^2}{2}\right)^2}{2\sigma^2}$.

The function v_∞ defined in (2.8) enjoys the following properties in the continuation region: v is decreasing, it dominates the immediate payoff, and the process $\left\{v_\infty(X(t))e^{-\rho t}\right\}_t$ is a local martingale. The condition (2.7) also empowers the supermartingality of the process $\left\{v_\infty(X(t))e^{-\rho t}\right\}_t$ in the early exercise region.

In Theorem 2.3 we study the American put option problem with $\rho < 0$ and $T < +\infty$. We analyze in detail the double free boundary, which is constituted by the upper free boundary (corresponding to the constant u_∞ in the perpetual case) and by the lower free boundary (corresponding to the constant l_∞ in the perpetual case). The upper free boundary enjoys all the properties it has in the standard case of non-negative interest rates: it is increasing, continuous and tends to the strike price at maturity. The lower free boundary is decreasing, continuous everywhere but at maturity, where it exhibits a discontinuity. We use the variational inequality approach to prove the continuity of the double free boundary, thus extending the standard-case results of Lamberton (1998).

Theorem 2.3 ***(Continuation region and free boundary characterization, finite-maturity put, negative interest rate)***

If conditions (2.6) *and* (2.7) *are verified, then for any* $t \in [0;T)$ *there exist*

$$\frac{\rho K}{\rho - \mu} \leq l(t) < u(t) \leq K \tag{2.12}$$

such that $(K-x)^+ = v(t,x)$ *for any* $x \in [l(t);u(t)]$ *and* $(K-x)^+ < v(t,x)$ *for any* $x \notin [l(t);u(t)]$.
The lower free boundary $l : [0;T] \to [0;l_\infty)$ *is decreasing, continuous for any* $t \in [0;T)$, $l(T^-) = \frac{\rho K}{\rho - \mu} > l(T) = 0$.
The upper free boundary $u : [0;T] \to (u_\infty;K]$ *is increasing, continuous for any* $t \in [0;T]$, *and* $u(T) = u(T^-) = K$.
The early exercise region is $\mathcal{ER} = \{(t,x) \in [0;T] \times [0;+\infty[\, : l(t) \leq x \leq u(t)\}$, *and the double continuation region is* $\mathcal{CR} = \{(t,x) \in [0;T] \times [0;+\infty[\, : 0 \leq x < l(t) \text{ or } x > u(t)\}$, *where* $\{(t,l(t))\,;(t,u(t)) : t \in [0;T]\}$ *is the double free boundary.*

Describing the free boundary close to maturity is of key importance for the American option holder. The asymptotic behavior of the free boundary of an American put option in the standard case of a positive interest rate and of a diffusive underlying has been studied by several authors, as Barles at al. (1995), and, more recently, by Evans et al. (2002), and by Lamberton and Villeneuve (2003). In a diffusive framework with stochastic volatility and stochastic interest rates, Medvedev and Scaillet (2010) derive an accurate approximation formula for the American put price, by first introducing an explicit proxy for the exercise rule based on the normalized moneyness, and then by using proper asymptotic expansions for short-maturities.

In Theorem 2.4 we study the asymptotic behavior of the double free boundary at maturity in the case of a negative interest rate. When the interest rate dominates the non-negative dividend yield of the underlying[4], Evans et al. (2002) show that the free boundary of an American put option tends at maturity to the strike price in a *parabolic-logarithmic* form. In the case of a negative interest rate the same asymptotic behavior at maturity is shown by the *upper free boundary.* As for the non standard *lower free boundary* we prove that it converges at maturity monotonically decreasingly to its left-limit[5] $l(T^-) = \frac{\rho K}{\rho - \mu}$ in a *parabolic* form.

Theorem 2.4 ***(asymptotic behavior of the free boundaries at maturity, put, negative interest rate)***
If conditions (2.6) *and* (2.7) *are verified, then for* $t \to T$ *the upper free boundary satisfies*

$$u(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi\,(T-t)\,\mu^2}}.$$

For $t \to T$*, the lower free boundary satisfies*

$$l(t) - \frac{\rho K}{\rho - \mu} \sim \frac{\rho K}{\rho - \mu}\left(-y^*\sigma\sqrt{(T-t)}\right),$$

where $y^* \in (-1;0)$*,* $y^* \approx -0.638$*, is the number such that* $\phi(y) = \sup_{0\le\Theta\le1}\mathbb{E}\left[\int_0^{\Theta}(y + B(s))\,ds\right] = 0$ *for all* $y \le y^*$ *and* $\phi(y) > 0$ *for all* $y > y^*$*.*

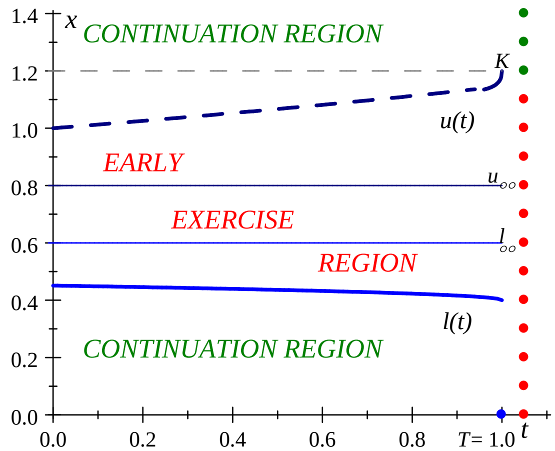
In Figure 2 we plot the double free boundary for an American put option with a negative interest rate. The dashed part of the upper free boundary is obtained via binomial approximation. The solid lines correspond to the asymptotic approximation (The binomial approximation of the lower free boundary coincides numerically with the parabolic asymptotic approximation for the entire life of the option). Red (green) circles indicate the

[4] The introduction of jumps can produce effects akin to an additional dividend rate. See e.g. Boyarchenko and Levendorskiĭ (2002a), Levendorskiĭ (2004), and Levendorskiĭ (2008).

[5] The discontinuity of our non-standard lower free boundary at T parallels the discontinuity of the (unique) free boundary at T in the standard case of a non-negative interest rate that is dominated by the underlying payout rate (see e.g. Evans, Kuske, and Keller (2002) and Ingersoll (1998)). We here adapt the covered-put argument of Ingersoll (1998). Assume tradability and consider the strategy of holding the underlying asset and the put at time $\tau = T - dt$ for a small positive dt. Recall that, in our non-standard case, the interest rate ρ and the underlying payout rate $\rho - \mu$ are negative. The critical (lower) price $x^* = l(\tau)$ is the indifference point such that the value of unwinding the strategy at τ equals the present value of waiting to do so at T: $K = Ke^{-\rho dt} + x^*(\rho - \mu)\,dt$. It follows that $\lim_{dt\to 0} x^* = K\frac{\rho}{\rho-\mu}$. Notice that the covered-put argument does not apply to the upper free boundary ($u(T^-) = u(T) = K$).

exercise (no exercise) region at T.

Figure 2: The double free boundary for a put $\rho = -4\%,\ K = 1.2,\ \sigma = 20\%,\ \mu = 8\%,\ T = 1$



Conditions (2.6) and (2.7) are sufficient but not necessary for the existence of the double free boundary. In the next proposition we provide a necessary time-dependent condition for the optimality of early exercise of the put option during the life of the option when interest rates are negative. As a consequence, this condition is also necessary for the existence of a double free boundary with negative interest rates.

Proposition 2.5 ***(necessary condition for early exercise, negative interest rate).*** *If $\rho < 0$ and $\mu > 0$ a necessary condition for the optimal exercise of the finite-maturity American put option at $t \in [0;T)$ is*

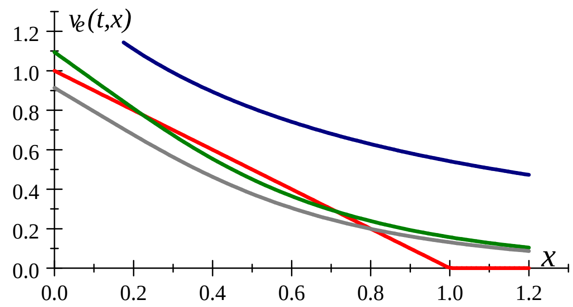
$$\mathcal{N}^{-1}\left(e^{\rho(T-t)}\right) - \mathcal{N}^{-1}\left(e^{(\rho-\mu)(T-t)}\right) \geq \sigma\sqrt{T-t}, \tag{2.13}$$

where $\mathcal{N}^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative distribution function.

Condition (2.13) requires μ, the growth rate of the underlying asset X, to be relatively high compared to the (negative) interest rate ρ, in such a way that the distance between the two quantiles $\mathcal{N}^{-1}\left(e^{\rho(T-t)}\right)$ and $\mathcal{N}^{-1}\left(e^{(\rho-\mu)(T-t)}\right)$ is at least as big as $\sigma\sqrt{T-t}$. While working towards the common objective of limiting the relative strength of ρ versus μ, condition (2.13) is a requirement milder than the sufficient conditions (2.6) and

(2.7).

Figure 3: The European finite-maturity put value $v_e(t,x)$ for $T-t=9$ and $K=1$



Gray: $\rho=1\%,\ \mu=3\%,\ \sigma=20\%,$ Green: $\rho=-1\%,\ \mu=3\%,\ \sigma=20\%,$ Blue: $\rho=-4\%,\ \mu=3\%,\ \sigma=40\%$

The intuition behind Proposition 2.5 is visualized in Figure 3: If the time t value of the European put option, $v_e(t,x)$, strictly dominates the immediate payoff function *for all* $x\geq 0$, then there is no optimal early exercise at t, since the time t value of the American put option dominates $v_e(t,x)$, that is $v(t,x)\geq v_e(t,x)>(K-x)^+$. If interest rates are non-negative, i.e. $\rho\geq 0$, this can never happen, because at $x=0$ we have that $v_e(t,0)=Ke^{-\rho(T-t)}\leq (K-0)^+=K$, and by continuity $v_e(t,x)$ lies below $(K-x)^+$ on an entire segment of non-negative underlying values (see the gray graph in Figure 3). On the contrary, when interest rates are negative, i.e. $\rho<0$, the time t value of the European put option when the underlying is 0 dominates the immediate payoff, because $v_e(t,0)=Ke^{-\rho(T-t)}>(K-0)^+=K$. Hence two alternatives are possible: Either $v_e(t,x)$ dominates the immediate payoff function *for all* $x\geq 0$ (the blue graph in Figure 3), and consequently early exercise is never optimal at date t. Or $v_e(t,x)<(K-x)^+$ for *some* $x>0$ (the green graph in Figure 3), and early exercise *might* be optimal at date t. When $\rho<0$, Equation (2.13) is equivalent to the existence of some $x>0$ such that $v_e(t,x)\leq (K-x)^+$. Equation (2.13) is therefore a *minimal* necessary condition for the possibility of optimal early exercise at date t in case of negative interest rates, that in turn implies the possible existence of a double continuation region.

3 The American call

We consider an American call option written on the log-normal asset X, whose drift under the valuation measure is positive and denoted with μ. We denote the volatility with σ, the strike with K, and the interest rate with ρ. The call value at time t is given by

$$\operatorname*{ess\,sup}_{t\leq\tau\leq T}\mathbb{E}\left[e^{-\rho(\tau-t)}\left(X(\tau)-K\right)^+\Big|\,\mathcal{F}_t\right]=v(t,X(t))$$

where

$$v(t,x)=\sup_{0\leq\Theta\leq T-t}\mathbb{E}\left[e^{-\rho\Theta}\left(x\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Theta+\sigma B(\Theta)\right)-K\right)^+\right]\tag{3.1}$$

and B is a standard Brownian motion under the valuation measure. We focus on the case $\rho < 0$.

If $\mu > 0$, the value of the perpetual call option is unbounded:

$$\begin{aligned} v(t,x) = v_\infty(x) &= \sup_{0\le\Theta} \mathbb{E}\left[e^{-\rho\Theta}\left(x\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Theta+\sigma\,B(\Theta)\right)-K\right)^+\right] \\ &\geq \sup_{0\le T} e^{-\rho T}\cdot\left(\mathbb{E}\left[x\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\,B(T)\right)\right]-K\right)^+ \\ &= \sup_{0\le T} e^{-\rho T}\left(x\cdot e^{\mu T}-K\right)^+ = +\infty \end{aligned}$$

by applying Jensen's inequality.

By contrast, for $\rho, \mu < 0$, the function v in (3.1) can be bounded also in the perpetual case, as we show in Proposition 3.2. In the finite-maturity case, v in (3.1) can be characterized as the solution of the variational inequality (2.5) with $\pi(x) = (x-K)^+$. Regardless of the sign of ρ, the function v in (3.1) dominates the call payoff ($0 \le (x-K)^+ \le v(t,x)$ for any $t \in [0;T]$ and $x \ge 0$) and is convex and increasing with respect to x for any $t \in [0;T]$. These properties are inherited from the convexity and the monotonicity of the call payoff. From the definition of v in (3.1) as a supremum on the set of stopping times from 0 up to time-to-maturity we can also deduce that, for any $x \ge 0$, the function $v(t,x)$ is decreasing with respect to t. Obviously, the finite-maturity option is dominated by the perpetual one: $v(t,x) \le v_\infty(x)$ for any $t \in [0;T]$ and $x \ge 0$. We also observe that the negative sign of ρ and μ (with the additional conditions (3.2) and (3.3)) prevents the function v_∞ to be dominated by the identity function, i.e. the standard inequality $v_\infty(x) \le x$ does not hold true, as opposite to the case depicted in Xia and Zhou (2007).

The mentioned properties of v in (3.1) imply that the early exercise region at time t is constituted by a connected segment defined by the extremes $l(t) \le u(t) \in [0;K]$ where

$$\begin{aligned} l(t) &= \inf\left\{x \ge 0 : v(t,x) = (x-K)^+\right\} \vee K \\ u(t) &= \sup\left\{x \ge 0 : v(t,x) = (x-K)^+\right\} \end{aligned}$$

such that $v(t,x) = (x-K)^+$ for $l(t) \le x \le u(t)$ and $v(t,x) > (x-K)^+$ for $x < l(t)$ and $x > u(t)$. This entails that the continuation region at time t is splitted in two segments. We characterize the double continuation region, the early exercise region and the double free boundary in Theorem 3.3. In Proposition 3.2 we give parameter value restrictions under which the American perpetual call option is finite even when interest rates are negative. We also provide explicit expressions for the constant double free boundary.

In the finite-maturity case the lower free boundary enjoys all the property it has in the standard case, where interest rates are positive: it is decreasing, continuous and tends to the strike price at maturity. The upper free boundary is increasing, continuous everywhere but at maturity, where it is infinite.

Proposition 3.2 and Theorem 3.3 are proved by building upon (respectively) Proposition 2.2 and Theorem 2.3 and by applying the American put-call symmetry (see Carr and Chesney (1996) and Schroder (1999)). The American put-call symmetry relates the price of an American call option to the price of an American put option by swapping the initial underlying price with the strike price and the dividend yield with the interest rate. As explained by Detemple (2001), such symmetry relies on the symmetry of the distribution of the log-price of X, and on the symmetry of call and put payoffs. The change of numeraire allows to derive such property also in our case, where *both* the *interest rate* ρ and the *'dividend yield'* $\delta = \rho - \mu$ are *negative.* The negativity of both ρ and δ is crucial to determine the presence of the double continuation region. For the ease of the reader, the following proposition remaps the American put-call symmetry to our framework.

Proposition 3.1 ***(American put-call symmetry)***

Consider the American call option with strike K, interest rate ρ, underlying drift μ, underlying volatility σ, and initial underlying value x, whose value at time $t \in [0;T]$ is denoted with $v(t,x) = v_{call}(t,x;K,\rho,\mu,\sigma)$ in (3.1).

Consider the symmetric *American put option with strike $K_{put} = x$, interest rate $\rho_{put} = \rho - \mu$, underlying drift $\mu_{put} = -\mu$, underlying volatility $\sigma_{put} = \sigma$ and initial underlying value $x_{put} = K$, whose value at time $t \in [0;T]$ is denoted with $v_{put}(t, x_{put}; K_{put},\ \rho_{put},\ \mu_{put},\ \sigma_{put}) = v_{put}(t, K; x,\ \rho - \mu,\ -\mu,\ \sigma)$.*

1. *The following conditions*
$$\rho < \mu < -\frac{\sigma^2}{2} < 0, \tag{3.2}$$
$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0, \tag{3.3}$$
for ρ, μ, σ in the American call problem are equivalent to conditions (2.6) *and* (2.7) *for parameters $\rho_{put}, \mu_{put}, \sigma_{put}$ in the symmetric American put problem.*

2. *(Carr and Chesney ((1996)); Detemple (2001); Detemple (2006)) The value of the American call coincides with the value of the symmetric American put*
$$v(t,x) = v_{call}(t,x;K,\rho,\mu,\sigma) = v_{put}(t,K;x,\ \rho-\mu,\ -\mu,\ \sigma) \tag{3.4}$$
for any $t \in [0;T]$.

3. *For any $t \in [0;T]$ let $l(t)$ (resp. $u(t)$) denote the lower (resp. upper) free boundary at time t for the American call option with strike K, and parameters ρ, μ, σ. Let $l_{put}(t)$ (resp. $u_{put}(t)$) denote the lower (resp. upper) free boundary at time t for the symmetric American put with strike $K_{put} = 1$, and parameters $\rho_{put}, \mu_{put}, \sigma_{put}$. If* (3.2) *and* (3.3) *are satisfied, then for any $t \in [0;T]$ we have*
$$l(t) = \frac{K}{u_{put}(t)}, \tag{3.5}$$
$$u(t) = \frac{K}{l_{put}(t)}. \tag{3.6}$$

We employ Proposition 3.1 to study the double free boundary for the American call option. Proposition 3.2 focuses on the perpetual case. Theorem 3.3 deals with the finite-maturity case and Theorem 3.4 provides the asymptotic behavior of the upper and lower free boundaries at maturity.

Proposition 3.2 ***(Perpetual call, negative interest rate)*** *If $T = +\infty$, and conditions (3.2) and (3.3) hold, then the perpetual American call option value is*

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0; l_\infty) \\ x - K & \text{for } x \in [l_\infty; u_\infty] \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty; +\infty) \end{cases}$$

where $\xi_l > \xi_u > 1$ are the positive solutions of the equation (2.9). The double free boundary is given by the constant l_∞, u_∞ defined in (2.10), and the constant A_l and A_u are given by equation (2.11).

Theorem 3.3 ***(Continuation region and free boundary characterization, finite-maturity call, negative interest rate)***

Under conditions (3.2) and (3.3), for any $t \in [0; T)$ there exist

$$l(t) \leq l_\infty < u_\infty \leq u(t)$$

such that $(x - K)^+ = v(t, x)$ for any $x \in [l(t); u(t)]$ and $(x - K)^+ < v(t, x)$ for any $x \notin [l(t); u(t)]$.

The lower free boundary $l : [0; T] \to [K; l_\infty)$ is decreasing, continuous for any $t \in [0; T]$, and $l(T) = l(T^-) = K$.

The upper free boundary $u : [0; T) \to \left(u_\infty; \frac{\rho K}{\rho - \mu}\right]$ is increasing, continuous for any $t \in [0; T)$, with $u(T^-) = \frac{\rho K}{\rho - \mu} > K$ and $u(T) = +\infty$.

The early exercise region $\mathcal{ER} = \{(t, x) \in [0; T] \times [0; +\infty[: l(t) \leq x \leq u(t)\}$ and the double continuation region is $\mathcal{CR} = \{(t, x) \in [0; T] \times [0; +\infty[: 0 \leq x < l(t) \text{ or } x > u(t)\}$, where $\{(t, l(t)); (t, u(t)) : t \in [0; T]\}$ is the double free boundary.

Theorem 3.4 ***(Asymptotic behavior of the free boundaries at maturity, call, negative interest rate)***

Under conditions (3.2) and (3.3), for $t \to T$ the upper free boundary satisfies

$$u(t) - \frac{\rho K}{\rho - \mu} \sim y^* \sigma \sqrt{(T - t)}.$$

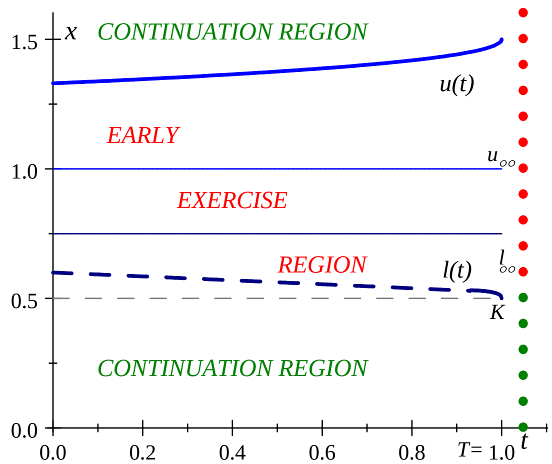
For $t \to T$, the lower free boundary satisfies

$$l(t) - K \sim K\sigma \sqrt{(T - t) \ln \frac{\sigma^2}{8\pi (T - t) \mu^2}},$$

where $y^* \approx -0.638$ is defined in Theorem 2.4.

In Figure 4 we plot the double free boundary for an American call option with a negative interest rate. The dashed part of the lower free boundary is obtained via binomial approximation. The solid lines correspond to the asymptotic approximation.

Figure 4: Double free boundary for a call with $\rho = -12\%,\ K = 0.5, \sigma = 20\%,\ \mu = -8\%,\ T = 1$



Red (green) circles indicate the exercise (no exercise) region at T.

Conditions (3.2) and (3.3) are sufficient but not necessary for the existence of a double free boundary for the call option. A necessary condition for optimal exercise at t is $\mathcal{N}^{-1}\left(e^{-(\mu-\rho)(T-t)}\right) - \mathcal{N}^{-1}\left(e^{\rho(T-t)}\right) \geq \sigma\sqrt{T-t}$, that can be derived by applying the put-call symmetry (Proposition 3.1) to the necessary condition for the early exercise of put options established in Proposition 2.5.

4 The gold loan

Collateralized borrowing has been seeing a huge increase after the financial crisis. Treasury bonds and stocks are the collateral usually accepted by financial institutions, but gold is increasingly being used around the world[6]. Major Indian non-banking financial companies like Muthoot Finance and Manappuram Finance have been quite active in lending against gold collateral. As Churiwal and Shreni (2012) report in their survey of the Indian gold loan market, gold loans tend to have short maturities and rather high spreads (borrowing rate minus riskfree rate), even if significantly lower than without collateral. The prepayment option is common, permitting the redemption of the gold at any time before maturity. We emphasize that gold loans noticeably differ from stock loans, because gold is a tradable investment asset with storage/insurance costs and without earnings. This can lead to peculiar redemption policies that constitute an interesting application of our results in Proposition 3.2 and Theorems 3.3 and 3.4.

In a gold loan, the borrower receives at time 0 (the date of contract inception) the loan amount $q > 0$ using one mass unit (one troy ounce, say) of gold as collateral. This amount grows at the rate γ, where γ is a

[6]For example see "J.P. Morgan Will Accept Gold as Type of Collateral" (Wall Street Journal, Commodities, February 8, 2011), reported by C. Cui and R. Hoyle.

constant borrowing rate (higher than the riskfree rate r) stipulated in the contract, and the cost of reimbursing the loan at time t is thus given by $qe^{\gamma t}$. When paying back the loan, the borrower regains the gold and the contract is terminated. We assume that the costs of storing and insuring gold holdings are $Gc > 0$ per unit of time, where G is the gold spot price. Consistently, the dynamics of G under the risk-neutral measure $\mathbf{Q}$ is assumed to be

$$\frac{dG(t)}{G(t)} = (r + c)\, dt + \sigma dW(t),$$

where r is the constant riskless rate, σ is the gold returns' volatility, and W is a Brownian motion under the risk-neutral measure $\mathbf{Q}$ (see for instance Hull (2011)). Given a finite maturity T, the value of the redemption option at date 0 is

$$\begin{aligned} C(0, G(0)) &= \sup_{0 \leq \tau \leq T} \mathbb{E}^{\mathbf{Q}} \left[e^{-r\tau} \left(G(\tau) - qe^{\gamma\tau} \right)^{+} \right] \\ &= \sup_{0 \leq \tau \leq T} \mathbb{E}^{\mathbf{Q}} \left[e^{-(r-\gamma)\tau} \left(X(\tau) - q \right)^{+} \right] \end{aligned}$$

where $X(t) = G(t)\, e^{-\gamma t}$ is the *gold price deflated at the rate* γ. Therefore, the initial value of the redemption option of a gold loan is the initial value of an American call option in (3.1) on the lognormal underlying X with parameters

$$\begin{aligned} \rho &= r - \gamma < 0 \\ \mu &= r + c - \gamma \\ K &= q. \end{aligned}$$

Similarly, the value of the redemption option at any date $t \in [0; T]$ can be computed as $C(t, G(t)) = v(t, X(t))$, with v defined in (3.1) . The percentage storage and insurance costs c are positive and usually below the spread $\gamma - r > 0$. Hence, we posit $\rho < \mu < 0$. If conditions (3.2) and (3.3) are also verified, i.e.

$$r - \gamma < r - \gamma + c < -\frac{\sigma^2}{2} \text{ and } \left(r - \gamma + c - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 (r - \gamma) > 0$$

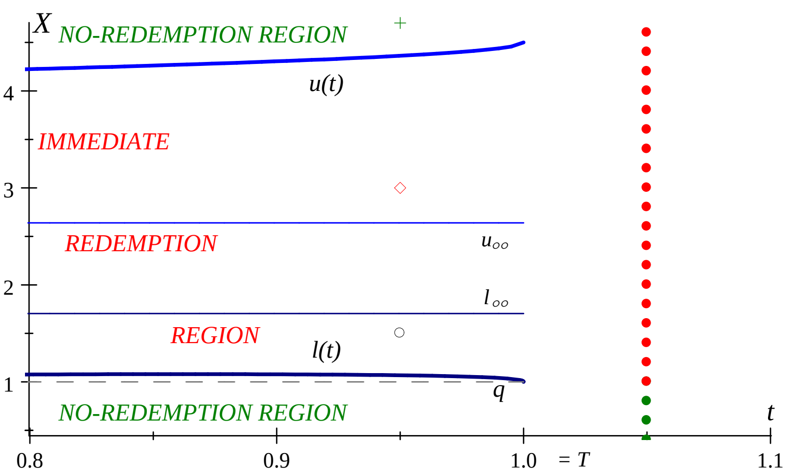
a *double no-redemption region* appears in the perpetual case, as by Proposition 3.2. Using the same proposition, we can compute the perpetual constant free boundaries l_∞ and u_∞ in terms of the deflated gold price process $X(t) = G(t)\, e^{-\gamma t}$. For finite-maturity contracts, Theorem 3.4 provides the asymptotic approximation of the double free boundaries near maturity. Churiwal and Shreni (2012) show that maturities for gold loans are generally within 36 months. Borrowing rates typically range from 12% to 16% for banks and from 12% to 24% for specialized institutions, whereas the yield on Indian short-term government bonds[7] has been hovering around 8%. Data from the Gold World Council[8] show that the daily log change in the gold spot price expressed in Indian rupees has registered an annualized historical volatility of 21.4% over the period from the 3rd of January

[7] The source is the Government Securities Market Section of the Reserve Bank of India DataBase on The Indian Economy (RBI's DBIE, http://dbie.rbi.org.in).

[8] http://www.gold.org/investment/statistics/.

1979 to the 5th of May 2013. Average storage/insurance costs are about[9] 2%. By fixing $r = 8\%$, $c = 2\%$, $\gamma = 17\%$, and $\sigma = 21.4\%$ the mentioned parametric restrictions are met. Given quantities normalized by the loan amount ($q = 1$), Figure 5 visualizes the perpetual double free boundary ($l_\infty = 1.70$ and $u_\infty = 2.64$) and the proxied finite-maturity double free boundary ($l(t)$ and $u(t)$ for t close to the expiry date $T = 1$ expressed in years), as by Theorem 3.4. Figure 5 highlights that the two perpetual free boundaries are a poor proxy for the two finite-maturity free boundaries near expiration. For instance, if at $t = 0.95$ the deflated gold price X is equal to 3 (the point denoted with a red diamond in Figure 5), the perpetual boundaries suggest to delay the gold loan redemption (the red diamond lies outside the *perpetual* immediate-redemption region), though the asymptotic approximation of the double free boundary implies optimal immediate redemption (the red diamond lies inside the immediate redemption region). Binomial-tree calculations show that the relative welfare loss associated to suboptimal delay is 3 basis points of the immediate-redemption value. A similar but lesser deep-in-the-money situation is represented in Figure 5 by the point denoted with a black circle ($X = 1.5$ at $t = 0.95$). The relative welfare loss from suboptimal delay in this case is of 23 basis points. Conversely, if the deflated gold price X is 4.7 at $t = 0.95$ (the point denoted with a green cross in Figure 5), it is optimal to postpone the gold redemption even though the redemption option is quite deep in the money and very short-lived.

Figure 5: Double no-redemption region of a gold loan near maturity



The parameter values are: $T = 1$, $r = 8\%$, $c = 2\%$, $\gamma = 17\%$, $\sigma = 21.4\%$, and $q = 1$.

Red (green) circles indicate the redemption (no redemption) region at T.

[9]The cost of leasing a bank safety locker and of insuring the jewellery kept in it adds up to about 2% of the sum assured ("Protect your riches", by Chandralekha Mukerji, *Money Today*, August 2012).

5 Capital investment options

This example closely follows the setup in Battauz et al. (2012), who consider exclusively the perpetual case. By contrast, we focus here on the finite-maturity case and on the behavior of the double free boundary near maturity. Uncertainty is described by the historical probability space $(\Omega, \mathbf{P}, (\mathcal{F}_t)_t)$. The present value V of the project and the investment cost I have lognormal dynamics under the historical probability measure $\mathbf{P}$ (see Dixit and Pindyck (1993) for a classical review of risky investment and Aase (2010) for a recent survey). The firm's management decides when to disburse the irreversible investment cost I to undertake the project. Risk adjustment corresponds to choosing the valuation measure $\hat{\mathbf{P}}$ (equivalent to $\mathbf{P}$) by the firm's management. The discount rate $\widehat{r}$ is also selected by the firm's management. The $\hat{\mathbf{P}}-$dynamics of V is

$$\mathrm{d}V_t = V_t \left(\widehat{\mu}_V \,\mathrm{d}t + \sigma_V \,\mathrm{d}W_t^{\hat{\mathbf{P}}} + \widetilde{\sigma}_V \,\mathrm{d}\widetilde{W}_t^{\hat{\mathbf{P}}} \right),$$

where $\widehat{\mu}_V$, σ_V, and $\tilde{\sigma}_V$ are real positive constants. The investment cost I has $\hat{\mathbf{P}}-$dynamics

$$\mathrm{d}I_t = I_t \left(\widehat{\mu}_I \,\mathrm{d}t + \sigma_I \,\mathrm{d}W_t^{\hat{\mathbf{P}}} \right),$$

where $\widehat{\mu}_I$ and σ_I, are real positive constants, and $W^{\hat{\mathbf{P}}}$, $\widetilde{W}^{\hat{\mathbf{P}}}$ are $\hat{\mathbf{P}}-$independent Brownian motions.

Access to the project is possible only up to the date T. Thus, at any date $t \in [0, T]$ the management evaluates the t-dated *value of the option to invest*

$$\operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}^{\hat{\mathbf{P}}} \left[e^{-\widehat{r}(\tau - t)} (V_\tau - I_\tau)^+ \middle| \mathcal{F}_t \right]. \tag{5.1}$$

The real option problem (5.1) can be reduced to a one-dimensional American put option by taking the process $V_t e^{\rho t}$ as numeraire (see Battauz (2002), Carr (1995), and Geman et al. (1995)), where

$$\rho = -\left(\widehat{\mu}_V - \widehat{r} \right)$$

is the opposite of V's expected growth rate (under $\hat{\mathbf{P}}$) in excess of the discount rate $\widehat{r}$. Indeed, denoting with $\mathbf{P}^V$ the probability measure associated to the numeraire $V_t e^{\rho t}$, whose Radon-Nikodym derivative with respect to the probability measure $\hat{\mathbf{P}}$ is $\frac{\mathrm{d}\mathbf{P}^V}{\mathrm{d}\hat{\mathbf{P}}} = \frac{V_T e^{\rho T}}{V_0 e^{\widehat{r} T}}$, the problem (5.1) can be written as

$$\operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}^{\hat{\mathbf{P}}} \left[e^{-\widehat{r}(\tau - t)} (V_\tau - I_\tau)^+ \middle| \mathcal{F}_t \right] = V_t \cdot v(t, X_t), \tag{5.2}$$

with

$$v(t, X_t) = \operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}^{\mathbf{P}^V} \left[e^{-\rho(\tau - t)} \left(1 - X_\tau \right)^+ \middle| \mathcal{F}_t \right] \tag{5.3}$$

and

$$X_t = \frac{I_t}{V_t}.$$

The underlying of the put option in (5.3) is the lognormal cost-to-value ratio X, that under the probability measure $\mathbf{P}^V$ can be written as

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right),$$

where B_t is a $\mathbf{P}^V$-Brownian motion, and where

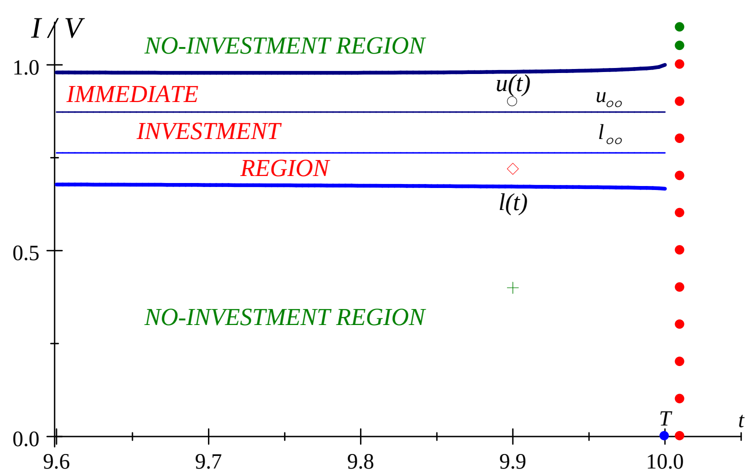
$$\sigma^2 = (\sigma_I - \sigma_V)^2 + \widetilde{\sigma}_V^2,$$
$$\mu = \widehat{\mu}_I - \widehat{\mu}_V.$$

The parameter $\rho = -(\widehat{\mu}_V - \widehat{r})$ plays in (5.3) the role of the interest rate. Consider now the case of a highly profitable project for which

$$\widehat{\mu}_V > \widehat{r}.$$

This case is usually neglected by the literature on real options, because it can lead to an explosive option value in the perpetual case (see Battauz at al. (2012) for a detailed discussion). In the finite maturity case, if $\mu = \widehat{\mu}_I - \widehat{\mu}_V < 0$, the option is optimally exercised only at maturity T. On the contrary, if $\mu = \widehat{\mu}_I - \widehat{\mu}_V > 0$, Theorem 2.3 shows that early exercise can be optimal, and that the early exercise region is surrounded by a double continuation region. Investments in nuclear plants are an interesting area of possible application. The business is extremely lucrative, but the overall cost of entering it is likely to increase markedly in the future (demand for nuclear plant safety is definitely rising). This may cause the cost of entering a nuclear energy project to grow at a higher expected rate than the value of the project itself, leading to $\mu = \widehat{\mu}_I - \widehat{\mu}_V > 0$.

Figure 6: Double free boundary for a capital investment option near maturity



The parameter values are: $T = 10$, $\widehat{r} = 3\%$, $\widehat{\mu}_V = 5\%$, $\sigma_V = 7\%$, $\widetilde{\sigma}_V = 3\%$, $\widehat{\mu}_I = 6\%$, $\sigma_I = 10\%$.

Red (green) circles indicate the investment (no investment) region at T.

For instance, with $\widehat{r} = 3\%$, $\widehat{\mu}_V = 5\%$, $\sigma_V = 7\%$, $\widetilde{\sigma}_V = 3\%$, $\widehat{\mu}_I = 6\%$, and $\sigma_I = 10\%$ (see[10] Figure 6), we get $\rho = -(\widehat{\mu}_V - \widehat{r}) = -2\%$, $\sigma = 4.242\%$, and $\mu = 1\%$. Conditions (2.6) and (2.7) are met, and Proposition 2.2 delivers the two perpetual free boundaries $l_\infty = 0.763$ and $u_\infty = 0.873$. Suppose that the option has

[10]The seminal work of McDonald and Siegel (1986) also deals with risk for both the value V and the cost I. With the key difference of a distinct hierarchy for the discount and growth rates, the parameter values for the risk-adjusted processes of V and I employed in Figure 6 are in the same range as those used by McDonald and Siegel (1986), see e.g. their Tables I and II at p. 720.

ten years to maturity, i.e. $T = 10$. Theorem 2.4 enables the investigation of the double free boundary near maturity. In Figure 6 the double free boundary is plotted for $t \in [9.6; 10]$, i.e. when only 4.8 months are left to expiration. At $t = 9.9$, if the cost-to-value ratio I/V is 0.72 (the red diamond in Figure 6), immediate investment is optimal. The perpetual double free boundary is a poor proxy for the double free boundary near expiration and implies a delayed investment (the red diamond lies outside the *perpetual* immediate investment region). Binomial-tree calculations show that the relative welfare loss associated to suboptimal adjournment is 1 basis points of the immediate-exercise value. A similar but lesser deep-in-the-money situation is depicted in Figure 6 by the black circle ($I/V = 0.9$ at $t = 9.9$). The relative welfare loss from suboptimal deferment in this case is of 15 basis points. Conversely, if the cost-to-value ratio I/V is 0.4 at $t = 9.9$ (the point green cross in Figure 6), the firm must postpone the investment, even if the investment option is quite deep in the money and definitely short-lived.

6 Conclusions

The significance of American option problems with an endogenous negative interest rate is conspicuous as they are reformulations of the option-like features of popular secured loans and of relevant capital budgeting problems. For finite-maturity and perpetual American puts and calls with a negative interest rate, we study in detail the conditions that bring about a non-standard double continuation region (option exercise is optimally delayed if moneyness is insufficient and, in a non-standard fashion, if it is overly sufficient) and comprehensively investigate the properties (existence, monotonicity, continuity, limits and behavior close to maturity) enjoyed by the double free boundary that separates the early-exercise region from the double continuation region.

Our study contributes a substantial extension of the standard optimal exercise properties for American options and covers the exact necessary/sufficient conditions that empower optimal early exercise of an American call with a negative underlying payout rate. We also contribute to the extant literature on the optimal redeeming strategy of tradable securities used as loan collateral as we characterize the double continuation region implicit in the gold loan, a thriving form of collateralized borrowing. Real options that combine strong expected growth for the project values with a marked escalation of the investment costs provide another distinct area of application for our results.

Several promising avenues of further research emerge, with an interesting mix of economic and technical challenges. They include studying the impact on non-standard optimal exercise policies of diffusive stochastic volatility, jump risk, and drift-parameter uncertainty. We plan to pursue them in future work.

7 Appendix

Proof of Proposition 2.1. See the proofs of Theorem 3.6 and of Corollary 3.7 in Jaillet et al. (1990) and note that, for $\rho < 0$, the discount factor is positive and bounded by $e^{-\rho T}$.■

Proof of Proposition 2.2.

The proof follows the same arguments of Battauz et al. (2012) while adapting them to our framework. The function v_∞ is retrieved by plugging into the differential equation in (2.5) for the continuation region the educated guess $v_\infty(x) \propto x^\xi$. This leads to equation (2.9) for the parameter ξ. Smooth pasting and value matching deliver the constants A_l, A_u and the two free boundaries l_∞, u_∞. The authors then verify that v_∞ defined in (2.8) and $\tau^* = \inf\{t \geq 0 : l_\infty \leq X_t \leq u_\infty\}$ satisfy

$$(a) \quad v_\infty(x) = \mathbb{E}\left[e^{-\rho\tau^*} v_\infty(X_{\tau^*})\right],$$
$$(b) \quad v_\infty(x) \geq \mathbb{E}\left[e^{-\rho\tau} v_\infty(X_\tau)\ \right] \quad \text{for any stopping time } \tau.$$

Such direct verification is needed because v_∞ violates the usual boundedness requirements (a typical boundedness assumption requires the existence of an integrable random variable H such that the inequality

$$e^{-\rho(\tau^* \wedge \tau \wedge t)} v_\infty(X_{\tau^* \wedge \tau \wedge t}) < H$$

holds almost surely for all $\mathcal{F}-$stopping times τ and for all $t > 0$). We look for negative values of the parameter ξ to capture the monotonicity property of v_∞. If the conditions (2.6) and (2.7) hold true, the equation (2.9) admits two negative solutions $\xi_{u,l} = \frac{-\left(\mu-\frac{\sigma^2}{2}\right) \mp \sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}}{\sigma^2}$, which define the two constant free boundaries $l_\infty = \frac{K\xi_l}{\xi_l - 1}$ and $u_\infty = \frac{K\xi_u}{\xi_u - 1}$. If the conditions (2.6) and $\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 = 0$ hold true, the equation (2.9) has a unique negative solution$\xi = \frac{-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}$, which defines the constant free boundary $l_\infty = u_\infty = \frac{K\xi}{\xi-1}$. In the early exercise region $\mathcal{ER}$ the function v_∞ defined in (2.8) coincides with the put payoff. It is important to verify that the process $\left\{v_\infty(X(t))e^{-\rho t}\right\}_t$ is a supermartingale also in $\mathcal{ER}$, because the variational inequality $\frac{\partial}{\partial t}v + \mathcal{L}v - \rho v \leq 0$ in (2.5) must hold on the whole $(0;T) \times \Re^+$. More precisely, on the early exercise region the variational inequality in (2.5) yields $\frac{\partial}{\partial t}v + \mathcal{L}v - \rho v = \frac{1}{2}\sigma^2 x^2 \cdot 0 + \mu x \cdot (-1) - \rho\,(K - x)^+ = x \cdot (\rho - \mu) - \rho K \leq 0$ for all $x \in [l_\infty\,; u_\infty]$. Since $\rho - \mu < 0$, the inequality is satisfied in $\mathcal{ER}$ if and only if $l_\infty \cdot (\rho - \mu) - \rho K \leq 0$. By substituting the expression for $l_\infty = \frac{K\xi_l}{\xi_l - 1}$, we see that the above inequality is satisfied if and only if $\frac{\xi_l}{\xi_l - 1} \cdot (\rho - \mu) - \rho \leq 0$, equivalent to $\xi_l \leq \frac{\rho}{\mu}$, and, by substituting the explicit expression for ξ_l, equivalent to $-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2} \leq \frac{\rho}{\mu}\sigma^2$. Under the condition (2.7) the last inequality is equivalent to the system

$$\begin{cases} \frac{\rho}{\mu}\sigma^2 + \left(\mu - \frac{\sigma^2}{2}\right) \geq 0 \\ \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 \leq \left(\frac{\rho}{\mu}\sigma^2 + \left(\mu - \frac{\sigma^2}{2}\right)\right)^2 \end{cases}.$$

The inequality in the second row is satisfied under our assumptions, because it is equivalent to: $2\rho\sigma^2 \leq \left(\frac{\rho}{\mu}\sigma^2\right)^2 + 2\frac{\rho}{\mu}\sigma^2\left(\mu - \frac{\sigma^2}{2}\right) \Leftrightarrow \left(\frac{\rho}{\mu}\sigma^2\right)^2 + 2\frac{\rho}{\mu}\sigma^2\left(\mu - \frac{\sigma^2}{2}\right) - 2\rho\sigma^2 \geq 0 \Leftrightarrow \rho\sigma^2 + 2\mu\left(\mu - \frac{\sigma^2}{2}\right) - 2\mu^2 \leq 0 \Leftrightarrow \rho\sigma^2 - \mu\sigma^2 \leq 0$ $\Leftrightarrow \rho \leq \mu$. We are left with verifying the inequality in the first row:

$$\frac{\rho}{\mu}\sigma^2 + \left(\mu - \frac{\sigma^2}{2}\right) \geq 0 \Longleftrightarrow \rho \geq -\frac{\mu}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right).$$

In fact, the condition (2.7) implies $\rho > -\frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{2\sigma^2} > -\frac{\mu}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)$, because $-\frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{2\sigma^2} > -\frac{\mu}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)$ is equivalent to $\frac{\mu-\frac{\sigma^2}{2}}{2} < \mu \Longleftrightarrow \mu - \frac{\sigma^2}{2} < 2\mu \Longleftrightarrow \mu + \frac{\sigma^2}{2} > 0$, which is the condition (2.6).■

Proof of Theorem 2.3. The two branches l and u of the double free boundary are defined in (2.3) and (2.4). We start by proving inequality (2.12). Under our assumptions, the function v_∞ and the constants l_∞ and u_∞ are well defined and the strict inequality $l_\infty < u_\infty$ holds because the inequality in (2.7) is strict, leading to $\xi_u < \xi_l < 0$ and therefore to $l_\infty < u_\infty$. The strict inequality $l(t) < u(t)$ for any $t \in [0;T]$ in (2.12) follows from the chain $l(t) \le l_\infty < u_\infty \le u(t)$.

To show that $l(t) \le l_\infty$ and that $u(t) \ge u_\infty$ for any $t \in [0;T]$ it is sufficient to observe that

$$\left\{x : v_\infty(x) > (K-x)^+\right\} \supset \left\{x : v(t,x) > (K-x)^+\right\}$$

for any fixed t, because $y \in \{x : v(t,x) > (K-x)^+\}$ implies $v_\infty(y) \ge v(t,y) > (K-x)^+$. Taking the complement sets, the above inclusion is reversed and we get

$$\left\{x : v_\infty(x) = (K-x)^+\right\} \subset \left\{x : v(t,x) = (K-x)^+\right\}.$$

By passing to the infimum, this inclusion leads to $l_\infty \ge l(t)$, and by passing at the supremum we get $u_\infty \le u(t)$. Next, we prove that $l(t) \ge \frac{\rho K}{\rho-\mu}$ for any $t \in [0;T)$. We observe that any (t,x) in the exercise region $\mathcal{ER}$ satisfies the inequality $\frac{\partial}{\partial t}v + \mathcal{L}v - \rho v \le 0$ in (2.5). Since for $(t,x) \in \mathcal{ER}$ we have $v(t,x) = K - x$, the inequality simplifies to $-\mu x - \rho(K-x) = (\rho-\mu)x - \rho K \le 0$, that is $x \ge \frac{\rho K}{\rho-\mu} > 0$ for any $(t,x) \in \mathcal{ER}$. By passing to the infimum over x for any fixed t in the previous inequality we get that $l(t) \ge \frac{\rho K}{\rho-\mu}$.

We now prove the monotonicity properties of l and u. We first show that l is decreasing. Let $t' < t''$. We have $(K - l(t'))^+ \le v(t'', l(t')) \le v(t', l(t')) = (K - l(t'))^+$, where the first inequality follows from $v(t'',\cdot) \ge (K-\cdot)^+$, the second one from the fact that $v(\cdot, l(t'))$ is decreasing, and the last equality from the definition of $l(t')$. As a consequence $v(t'', l(t')) = (K - l(t'))^+$, and therefore $l(t'') \le l(t')$.

To show that u is increasing, let $t' < t''$. We exploit the monotonicity properties of v and we get $(K - u(t'))^+ = v(t', u(t')) \ge v(t'', u(t')) \ge (K - u(t'))^+$. Therefore $v(t'', u(t')) = (K - u(t'))^+$, and, consequently, $u(t'') \ge u(t')$.

The next step is to prove that at maturity $l(T) = 0$ and $u(T) = K$. To show that $l(T) = 0$, we observe that, since $v(T,x) = (K-x)^+$ for all $x \ge 0$, $l(T) = \inf\left\{x \ge 0 : v(T,x) = (K-x)^+\right\} = \inf\{x \ge 0\} = 0$. The equality $u(T) = K$ follows from $u(T) = \sup\left\{x \ge 0 : v(T,x) = (K-x)^+\right\} \wedge K = \sup\{x \ge 0\} \wedge K = K$.

We now show that $u(T^-) = K = u(T)$ and $l(T^-) = \frac{\rho K}{\rho-\mu} > 0 = l(T)$. By construction $u(t) \le K$ for all $t \in [0;T]$, and hence $u(T^-) \le K$. Suppose by contradiction that $u(T^-) < K$. The set $(0;T) \times (u(T^-);K) \subset \mathcal{CR}$ and therefore $(\mathcal{L} - \rho)v = -\frac{\partial}{\partial t}v \ge 0$. As $t \uparrow T$ we have

$$(\mathcal{L}-\rho)v \to (\mathcal{L}-\rho)(K-x) = (\rho-\mu)x - \rho K \text{ for } x \in \left(u\left(T^-\right);K\right).$$

But then we have $(\rho-\mu)x - \rho K \ge 0$ for $x \in (u(T^-);K)$ and therefore $(\rho-\mu)u(T^-) - \rho K \ge 0 \Longrightarrow u(T^-) \le \frac{\rho K}{\rho-\mu}$, delivering the contradiction.

Suppose now (by contradiction) that $l(T^-) > \frac{\rho K}{\rho-\mu}$. The set $(0;T) \times (0; l(T^-)) \subset \mathcal{CR}$ and hence

$$(\mathcal{L}-\rho)v = -\frac{\partial}{\partial t}v \ge 0 \text{ for } x \in \left(\frac{\rho K}{\rho-\mu}; l\left(T^-\right)\right) \subset \left(0; l\left(T^-\right)\right).$$

As $t \uparrow T$ we have

$$(\mathcal{L}-\rho)\,v \to (\mathcal{L}-\rho)\,(K-x) = (\rho-\mu)\,x - \rho K \text{ for } x \in \left(\frac{\rho K}{\rho-\mu}; l\left(T^{-}\right)\right),$$

where the limit is in the sense of distributions. We hence have $(\rho-\mu)\,x-\rho K \geq 0$ for $x \in \left(\frac{\rho K}{\rho-\mu}; l\left(T^{-}\right)\right)$, that is

$$(-\rho+\mu)\,x + \rho K \leq 0 \text{ for } x \in \left(\frac{\rho K}{\rho-\mu}; l\left(T^{-}\right)\right).$$

which delivers the contradiction because $x \geq \frac{\rho K}{\rho-\mu}$ implies $(-\rho+\mu)\,x+\rho K \geq (-\rho+\mu)\,\frac{\rho K}{\rho-\mu}+\rho K = 0$.

We finally deal with the continuity of the l and u. The argument for u is the same as the one used by Lamberton and Mikou (2008), so that we omit it. We show instead how to prove the continuity of l. The *right* continuity of l follows from the monotonicity of l, and the continuity of v and $(K-\cdot)^{+}$. Indeed, since l is decreasing, we have, for any sequence $t_n \downarrow t \in [0;T]$, that $\lim_{t_n \downarrow t} l(t_n) \leq l(t)$. Because of the definition of l, for any t_n we have the equality $v\,(t_n, l(t_n)) = (K-l(t_n))^{+}$. By the continuity of v and of the put payoff we pass to the limit and we get $v\,(t, \lim_{t_n \downarrow t} l(t_n)) = (K-\lim_{t_n \downarrow t} l(t_n))^{+}$. This equality implies that $\lim_{t_n \downarrow t} l(t_n) \geq l(t)$, and right continuity is proved.

The *left* continuity follows from the system of variational inequalities (2.5) satisfied by v. First of all we observe that if for some $\bar{t} \in [0;T)$ we have $l(\bar{t}) = \frac{\rho K}{\rho-\mu}$, then $l(t) = \frac{\rho K}{\rho-\mu}$ for all $t \in \left[\bar{t};T\right)$, because l is decreasing and bounded from below by the constant $\frac{\rho K}{\rho-\mu}$. With a small abuse of notation we denote with $\left[\bar{t};T\right)$ the (possibly empty) set in which $l(t) = \frac{\rho K}{\rho-\mu}$. On $\left[\bar{t};T\right)$ the function l is constant and therefore continuous. Let $t \in \left(0;\bar{t}\right]$ and take a generic sequence $t_n \uparrow t$. Since l is monotonically decreasing, the limit $l(t^{-}) = \lim_{t_n \uparrow t} l(t_n)$ exists and $l(t^{-}) \geq l(t)$. Suppose by contradiction that the inequality is strict, i.e. $l(t^{-}) > l(t)$. Then the open set $(0;t) \times (l(t);l(t^{-})) \subset \mathcal{CR}$ and therefore (2.5) implies $\frac{\partial}{\partial t}v + \mathcal{L}v - \rho v = 0$, that is $\mathcal{L}v - \rho v = -\frac{\partial}{\partial t}v \geq 0$ on $(0;t) \times (l(t);l(t^{-}))$ where the inequality holds because v is decreasing with respect to t.

Conversely the open set $(t;T) \times (l(t);l(t^{-})) \subset \mathcal{ER}$ and therefore (2.5) implies $0 \geq \frac{\partial}{\partial t}v + \mathcal{L}v - \rho v = \mathcal{L}v - \rho v = (\rho-\mu)\,x - \rho K$ on $(t;T) \times (l(t);l(t^{-}))$, where the equalities follow from $v(t,x) = K-x$ on $\mathcal{ER}$.

Hence by continuity we get $\mathcal{L}v - \rho v = (\rho-\mu)\,x - \rho K = 0$ for any $x \in (l(t);l(t^{-}))$, that is satisfied only for $l(t) = l(t^{-}) = x = \frac{\rho K}{\rho-\mu}$, delivering the contradiction.■

Proof of Theorem 2.4. To prove the asymptotic behavior of the upper free boundary, we exploit formula (1.5) at page 221 in Evans et al. (2002) with interest rate $r=\rho$ and dividend yield $D = \rho-\mu < \rho = r < 0$. Their formula relies only on the satisfied inequality $D<r$, and is not affected by the negativity of the parameters r and D. Hence we get $u(t) \sim K - K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}}$, as $t \to T$. To prove the asymptotic behavior of the lower free boundary we exploit Remark 2 in Lamberton and Villeneuve (2003), that in our framework, applied at $-y$, and with $\vartheta = T-t$, and $\lambda := l\left(T^{-}\right)e^{-\sigma y\sqrt{\vartheta}}$, implies

$$v\left(T-\vartheta;\lambda\right) = (K-\lambda)^{+} + \vartheta^{\frac{3}{2}}\left|\rho\right| K\sigma\phi\left(y\right) + o\left(\vartheta^{\frac{3}{2}}\right)$$

for $y > y^{*}$, since $\frac{\partial}{\partial x}\left(-\rho K e^{-\rho t} + (\rho-\mu)\,e^{-\left(\rho-\mu+\frac{\sigma^2}{2}\right)t+\sigma x}\right)\Big|_{\left(0;\frac{1}{\sigma}\ln\frac{\rho K}{\rho-\mu}\right)} = \rho K\sigma < 0$. (This equation substitutes equation (2) in the proof of Theorem 2 in Lamberton and Villeneuve (2003). For convenience of the reader, their notation is: $r = \rho$, $\delta = \rho - \mu$, $t_0 = 0$, $x_0 = \frac{1}{\sigma} \ln \frac{\rho K}{\rho - \mu}$, $f(t,x) = e^{-\rho t} \left(K - e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x} \right)^+$, $\mathcal{D}f(t,x) = -\rho K e^{-\rho t} + (\rho - \mu) e^{-\left(\rho - \mu + \frac{\sigma^2}{2}\right)t + \sigma x}$. Theorem 2 in Lamberton and Villeneuve (2003) cannot be used in our case to find out the beahviour of the lower free boundary, since it relies on the non-negativity of interest rates). Since $\phi(y) > 0$, it follows that $v(T - \vartheta; \lambda) > (K - \lambda)^+$. Hence $(T - \vartheta; \lambda) = \left(T - \vartheta; l(T^-) e^{-\sigma y \sqrt{\vartheta}}\right) \in \mathcal{CR}$ and for ϑ small enough this is equivalent to say that

$$\lambda = l\left(T^-\right) e^{-\sigma y \sqrt{\vartheta}} < l(T - \vartheta).$$

Note that the inequality is here reversed with respect to the standard case of a unique (upper) free boundary. Passing to the log we get

$$\ln l\left(T^-\right) - \sigma y \sqrt{\vartheta} < \ln l(T - \vartheta)$$
$$\ln l\left(T^-\right) - \ln l(T - \vartheta) < \sigma y \sqrt{\vartheta}$$

and therefore

$$\limsup_{t \to T} \frac{l(T^-) - l(t)}{l(T^-)\, \sigma \sqrt{(T - t)}} \leq y.$$

Since the inequality holds for all $y > y^*$, we get

$$\limsup_{t \to T} \frac{l(T^-) - l(t)}{l(T^-)\, \sigma \sqrt{(T - t)}} \leq y^*.$$

We now prove the opposite inequality for $y \leq y^*$. If for all $y \leq y^*$ and $\vartheta = T - t \to 0$

$$l(T - \vartheta) \leq l\left(T^-\right) e^{-\sigma y \sqrt{\vartheta}} \approx l\left(T^-\right) \left(1 - y \sigma \sqrt{\vartheta}\right)$$

the proof is complete. Hence, suppose now that

$$l(T - \vartheta) > \lambda = l\left(T^-\right) e^{-\sigma y \sqrt{\vartheta}}.$$

This means that $(T - \vartheta; \lambda) \in \mathcal{CR}$. We exploit again Remark 2 in Lamberton and Villeneuve (2003) applied to $-y$ (instead of y) that implies

$$\varphi(\vartheta) = v(T - \vartheta; \lambda) = (K - \lambda)^+ + g(\vartheta) \text{ with } g(\vartheta) = o\left(\vartheta^{\frac{3}{2}}\right) > 0,$$

where the positivity of $g(\vartheta)$ follows from the fact that $\lambda \in \mathcal{CR}$. The *smooth fit property* (Proposition 2.1) allows to find $\zeta \in (\lambda; l(T - \vartheta))$ such that

$$v(T - \vartheta; \lambda) - (K - \lambda) = \frac{(l(T - \vartheta) - \lambda)^2}{2} \frac{\partial^2 v}{\partial x^2} (T - \vartheta; \zeta). \tag{7.1}$$

Indeed, since v admits continuous first order derivative w.r.t. x and there exists $\frac{\partial^2 v}{\partial x^2}(T - \vartheta; x)$ for all $x \in (\lambda; l(T - \vartheta))$, we can apply a Taylor expansion with the Lagrange remainder for $x = \lambda$ and $\widehat{x}_0 = l(T - \vartheta)$ to conclude that

$$v(T - \vartheta; x) = v(T - \vartheta; \widehat{x}_0) + \frac{\partial}{\partial x} v(T - \vartheta; \widehat{x}_0)(x - \widehat{x}_0) + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} (T - \vartheta; \zeta)(x - \widehat{x}_0)^2$$

for some $\zeta \in (x; \widehat{x}_0) = (\lambda; l(T-\vartheta))$. Since

$$v(T-\vartheta; \widehat{x}_0) = v(T-\vartheta; l(T-\vartheta)) = K - l(T-\vartheta)$$
$$\frac{\partial}{\partial x} v(T-\vartheta; \widehat{x}_0) = \frac{\partial}{\partial x} v(T-\vartheta; l(T-\vartheta)) = -1,$$

the Taylor expansion delivers (7.1).

As $\zeta \in (\lambda; l(T-\vartheta))$, we have that $(T-\vartheta, \zeta) \in \mathcal{CR}$ and therefore

$$-\frac{\partial}{\partial \vartheta} v + \mathcal{L}v - \rho v = 0 \text{ for } (t,x) = (T-\vartheta; \zeta)$$

with $(\mathcal{L}v)(t,x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} v(t,x) + \mu x \frac{\partial}{\partial x} v(t,x)$. From the PDE at $(t,x) = (T-\vartheta; \zeta)$ we derive that

$$\begin{aligned}
\frac{1}{2}\sigma^2\zeta^2 \frac{\partial^2 v}{\partial x^2}(T-\vartheta;\zeta) &= \frac{\partial}{\partial \vartheta} v(T-\vartheta;\zeta) - \mu\zeta \frac{\partial}{\partial x} v(T-\vartheta;\zeta) + \rho v(T-\vartheta;\zeta) \\
&\geq 0 - \mu\zeta \frac{\partial}{\partial x} v(T-\vartheta;\zeta) + \rho v(T-\vartheta;\zeta) \quad \text{because } v \text{ increasing w.r.t. } \vartheta \\
&> -\mu\zeta\,(-1) + \rho v(T-\vartheta;\zeta) \ \text{ because } \frac{\partial}{\partial x} v(T-\vartheta;\zeta) \leq -1 \\
&> \mu\lambda + \rho v(T-\vartheta;\lambda) \ \text{ because } \zeta > \lambda \text{ and } v(T-\vartheta;\zeta) < v(T-\vartheta;\lambda).
\end{aligned}$$

The quantity $\mu\lambda + \rho v(T-\vartheta;\lambda)$ is positive, since

$$\begin{aligned}
\mu\lambda + \rho v(T-\vartheta;\lambda) &= \mu\lambda + \rho\left((K-\lambda) + g(\vartheta)\right) \\
&= \rho K\left(1 - e^{-\sigma y\sqrt{\vartheta}}\right) + \rho g(\vartheta) \sim \rho K \sigma y \sqrt{\vartheta} + o\left(\sigma y \sqrt{\vartheta}\right) > 0.
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
(l(T-\vartheta) - \lambda)^2 &= \frac{(v(T-\vartheta;\lambda) - (K-\lambda))}{\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(T-\vartheta;\zeta)} \\
&< \frac{g(\vartheta)}{\frac{\mu\lambda + \rho v(T-\vartheta;\lambda)}{\sigma^2\zeta^2}} = \frac{\sigma^2\zeta^2 g(\vartheta)}{\mu\lambda + \rho\left((K-\lambda) + g(\vartheta)\right)} < C\frac{g(\vartheta)}{\mu\lambda + \rho\left((K-\lambda) + g(\vartheta)\right)}
\end{aligned}$$

where $C > 0$. Hence

$$\begin{aligned}
(l(T-\vartheta) - \lambda)^2 &< C\frac{g(\vartheta)}{\rho K\left(1 - e^{-\sigma y\sqrt{\vartheta}}\right) + \rho g(\vartheta)} \\
&\sim C\frac{o\left(\vartheta^{\frac{3}{2}}\right)}{\rho K \sigma y\sqrt{\vartheta} + o\left(\sigma y\sqrt{\vartheta}\right)} = C'\frac{o\left(\vartheta^{\frac{3}{2}}\right)}{-\sigma y\sqrt{\vartheta} + o\left(\sigma y\sqrt{\vartheta}\right)} = C' o\left(\sigma^2 y^2 \vartheta\right)
\end{aligned}$$

where $C' > 0$. This implies that

$$(l(T-\vartheta) - \lambda) < o\left(-\sigma y\sqrt{\vartheta}\right) \quad \text{as } \vartheta \to 0$$

But then

$$l(T-\vartheta) - \lambda = l(T-\vartheta) - l\left(T^-\right)e^{-\sigma y\sqrt{\vartheta}} < o\left(-\sigma y\sqrt{\vartheta}\right) \quad \text{as } \vartheta \to 0$$

i.e.

$$l(T-\vartheta) \leq l\left(T^{-}\right)\left(1-\sigma y\sqrt{\vartheta}\right)+o\left(-\sigma y\sqrt{\vartheta}\right) \text{ as } \vartheta \to 0$$

for $y \leq y^*$. In other words

$$l\left(T^{-}\right)-l(t) \geq l\left(T^{-}\right)-\left(l\left(T^{-}\right)\left(1-y\sigma\sqrt{(T-t)}\right)\right)=l\left(T^{-}\right) y\sigma\sqrt{(T-t)},$$

for all $y \leq y^*$, and hence

$$l\left(T^{-}\right)-l(t) \geq l\left(T^{-}\right) y^{*}\sigma\sqrt{(T-t)}.$$

Therefore we get

$$\liminf_{t\to T} \frac{l\left(T^{-}\right)-l(t)}{l\left(T^{-}\right)\sigma\sqrt{(T-t)}} \geq y^{*}$$

and thus our proof is complete.■

Proof of Proposition 2.5 If the European put option v_e dominates the immediate payoff at t for all values of the underlying x, then there is no optimal exercise for the American option at t. The distance between the European put option and the immediate payoff at (t,x) is $f(t,x)=v_e(t,x)-(K-x)^{+}$, where

$$v_e(t,x)=Ke^{-\rho(T-t)}\mathcal{N}(\overline{z})-xe^{(\mu-\rho)(T-t)}\mathcal{N}\left(\overline{z}-\sigma\sqrt{(T-t)}\right), \tag{7.2}$$

with $\mathcal{N}(y)$ denoting the distribution function of a standard normal random variable, and $\overline{z}=\left(\ln\frac{K}{x}-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}$. For any $t \in [0;T]$, the function $f(t,\cdot)$ is convex, reaching its minimum at $0 < x_m < K$ such that $\frac{\partial}{\partial x} f(t,x_m)=0$. Hence $f(t,x_m) > 0$ is equivalent to the fact that the European option $v_e(t,x)$ dominates at t the immediate payoff *for any* $x > 0$. Therefore x_m is the solution of the equation $\frac{\partial}{\partial x} f(t,x)=0$ or $\frac{\partial}{\partial x} v_e(t,x)=-1$. We compute $\frac{\partial}{\partial x} v_e(t,x)=Ke^{-\rho(T-t)}f_{\mathcal{N}}(\overline{z})\frac{\partial \overline{z}}{\partial x}-e^{(\mu-\rho)(T-t)}\mathcal{N}\left(\overline{z}-\sigma\sqrt{(T-t)}\right)-xe^{(\mu-\rho)(T-t)}f_{\mathcal{N}}\left(\overline{z}-\sigma\sqrt{(T-t)}\right)\frac{\partial \overline{z}}{\partial x}$, where $f_{\mathcal{N}}$ denotes the density of a standard normal random variable and $\frac{\partial \overline{z}}{\partial x}=-\frac{1}{x\sigma\sqrt{T-t}}$. Hence

$$\frac{\partial}{\partial x} v_e(t,x)=\frac{e^{-\rho(T-t)}}{\sigma\sqrt{T-t}}\left(-\frac{K}{x}f_{\mathcal{N}}(\overline{z})+\underbrace{e^{\mu(T-t)}f_{\mathcal{N}}\left(\overline{z}-\sigma\sqrt{(T-t)}\right)}_{\frac{K}{x}f_{\mathcal{N}}(\overline{z})}\right)-e^{(\mu-\rho)(T-t)}\mathcal{N}\left(\overline{z}-\sigma\sqrt{(T-t)}\right),$$

delivering $\frac{\partial}{\partial x} v_e(t,x)=-e^{(\mu-\rho)(T-t)}\mathcal{N}\left(\overline{z}-\sigma\sqrt{(T-t)}\right)$. Therefore x_m is defined via the following equation $\mathcal{N}\left(\overline{z}_m-\sigma\sqrt{T-t}\right)=e^{-(\mu-\rho)(T-t)}$, where $\overline{z}_m=\left(\ln\frac{K}{x_m}-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}$. Finally

$$v_e(t,x_m)=Ke^{-\rho(T-t)}\mathcal{N}(\overline{z}_m)-x_m e^{(\mu-\rho)(T-t)}e^{-(\mu-\rho)(T-t)}=Ke^{-\rho(T-t)}\mathcal{N}(\overline{z}_m)-x_m$$

and hence $f(t,x_m)=v_e(t,x_m)-(K-x_m)=e^{-\rho(T-t)}K\mathcal{N}(\overline{z}_m)-K > 0$ if and only if $e^{-\rho(T-t)}\mathcal{N}(\overline{z}_m)-1 > 0$. Therefore the necessary condition for possible optimal exercise at t is $e^{-\rho(T-t)}\mathcal{N}(\overline{z}_m)-1 \leq 0$, i.e. $\overline{z}_m \leq \mathcal{N}^{-1}\left(e^{\rho(T-t)}\right)$. Since z_m is defined via $\mathcal{N}\left(\overline{z}_m-\sigma\sqrt{T-t}\right)=e^{-(\mu-\rho)(T-t)}$, we get $\overline{z}_m=\sigma\sqrt{T-t}+\mathcal{N}^{-1}\left(e^{-(\mu-\rho)(T-t)}\right)$, that delivers (2.13).■

Proof of Proposition 3.1. The proof of Points 1 and 2 relies on the change of numeraire, as explained by Detemple (2001). In particular, we refer to Theorem 6, page 76 in Detemple (2001) extended to the case of a negative interest rate ρ as well as a negative 'dividend yield' $\delta = \rho - \mu < 0$ for the call's underlying asset. Denote with $\rho_{put} = \rho - \mu$ and $\mu_{put} = -\mu$. Conditions (3.2) and (3.3) for ρ, μ are *equivalent* to conditions (2.6) and (2.7) in Proposition 2.2 and in Theorem 2.3 for $\rho_{put} = \rho - \mu$ and $\mu_{put} = -\mu$. In fact, (2.6) follows immediately from (3.2), and (2.7) becomes $\left(\mu_{put} - \frac{\sigma^2}{2}\right)^2 + 2\rho_{put}\sigma^2 = \left(-\mu - \frac{\sigma^2}{2}\right)^2 + 2(\rho - \mu)\sigma^2 = \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0$, which is true by condition (3.3) (the converse implication is also readily verified). This completes the proof of Points 1 and 2.

To prove Point 3 and derive formulae (3.5) and (3.6), we use the put-call symmetry provided by Carr and Chesney (1996). If the call and the put option have the same moneyness, $\frac{x}{K} = \frac{\widehat{K}_{put}}{\widehat{x}_{put}}$ formula (5) in Section III of Carr and Chesney (1996) implies that

$$v_{call}(t, x; K, \rho, \mu, \sigma) = \sqrt{xK}\frac{v_{put}\left(t, \widehat{x}_{put}; \widehat{K}_{put},\ \rho - \mu,\ -\mu,\ \sigma\right)}{\sqrt{\widehat{x}_{put}\widehat{K}_{put}}}.$$

This formula derives from (3.4) by exploiting the homogeneity property of the put option. In fact, take a $\beta > 0$ such that $\widehat{K}_{put} = \frac{x}{\beta}$, is an unconstrained strike for the put option, and let $\widehat{x}_{put} = \frac{x_{put}}{\beta} = \frac{K}{\beta}$. The put option with parameters $\widehat{x}_{put}, \widehat{K}_{put}$ (and $\rho_{put},\ \mu_{put},\ \sigma_{put}$ as before) has the same moneyness of the call option, because $\frac{\widehat{K}_{put}}{\widehat{x}_{put}} = \frac{x}{K}$. Moreover

$$\begin{aligned}
v_{call}(t, x; K, \rho, \mu, \sigma) &= v_{put}(t, x_{put}; K_{put},\ \rho_{put},\ \mu_{put},\ \sigma_{put}) \quad \text{by formula (3.4)} \\
&= v_{put}(t, K; x,\ \rho - \mu,\ -\mu,\ \sigma) \\
&= \beta \cdot v_{put}\left(t, \frac{K}{\beta}; \frac{x}{\beta},\ \rho - \mu,\ -\mu,\ \sigma\right) \quad \text{by the homogeneity property of put} \\
&= \beta \cdot v_{put}\left(t, \widehat{x}_{put}; \widehat{K}_{put},\ \rho_{put},\ \mu_{put},\ \sigma_{put}\right).
\end{aligned}$$

Writing $\beta = \sqrt{\beta \cdot \beta} = \sqrt{\frac{x}{\widehat{K}_{put}} \cdot \frac{K}{\widehat{x}_{put}}}$, we arrive at formula (5) in Section III of Carr and Chesney (1996). We apply now this formula to derive the expression of the upper free boundary as in formula (3.6). Since (2.6) and (2.7) in Proposition 2.2 and in Theorem 2.3 are satisfied, there exist *two* critical prices at time $t \in (0; T)$ for the American put option $v_{put}\left(t, \widehat{x}_{put}; \widehat{K}_{put},\ \rho_{put},\ \mu_{put},\ \sigma_{put}\right)$. Let $\widehat{K}_{put} = 1$ and denote with $0 < l_{put}(t) < u_{put}(t)$ the lower and upper free boundary of the American put option $v_{put}(t, \widehat{x}_{put}; 1,\ \rho_{put},\ \mu_{put},\ \sigma_{put})$. The parameters x, K, and $\widehat{x}_{put}$ are constrained by the equality $\frac{x}{K} = \frac{1}{\widehat{x}_{put}}$. The Carr and Chesney (1996) version of the American put-call symmetry allows to write

$$v_{call}(t, x; K, \rho, \mu, \sigma) = \sqrt{xK}\frac{v_{put}(t, \widehat{x}_{put}; 1,\ \rho - \mu,\ -\mu,\ \sigma)}{\sqrt{1 \cdot \widehat{x}_{put}}}.$$

The time-t upper free boundary for the call can be written as

$$
\begin{aligned}
u(t) &= \sup\left\{x \geq 0 : v_{call}(t,x) = (x-K)^{+}\right\} \\
&= \sup\left\{\frac{K}{\widehat{x}_{put}} \geq 0 : \sqrt{xK}\frac{v_{put}\left(t,\widehat{x}_{put};1,\ \rho-\mu,\ -\mu,\ \sigma\right)}{\sqrt{\widehat{x}_{put}}} = \left(\frac{K}{\widehat{x}_{put}}-K\right)^{+}\right\} \\
&= K\cdot\left(\inf\left\{\widehat{x}_{put} \geq 0 : \sqrt{\frac{K}{\widehat{x}_{put}}K}\frac{v_{put}\left(t,\widehat{x}_{put};1,\ \rho-\mu,\ -\mu,\ \sigma\right)}{\sqrt{\widehat{x}_{put}}} = \left(\frac{K}{\widehat{x}_{put}}-K\right)^{+}\right\}\right)^{-1} \quad \text{because } \frac{x}{K} = \frac{1}{\widehat{x}_{put}} \\
&= K\cdot\left(\inf\left\{\widehat{x}_{put} \geq 0 : \frac{K}{\widehat{x}_{put}}v_{put}\left(t,\widehat{x}_{put};1,\ \rho-\mu,\ -\mu,\ \sigma\right) = \frac{K}{\widehat{x}_{put}}\left(1-\widehat{x}_{put}\right)^{+}\right\}\right)^{-1} \\
&= K\cdot\left(\inf\left\{\widehat{x}_{put} \geq 0 : v_{put}\left(t,\widehat{x}_{put};1,\ \rho-\mu,\ -\mu,\ \sigma\right) = \left(1-\widehat{x}_{put}\right)^{+}\right\}\right)^{-1} \\
&= K\cdot\left(l_{put}\left(t\right)\right)^{-1},
\end{aligned}
$$

which gives formula (3.6) . Formula (3.5) follows by similar arguments. Formulae (3.5) and (3.6) extend formula (6) in Section III of Carr and Chesney (1996) to account for the double free boundary.■

Proof of Proposition 3.2. By Point 1 of Proposition 3.1, $\rho_{put} = \rho - \mu$ and $\mu_{put} = -\mu$ satisfy conditions (2.6) and (2.7) in Proposition 2.2. Therefore, for the *symmetric* perpetual put option with $K_{put} = 1$, there exist *two* constant free boundaries

$$0 < l_{\infty}^{put} < u_{\infty}^{put}\ .$$

The boundaries $l_{\infty}^{put} < u_{\infty}^{put}$ lead to $u_{\infty} > l_{\infty}$ for the call option via equations (3.6) and (3.5) .■

Proof of Theorem 3.3. Under assumptions (3.3) and (3.2), the function v_{∞} and the constants l_{∞} and u_{∞} are well defined. The monotonicity and the continuity of the free boundary follow by arguments similar to those used for the put case. The results can also be derived from Theorem 2.3 by means of the American put-call symmetry as explained in Proposition 3.1. We focus here on the non-standard upper free boundary for the call option, whose existence is implied by the negative interest rate $\rho < 0$. For $t = T$, the definition of u as

$$u(T) = \sup\left\{x \geq 0 : v(T,x) = (x-K)^{+}\right\}$$

implies $u(T) = +\infty$.

For any $t \in (0;T)$, equation (3.6) in Proposition 3.1 yields that $u\left(t\right) = \frac{K}{l_{put}(t)}$ is positive, increasing and continuous, because by Theorem 2.3 the non-standard lower free boundary l_{put} of the put option is positive, decreasing and continuous on $(0;T)$. In particular, the limit of u as $t \to T^{-}$ is $u\left(T^{-}\right) = \frac{K}{l_{put}(T^{-})} = \frac{K}{\frac{\rho_{put}}{\rho_{put}-\mu_{put}}} = \frac{K}{\frac{(\rho-\mu)}{\rho-\mu+\mu}} = \frac{K\cdot\rho}{\rho-\mu} > K$ and this concludes the proof.■

Proof of Theorem 3.4. The asymptotic expressions of u and l at maturity derive from formulae (3.5) and (3.6) applied to the asymptotic expression found in Theorem 2.4 for the symmetric put with parameters as defined in Proposition 3.1. A Taylor approximation of the first order delivers the final expression.■

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