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A Multivariate Model of Strategic Asset Allocation with Longevity Risk

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Abstract

This paper proposes a framework to evaluate the impact of longevity-linked securities on the risk-return trade-off for traditional portfolios. Generalized unexpected raise in life expectancy is a source of aggregate risk in the insurance sector balance sheets. Longevity-linked securities are a natural instrument to reallocate these risks by making them tradable in the financial market. This paper extends the strategic asset allocation model of Campbell and Viceira (2005) to include a longevity-linked investment in addition to equity and fixed income securities and describe the resulting term structure of risk-return trade-offs. The model highlights an unexpected predictability pattern of the survival probability estimates and gives an empirical valuation of the market price of longevity risk based on the Lee and Carter (1992) mortality model and on the time series of prices for standardized annuities publicly offered by US insurance companies.

Keywords: Longevity Risk, Strategic Asset Allocation
JEL Classification: [G11, G12, G22]

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1 Introduction

This paper proposes a framework to evaluate the impact of longevity-linked securities on the risk-return trade-off for traditional portfolios. Longevity-linked securities are instruments designed to reduce the impact of undiversifiable longevity risk on public and private balance sheets. Longevity risk is the risk that an annuitant lives more than forecasted by the annuity provider, so that the company has to pay the annuity for a longer-then-expected period after her retirement. Longevity risk can be decomposed in two underlying components: an idiosyncratic random variation risk and a common trend risk. Random variation risk is the risk that mortality rates differ from their expected outcome as a result of chance or individual-specific characteristics. Trend risk, on the other hand, is the risk that unanticipated changes in lifestyle behavior or medical advances significantly improve longevity for the population as a whole. Idiosyncratic risk is dealt with by pooling a large number of different individuals. Trend risk, similarly to any macroeconomic risk, is on the other hand an "aggregate risk" that cannot be diversified away by pooling. One path toward the reduction of the impact of longevity risk on the balance sheets of public and private insurance providers passes through the creation of a market for longevity-linked securities, both to enhance risk sharing among different categories of financial investors and insurance sellers and to produce an efficient valuation of the cost of longevity risk[1]. An important step to understand the potential of such a market is the evaluation of the impact of longevity-linked securities on the risk return trade-off for traditional portfolios.

The notion of risk term structure is the major innovation of our approach to longevity risk security valuation models, and represents our first contribution to the literature. A risk term structure describes the dependence of the risk-return tradeoff on the investor's holding period. Its importance is well known in the analysis of financial markets, where the set of investment opportunities includes a number of securities with different degree of persistence and predictability: equities are traded on high frequency markets and bonds are traded on a wide range of maturities, thus making optimal blend of financial securities vary with the investor's holding period. Quite surprisingly, most of the financial research on securitization of longevity risks relies on valuation models where uncertainty is simply accounted by their period volatility (e.g. yearly). This assumption is clearly unrealistic, as longevity

[1] An early proposal to design of financial instruments for hedging longevity risk is done in Blake and Burrows (2001).

shocks are known to be driven by low frequency trends on time scales comparable to business cycles. Their impact is negligible for short-term investors but can be substantial for a long term investor like a pension fund, which may be institutionally constrained and unable to periodically rebalance its portfolio of annuitants. As a consequence, the impact of longevity risk on financial portfolios is expected to depend heavily on the holding period, and a proper description of the risk-return tradeoff term structure is therefore crucial to promote an effective integration of longevity risk management in the insurance industry with the maturity transformation and risk diversification business in the financial sector.

To investigate the impact of longevity risks on the term structure of risk return trade-offs and of optimal investment allocations we extend the Vector Autoregression (VAR) framework originally proposed by Campbell and Viceira (2005) to estimate the term structure of the Markowitz (1952) risk-return efficient frontier generated by US stocks, Treasury bonds and Treasury bills. In their model, Campbell and Viceira (2005) estimate a VAR model including returns on US stocks, Treasury bonds and bills and a set of associated predictors, namely the dividend-price ratio, the spread between long-term and short-term bonds and the nominal T-bill yield. We extend this framework by assuming that the set of investment opportunities also includes a synthetic financial security exposed to aggregate longevity risk and that the set of predictors includes a mortality predictor. In particular, the set of returns is extended by considering the prices of standardized immediate annuities offered and publicly reported by North American insurance companies. The mortality predictor, on the other hand, is delivered by the popular Lee and Carter (1992) stochastic mortality model and is used to describe the aggregate actuarial uncertainty driving the evolution of annuity prices.

The second contribution of the paper characterizes the efficient mean-variance allocations for investors with holding period ranging from 1 to 40 years when longevity-linked securities can be included in their portfolios. This evaluation offers a natural benchmark for longevity risk pricing and is a necessary step for the development of a liquid market for longevity-linked securities. From a portfolio allocation point of view, we build on Cocco and Gomes (2012), who analyze the portfolio choice problem of an agent investing in financial assets whose returns are correlated with the shocks to survival probabilities and which can be used to buy insurance against aggregate longevity risk. In particular, the authors study both the portfolio allocation between these bonds and risk-free assets and how their demand changes over the life-cycle depending on individual characteristics. Long

horizon mean-variance allocations share many properties with the strategic asset allocations chosen by intertemporal utility maximizing investors, but they are much easier to compute (Campbell and Viceira (2005)). In this respect, our VAR is very close in spirit to the Campbell, Chan, and Viceira (2003) strategic asset allocation model but allows to extend the analysis of Cocco and Gomes (2012) to analyze allocation choices for an investor who can choose between a larger set of securities, namely equity and long term bonds. Along this direction, our precise description of the term structure of longevity risk-return trade-offs illustrates that the creation of liquid longevity-linked securities with a stable demand critically hinges on an efficient maturity transformation activity. In fact, only with the support of the latter it is possible to diversify longevity risks among investors with short holding periods and averse to liquidity and credit risks of long-duration bonds.

The results we get from this modeling and estimation exercise are promising: our investigation identifies an interesting and unexpected predictability pattern on the estimated survival probability. Moreover, the joint information provided by the evolution of survival probabilities and annuity prices offers a simple quantification of the compensation of aggregate longevity risk exposure which is robust and not far from common expectations in the industry. Last but not least, the maturity profiles of longevity risk offers a clear and natural indication on the stochastic asset (and liability) management policies which can be used to improve diversification and fully exploit the benefits from financial innovation. In particular, the analysis of Sharpe Ratios and durations of longevity liabilities is bridged with effective hedging policies and is a natural point of departure for the design of synthetic longevity-linked securities.

While annuity prices are used to extract information about the price of longevity risk implicit in the annuity contracts offered by US insurance companies, it is important to draw a clear distinction between annuities and the synthetic tradable security included in the VAR. The prices used in our empirical estimation refer to standardized retail insurance contracts which significantly differ from tradable financial securities: they are individual-specific and their purchase is irreversible. The synthetic security introduced in the VAR has a payoff which only depends on aggregate longevity risk - as measured by the (ideally publicly available) index of our construction - and offers a stylized example of a longevity-linked security. The distinction between annuities and longevity-linked securities is particularly important as the valuations of financial and actuarial contracts differ in a significant way, as previously highlighted in the actuarial literature.

From an empirical point of view Mitchell, Poterba, Warshawsky, and Brown (1999), Poterba (2001), Horneff, Mitchell, and Stamos (2009) among others, study the welfare benefits from purchasing annuities and discuss the well-known under-annuitization puzzle. On the theoretical side, on the other hand, the actuarially fair pricing of annuities is a well known and analyzed problem (see e.g. Pitacco, Denuit, Haberman, and Olivieri (2009) and Milevsky (2006) and references therein). The point of contact between the valuation of an annuity as from the above literature and that of the longevity-linked security we employ in this paper is a "fair pricing" argument of widespread use in actuarial science (Milevsky (2006)): rational agents decide whether to annuitize or to defer the purchase of the annuity for a given period of time by comparing the return offered by the annuity with the potential return from self-annuitization during the deferral period. As the efficiency of self-annuitization depends on the alternative financial investments available in the market, it is reasonable to expect that an annuity contract will offer a return that is both competitive as compared to that of similar financial securities and includes a specific mortality credit component.

Biffis, Denuit, and Devolder (2010) point our three sources of risk affecting insurance securities: basis risk, i.e. the risk that the population from which the survival probabilities were estimated differs from the insurer's cohort, systematic mortality risk, and unsystematic mortality risk. We focus on the systematic mortality risk component, the only one which should be priced by rational agents willing to share undiversifiable longevity risk. A central result of our modelling procedure is the precise quantification of the potential benefits for investors and insurance providers generated by the creation of an integrated market for longevity risk sharing. In this context, it is important to remark that a transparent and accessible evaluation of the price of aggregate risk is necessary not only for financial longevity risk sharing schemes but also for actuarial ones.

An approach similar to ours for the pricing of longevity risk is followed by Lin and Cox (2005) and Lin and Cox (2008), who apply the 1-factor and 2-factor Wang transform to estimate longevity premia from annuity prices. While we retain their main actuarial valuation principles, we avoid the use of the Wang transform which, as pointed out by Pelsser (2008), "does not lead to a price which is consistent with the arbitrage-free price" and, therefore, "cannot be a universal framework for pricing financial and insurance risks" (Wang (2002)). Milevsky, Promislow, and Young (2006) and Bayraktar, Milevsky, Promislow, and Young (2009) develop a theory for pricing undiversifiable mortality risk in an incomplete market. They postulate that

an issuer of a life contingency requires compensation for this risk according to a pre-specified instantaneous Sharpe ratio (see also Bayraktar and Young (2008)). Within our model, the incompleteness generated by demographic uncertainty is accounted for by including an additional state variable which is extracted from the Lee and Carter (1992) model for stochastic mortality. Previous attempts to quantify the impact of longevity risk on market prices, like Friedberg and Webb (2007), who apply the Capital Asset Pricing Model (CAPM) and the Consumption Capital Asset Pricing Model (CCAPM) to quantify risk premia for potential investors in longevity bonds, produce very low estimates of such a premium. The authors acknowledge that there is likely to exist a "mortality premium puzzle" similar to the well-known "equity premium puzzle" (Mehra and Prescott (1985)) driving higher mortality risk premia in the data than those economic models would suggest.

The paper is organized as follows: Section 2 describes the construction of the aggregate longevity risk state variable and the estimation of an extended VAR that includes the aggregate longevity risk shock and annuity price changes. Section 3 describes the optimal allocation for investors who have the opportunity to invest in a synthetic longevity-linked security with short duration and discusses the normative implications for the design of an efficient market for longevity risk transfer. Section 4 defines an hedging portfolio for aggregate longevity risk and quantifies longevity risk compensation as measured by the Sharpe Ratio of this hedging portfolio. Section 5 concludes.

2 Risk and returns in a VAR model for financial securities and annuity prices

Our empirical strategy follows the approach to the optimal portfolio choice problem under return predictability proposed by Campbell, Chan, and Viceira (2003) and Campbell and Viceira (2005). However, we extend the traditional investment opportunity set made by stocks, bonds and T-bills with a longevity-linked security and derive the optimal portfolio allocation at different horizons based on a Vector Autoregressive (VAR) specification for returns and their predictors. As we include a longevity-linked security in the investment opportunity set, we build an appropriate associated predictor from the estimation of a stochastic mortality model. We therefore first illustrate how annuity valuation implies that the unexpected generalized mortality innovation as from the popular Lee and Carter (1992) model can be used as a predictor of the return of a longevity-linked security. We

then propose a VAR model of the joint dynamics of the returns on stocks, bills, bonds, longevity-linked securities and their predictors. In the context of the VAR dynamics, we build the optimal intertemporal hedging portfolio and address the issue of pricing longevity risk.

2.1 A reduced form model for annuity valuation

Our estimate of the longevity risk-return trade-off is based on the historical time series of observed prices for standardized annuity contracts offered by insurance companies to voluntary individual annuitants. By "standardized" annuities we mean single premium (involving a one-time investment) immediate (commencing regular income payments one period after the premium has been paid) single life (guaranteeing to make payments only to a single beneficiary until her death) fixed (providing fixed payments) annuities.

It is important to remark that annuity contracts significantly differ from tradable financial contracts, as they are individual-specific and their purchase is irreversible. Moreover, insurance companies cannot liquidate the subscribers and annuities cannot be replicated or sold short[2]. Finally, informational asymmetry between the subscriber and the insurance company is known to affect annuity pricing: it is documented that voluntary subscribers of life annuities live longer than average population[3].

Despite these differences, the following simple argument proves that returns from annuities are not independent from those of alternative financial investments. Assume that a rational agent of age x at time t faces the alternative between immediate annuitization at price $P_{x,t}$ or a deferral of the entry decision by one year, buying the annuity at time $t+1$ and age $x+1$ at a price $P_{x+1,t+1}$. In order for the agent to opt for immediate annuitization, the return provided by the annuity should be at least as large as the one she would get from a financial investment with a similar risk-return profile plus a *mortality credit*, the extra return required by the annuitant as a compensation for the risk of a premature death between x and age $x+1$. To precisely quantify such risk, let $q_{x,t}$ denote the mortality rate for individuals

[2] An important element distinguishing insurance companies from several other financial intermediaries is the lack of a secondary market where the contracts written by insurance companies may be traded. The holder cannot sell her insurance policy to a third final investor, albeit, in recent years, secondary markets for some insurance contracts have developed.

[3] The adverse selection problem in annuity pricing has been discussed, among others, by Mitchell, Poterba, Warshawsky, and Brown (1999) and Finkelstein and Poterba (2004). More recently the role of informational asymmetries in longevity markets has been analyzed in Biffis and Blake (2010)

of age x in year t, i.e. the probability that a person aged x and alive at the beginning of year t dies within the end of the year. We define by $s_{\overline{x}+i,t+i}$ the probability to be alive in year $t+i$, of age $\overline{x}+i$, conditional on being alive at age $\overline{x}$, so that [4]

$$\begin{aligned} s_{\overline{x},t} &= 1, \\ s_{\overline{x}+i,t+i} &= s_{\overline{x}+i-1,t+i}[1-q_{x+i,t+i}] \quad i=1,\ldots,\infty. \end{aligned}$$

Life expectancy for a person aged x at time t is defined as $e_{x,t} = \sum_{i=1}^{\infty} s_{x+i,t+i}$. Survival probabilities tend to zero as time increases, given that mortality rates $q_{x,t}$ increase with age x, and the probability of a premature death between x and age $x+1$ is then quantified by $1 - s_{x+1,t+1}/s_{x,t}$.

Assuming that annuities are offered to rational agents in a competitive market, prices set by insurance companies should correspond to the lowest return making the investor indifferent between immediate annuitization or deferral. Given this premise and the previous definitions, we derive an approximate accounting identity providing an explicit expression of these contributions.

As a starting point consider the definition of the one-period holding return for an annuity paying a coupon C in each period (year) to a person aged x at time t as follows:

$$R^A_{t+1} = \frac{(P_{x+1,t+1}+C)s_{x+1,t+1}}{P_{x,t}s_{x,t}} - 1. \tag{1}$$

Dividing both sides of (1) by $\left(1+R^A_{t,t+1}\right)$ and multipling both sides by $\frac{P_{x,t}}{C}$, we have:

$$\frac{P_{x,t}}{C} = \frac{1}{\left(1+R^A_{t,t+1}\right)}\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right)\left(1+\frac{P_{x+1,t+1}}{C}\right).$$

Denoting with lowercase letters the natural logarithms of uppercase letters we have:

$$p_{x,t} - c = -r^A_{t+1} + \ln\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right) + \ln\left(1+e^{p_{x+1,t+1}-c}\right).$$

Finally, taking a Taylor expansion of the last term about the average log price-coupon ratio, $\frac{P}{C} = e^{\overline{p-c}}$, we have

[4] The common actuarial notation for the survival probability $s_{\overline{x}+i,t+i}$ would be ${}_ip_{\overline{x},t}$. It is modified in order to keep using the common financial convention where p indicates the logarithmic price of a risky security, e.g. $p-d$ will indicate the price-dividend ratio.

$$
\begin{aligned}
p_{x,t} - c &\simeq -r^A_{t+1} + \ln\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right) + \ln\left(1 + \frac{P}{C}\right) + \\
&\quad + \frac{P/C}{1 + P/C}\left(p_{x+1,t+1} - c - \overline{p - c}\right) \\
&= -r^A_{t+1} + \ln\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right) + k^s + \rho\left(p_{x+1,t+1} - c\right),
\end{aligned}
$$

where $\rho \equiv \frac{e^{\overline{p-c}}}{1+e^{\overline{p-c}}}$. Therefore annuity returns can be written as

$$
r^A_{t+1} = k + \rho\left(p_{x+1,t+1}\right) - p_{x,t} + (1-\rho)\, c + \ln\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right),
$$

and recalling that

$$
\ln\left(\frac{s_{x+1,t+1}}{s_{x,t}}\right) = \ln\left(1 - q_{x,t}\right),
$$

we get by rearranging that

$$
p_{x,t} = k + (1-\rho)\, c + \rho\left(p_{x+1,t+1}\right) - r^A_{t+1} + \ln\left(1 - q_{x,t}\right).
$$

Taking the term-by-by-term difference of the valuation equation between time $t+1$ and t, the value of the coupon (nominal payment is fixed) disappears from the valuation equation, so that

$$
\Delta p_{x,t} = \rho\left(p_{x+1,t+1} - p_{x+1,t}\right) - \Delta r^A_{t+1} + \Delta \ln\left(1 - q_{x,t}\right), \tag{2}
$$

with $\Delta p_{x,t} \equiv p_{x,t} - p_{x,t-1}$.

Consider now the the popular Lee and Carter (1992) model for stochastic mortality. This model has both strong within-sample fitting properties and remarkable out-of-sample predictive power. Together with the relative ease of its computation, these characteristics have made it the standard mortality forecasting model among practitioners and academics. The model consists of a system of equations for the logarithm of the mortality rate of each age cohort x at time t, $q_{x,t}$ and a time-series equation for an unobservable time-varying mortality index k_t, common among all age cohorts. In particular, we have

$$
\begin{aligned}
\ln\left(q_{x,t}\right) &= a_x + b_x k_t + \epsilon_{x,t}, &(3)\\
k_t &= c_0 + c_1 k_{t-1} + e_t, &(4)\\
\epsilon_{x,t} &\sim NID\left(0, \sigma^2_\epsilon\right), \\
e_t &\sim NID\left(0, \sigma^2_e\right),
\end{aligned}
$$

where a_x and b_x are age-specific constants. The error term $\epsilon_{x,t}$ captures cross-sectional errors in the model-based prediction of mortality for different cohorts, while the error term e_t captures random fluctuations in the time series of the common factor k_t driving mortality at all ages. This common factor evolves over time as an auto-regressive process and the favorite Lee-Carter specification makes it a unit-root process by setting $c_1 = 1$. Identification is achieved by imposing the restrictions $\sum_t k_t = 0$ and $\sum_x b_x = 1$, so that the unobserved mortality index k_t is estimated through Singular Value Decomposition[5].

Under the Lee-Carter specification we have that the revision of the mortality credit contribution is linear in the innovation to the unobserved common factor component k_t. In fact,

$$\begin{aligned} q_{x,t} &= \exp\left(a_x + b_x k_t + \varepsilon_{x,t}\right), \\ k_t &= c_0 + c_1 k_{t-1} + e_t, \end{aligned}$$

so that for small variations in mortality rates

$$\begin{aligned} &\ln\left(1 - \exp\left[a_x + b_x\left(k_t\right) + \varepsilon_{x,t}\right]\right) - \ln\left(1 - \exp\left[a_x + b_x\left(k_{t-1}\right) + \varepsilon_{x,t-1}\right]\right) \\ &\simeq -\left[b_x e_t + \varepsilon_{x,t+1} - \varepsilon_{x,t}\right]. \end{aligned}$$

A similar measure has already been introduced and discussed in Friedberg and Webb (2007). Assuming that agents will compensate only aggregate risk, the priced contribution to mortality credit is given by e_{t+1}, and we can re-write (2) as

$$\Delta p_{x,t} = \rho\left(p_{x+1,t+1} - p_{x+1,t}\right) - \Delta r^A_{t+1} + b_x\left(-e_t\right). \tag{5}$$

The last equation shows that the innovations in the common mortality factor in the Lee-Carter model can be taken as a predictor for the change in the price of a longevity-linked security.

Solving this relation forward up to period $t+m$ and taking expectations given the information set available at time t, we have:

$$\Delta p_{x,t} \simeq -\sum_{j=0}^{m} \rho^j E_t \Delta r^A_{t+1+j} + \sum_{j=0}^{m} \rho^j b_x E_t\left(-e_{t+j+1}\right),$$

which shows that the annuity price variation is determined by future changes in the holding period returns and by expected revisions in mortality rates.

[5]See Appendix A for a full description of the adopted identification and estimation strategy.

2.2 A model for stochastic mortality and its performance on the US data

We apply the Lee and Carter (1992) model to estimate shocks to mortality for cohorts in the age interval between 65 and 110. We restrict the estimation to the cohorts of retired population for several reasons. First, the active rebalancing of the contributions is not feasible for these cohorts, as they typically consist of people who left the accumulation phase and entered the decumulation phase. Hence, reallocation via securitization or resinsurance is the only viable strategy that insurance companies can pursue to hedge the associated longevity risk associated. Second, the largest publicly available empirical data sets on annuity prices apply to annuitants belonging to these cohorts. Third, limiting the specification to retired cohorts alleviates some well-known limitations of the Lee-Carter model when applied to the entire population (see Lee (2000)). On the other hand, the approach we propose is fully flexible and we do not see specific obstacles to extend it to any other (log-linearized version) of stochastic mortality models.

In Figure 1 we report evidence on the performance of the Lee-Carter model in fitting US mortality rates. Our data come from the Human Mortality Database of the University of Berkeley[6]. In Figure 1.1 we plot realized mortality at age 65 throughout the period 1952-2007 (red dashed line) against its Lee-Carter fitted value (blue continuous line). The model is estimated on cohorts aged 20 to 110. In Figure 1.2, we report the cross-sectional R^2 of the estimate for all age cohorts in the same period. The model performs very well in fitting mortality rates at all ages but those greater than 95, where the volatility of mortality is high: for more than fifty percent of ages, the R^2 is above 95%, and for more than seventy-five percent of ages it is above 80%. Figure 1.3 reports the estimated unobservable common mortality index k from Equation (4), which clearly features a negative trend. The autoregressive coefficient c_1 from Equation (4) is equal to 0.977 with an associated standard deviation of 0.015, and this persistence allows to make strong predictions about future mortality. Figure 1.4 reports the innovations in the unobservable mortality index, which has a persistent autoregressive structure: this variable will be our measure of the relevant uncertainty on mortality and the predictor of returns of longevity-linked securities.

Insert Figure 1 about here

We use the Lee-Carter model to derive an observable counterpart of the priced contribution to mortality credit. This measure, called qk_t, is very

[6]The data are publicly available at http://www.mortality.org/

similar to the one discussed in Friedberg and Webb (2007), and coincides with the unexpected variation in the survival rate pooled over all retired cohorts[7]:

$$qk_{t+1} \simeq \sum_{x=65}^{110} \left\{ \ln\left(1 - q_{x,t+1}\right) - E_t\left[\ln\left(1 - q_{x,t+1}\right)\right] \right\}.$$

Using the Lee Carter specification, the observable index qk_{t+1} can be approximated by:

$$\begin{aligned} qk_{t+1} &\simeq -\sum_{x=65}^{110} \left(\alpha_x + b_x k_{t+1} + \varepsilon_{x,t+1}\right) - \left(\alpha_x + b_x E_t\left[k_{t+1}\right]\right) \\ &= -\sum_{x=65}^{110} \left(\alpha_x + b_x\left(k_t + e_{t+1}\right) + \varepsilon_{x,t+1}\right) - \left(\alpha_x + b_x k_t\right) \\ &= -\sum_{x=65}^{110} \left(b_x e_{t+1} + \varepsilon_{x,t+1}\right) \end{aligned}$$

and, taking into account the normalization condition $\sum_{x=65}^{110} b_x = 1$ and the assumption that the non-systematic component vanishes when pooling cohorts, i.e. $\sum_{x=65}^{110} \varepsilon_{x,t+1} \simeq 0$, we have

$$qk_{t+1} \simeq -\sum_{x=65}^{110} \left(b_x e_{t+1} + \varepsilon_{x,t+1}\right) \simeq -e_{t+1}.$$

This shock describes the time evolution of the unexpected variation in mortality rates which has a uniform impact across cohorts, and is estimated by applying the Lee and Carter model only to retired cohorts. Then the filtered innovation for the unobserved mortality index in the (restricted) Lee Carter model is included in the vector of autoregressive variables to describe unexpected variation in mortality rates. This variable offers a publicly available, cohort-independent, information which investors can observe and use to quantify variability of systematic longevity risk.

2.3 VAR dynamics with annuity price changes

In this subsection we show that, using the above reduced valuation approach, it is possible to model the stochastic evolution of annuity prices $\Delta p_{x,t}$ using a VAR specification which extends that of Campbell and Viceira (2005)

[7] We estimate the model on retired cohorts, only.

(hereinafter CV-VAR(1)). Following Barberis (2000), Campbell and Viceira (2002, 2005), we describe dynamics of asset returns and relevant predictors using a VAR(1) model:

$$\mathbf{z}_t^{Mkt} = \Phi_0^{Mkt} + \Phi_1^{Mkt}\mathbf{z}_{t-1}^{Mkt} + \nu_t^{Mkt} \tag{6}$$

where

$$\mathbf{z}_t^{Mkt} = \begin{bmatrix} r_{0t} \\ \mathbf{x}_t^{Mkt} \\ \mathbf{s}_t^{Mkt} \end{bmatrix}$$

is a $m \times 1$ vector, with r_{0t} being the log real return on the asset used as a benchmark to compute excess returns on all other asset classes, $\mathbf{x}_t$ being the $n \times 1$ vector of log excess returns on all other asset classes with respect to to the benchmark, and $\mathbf{s}_t$ is the $(m - n - 1) \times 1$ vector of returns predictors. The exact specification and its estimation results are reviewed in Appendix B.

As described in the previous subsections, although an annuity is not a financial security and cannot be priced accordingly, rationality of the annuitant forces the (log) holding period return r_t^A to be comparable (but for the mortality credit) to the compensation one would get by investing in a portfolio of traded financial securities with similar risk and return characteristics while deferring by one year the annuitization. Hence we claim, and later show empirically, that the financial component of the return r_t^A can be replicated using a portfolio of securities whose evolution is described by the CV-VAR(1) model. Moreover, assuming a stationary evolution for r_t^A, the VAR(1) specification implies that also $-\sum_{j=0}^{m} \rho^j E_t \Delta r_{t+1+j}^A$ is a linear function of the state variables $\mathbf{z}_t^{Mkt}$. This is equivalent to assuming

$$-\sum_{j=0}^{m} \rho^j E_t \Delta r_{t+1+j}^A = \phi_0^A + \phi_1^{A,Mkt}\mathbf{z}_t^{Mkt}.$$

If innovations in the common mortality trend are persistent, we can express the variation in the price of the annuity as follows:

$$\Delta p_{x,t} \simeq \phi_0^A + \phi_1^{A,Mkt}\mathbf{z}_t^{Mkt} + \phi_2 \left(-e_t\right),$$

and therefore the standard CV-VAR(1) model can be augmented to include the evolution of the annuities' (log) price growth in excess to the return of the safe asset, $x\Delta p_{x,t}$, following the specification

$$x\Delta p_{x,t} = \phi_0^A + \phi_1^{A,Mkt}\mathbf{z}_{t-1}^{Mkt} + \phi_2 \left(-e_t\right) + \nu_{t+1}^A,$$

where ν_{t+1}^A is the combination of all shocks in the state variables and idiosyncratic mortality shocks. Excess changes in annuity prices are determined by a combination of market returns, market return predictors and the aggregate longevity predictor. We now analyze the effect of extending the traditional portfolio to include excess annuity prices by considering the following augmented VAR specification:

$$\begin{aligned} \mathbf{z}_t &= \Phi_0 + \Phi_1 \mathbf{z}_{t-1} + \nu_t \\ \nu_t &\sim \mathcal{N}(0, \Sigma_\nu), \end{aligned} \tag{7}$$

where

$$\mathbf{z}_t = \begin{bmatrix} r_{0t} \\ \mathbf{x}_t^{Mkt} \\ \Delta p_{x,t} \\ \mathbf{s}_t^{Mkt} \\ -e_t \end{bmatrix}$$

and Σ_ν is the $(m+2)\times(m+2)$ variance-covariance matrix of the returns on financial assets, the annuity prices and their associated predictors.

2.4 The dynamics of returns of US bonds, bills, stocks and annuities

To evaluate how the inclusion of annuities and a predictor for the change in their prices modifies the optimal portfolio allocation at different horizons, we compare the results obtained from the CV-VAR(1) estimation over the yearly sample 1953-2007 to those obtained from our extended VAR. The first model includes six variables: the ex-post real T-bill rate, the annual excess returns on stocks, the annual excess returns on long-term (20-year) bonds, the log yield on a 90-day T-bill, the log dividend-price ratio and the yield spread (defined as the difference between the 20-year zero-coupon bond yield from the CRSP Fama-Bliss data file - the longest yield available in the file - and the T-bill rate). The second model is an eight-variables VAR obtained by adding to the standard CV-VAR(1) the log difference in the annuity premium minus the risk-free rate (which extends the set of excess returns) and the aggregate mortality shock (which extends the set of predictors). Table 1 shows sample statistics for all variables. Our sample which includes observations up to the most recent update of mortality data, compares well with the annual sample used in previous studies. Only the statistics on long term bond indicate a lower expected return, a result which is clearly driven by the recent trends in interest rate policy.

Table 2 shows the results for the standard CV-VAR(1). Table 3 shows the results for the augmented VAR. Both estimated VAR's include constants in each equation. Detailed information on the data used in the estimation are reported in Appendix C. The results from the standard model are well in line with those reported in Campbell, Chan, and Viceira (2003). When the extended VAR is estimated, no major changes take place in the coefficients attached to the six financial variables in the original model. For those variables whose explanatory power is significantly different from zero, the impulse response coefficients are qualitatively similar between the two tables and confirm all the stylized properties found in the original estimation: real T-bill, stock and bond returns are predicted by nominal short rate, dividend-price and term spread. The longevity shock is persistent and helps to predict the change in annuity prices (a positive shock to longevity increases the price of annuities). It also has some significance in predicting excess returns on bonds. The two new equations included in the extended VAR describe the evolution of the aggregate longevity shock and the logarithmic yearly change of the annuity price in excess to the nominal T-bill. The estimated aggregate longevity shock dynamics qk_t is substantially a univariate mean reverting with a persistence of 0.75 with substantially zero expectation (estimation provides a value $E_{t-1}[qk_t] = 0.0005$) confirming both that the information conveyed by the aggregate longevity shock is not spanned by other variables and that the spread between expected and realized aggregate longevity is mean reverting and thus may be used to forecast aggregate longevity growth.

The aggregate longevity shock is a significant predictor for $x\Delta pr_{t+1}$ (the log difference in the annuity premium minus the risk-free rate), which is also significantly predicted by past real and nominal T-bill rates and the excess returns on long-term bonds. Annuity price growth in excess to the T-bill rate have a positive loading on the real rate and a negative one on the nominal T-bill rate, a negative loading on long term excess bond rate returns and a positive dependence on the aggregate longevity shock. In Figure 2 the time series of historical (real) logarithmic price changes is compared to the replication as operated by the VAR dynamic model aggregating the information of financial securities returns and of the forecasting variables, including the aggregate longevity shock.

Insert Figure 2 about here

The good fit indicates that the VAR estimation produces a realistic "reduced form" pricing model for the annuity contract offered by insurance

companies to annuitants. We underline once more that this approach is tailored to capture the main stylized features of the interaction between financial markets and annuity risk management in the insurance industry. This approach accounts only for aggregate longevity risk and does not account for the actuarial components of insurance premia which must be included to hedge basis risks or adverse selection effects, which in turn require a discussion of the specific characteristics of annuitants.

3 The impact of longevity securitization on optimal allocations

The possibility to trade longevity-linked securities extends the set of investment opportunities and offers a new diversification dimension. Following Campbell and Viceira (2005), we compute the optimal portfolios for investors adopting a buy-and-hold strategy with holding period between 1 and 40 years. The set of investment opportunities is composed by T-bills, equity, a rolling strategy in a long term bond and the Annuity Linked Security (ALS henceforth). This security grants to its holder a yearly return equal to Δpr_t the variation of the mean (logarithmic) price observed on the US insurance market for a standardized annuity contract. Clearly the return from this contract will raise if aggregate longevity is raising and will decrease if aggregate mortality increases. A long position in the ALS corresponds to an unavailable tontine insurance in which contracts are terminated and then possibly renegotiated every year (for an actuarial discussion of such synthetic contracts see Milevsky (2006), p. 224). A short position in the ALS allows the investor to sell protection on the longevity risk of the cohort of 65-years-old US annuitants.

A term structure of conditional volatilities at different horizons can be naturally derived from the estimation of our VAR process for returns and predictors.

Insert Figure 3 about here

In Figure 3, we compare the term structures of the standard deviation of the ALS and of traditional financial securities for an horizon ranging from 1 to 40 years. Notice that the annualized volatility increases with the holding period, the clear sign of the long-term nature of the risks underlying annuity prices. For holding periods shorter than 10 years, ALS is less risky than equity and (rolling) bond investments, while on longer horizons its

risk exceeds that of other securities: the irreversible nature of annuitization implies that, from a pure financial point of view, this contract has a risk-return profile similar to the one by a buy-and-hold strategy on a long term bond. ALS price fluctuations reflect changes in the long-run expectations about inflation and longevity trends, and a small persistent change to future expectations can have a relevant impact on the current evaluation of the annuity contract, making prices fluctuate accordingly. Note that an ALS is not the contract that an insurance company would like to use to reinsure aggregate longevity risk. In fact, an insurance company with a portfolio of annuities under management would be willing to reinsure only aggregate longevity risk but would prefer to retain the residual difersifiable risk, as pooling different cohorts of annuitants is the remunerative core business of the industry.

Figures 4, 5 plot the correlations among the extended set of securities as a function of the holding period.

Insert Figure 4, 5 about here

The correlations between the ALS and financial securities have a sharp decline with the holding period. These correlations, and more in general the term structure of assets' risk and returns, determine the weight that each asset receives in the portfolio allocation of an investor with mean-variance preferences for any given horizon. To have a sense of how this portfolio allocation changes by including the possibility to invest in the ALS we again follow Campbell and Viceira Campbell and Viceira (2005) and first consider the generalized absolute minimum variance portfolio (henceforth GMV), the portfolio with the lowest variance on the mean-variance efficient frontier. For each holding period this portfolio is described in Figure 6.

Insert Figure 6 about here

Figure 6 shows that the investor overweights the allocation in the T-bill to buy a combination of ALS and long term bond independently of the investment horizon. This combination is a long position in the bond and a short one in the ALS when the holding period is below 10 years, while for longer horizons the two positions are switched. Hence, over periods of time smaller than a decade, risk exposure is minimized by selling protection to longevity while the same investment becomes speculative over longer holding periods. As a consequence, it is expected that demand for longevity exposure and the liquidity of longevity-linked securities can be considerably increased by offering products with short durations.

Figure 7 compares the term structure of risk of the extended GMV with that of the "Campbell-Viceira" GMV and that of a T-bill. Inclusion of the ALS reduces risk for all holding periods with the exception of the interval between 10 and 15 years, where the optimal allocation in the ALS shifts from negative to positive values and the rolling position in long term bonds from positive to negative.

Insert Figure 7 about here

Roughly speaking, the allocation and the risk profile of the GMV confirm that a long position in aggregate longevity is financially appealing, with low risk and good diversification properties, only for an investor with an horizon longer than 12-13 years. This result is consistent with the hypothesis that annuity prices offer a return which is competitive with the alternative investments available to the investor: annuities for a 65 year old investor have an effective duration around 12 years (see, for example, Loeys, Panigirtzoglou, and Ribeiro (2007)). Note that the 12-year minimum variance portfolio corresponds to an allocation in the ALS which is essentially zero.

Based on these observations it is possible to conclude that the creation of short-duration (less than 10 years) longevity-linked securities is the key step for an efficient securitization of longevity risk. These securities offer a stochastic liability which can be efficiently used to finance investments with good diversification properties. The risk-return analysis of the ALS shows that securitization of longevity, which is a long run risk in the sense of carrying a small but persistent component, does not necessarily require the use of long-duration securities. On the contrary, upon a precise quantification of the term structure of longevity risk exposures, a more efficient management of maturity transformation can be realized using structured securities, like for example swaps. These findings indicate that the problems which affected early longevity-indexed security issuances were clearly determined by long durations giving rise to liquidity and credit risk components so large as to overwhelm the effect of longevity risk both for pricing and hedging.

As a second illustration of the optimal mean-variance allocations including a position in the short-term ALS, in Figure 8 we plot the optimal allocations for a portfolio with an expected return of 10% as a function of the holding period returns. As expected, the ALS short position is used to leverage a portfolio of T-bills, equity and long-term bonds.

Insert Figure 8 about here

4 Longevity securitization and inter-temporal hedging of the aggregate longevity risk

The results from the estimation of the extended VAR provide evidence of a significant response of annuity prices to variations in aggregate longevity rates. Quoted annuity prices are expected to include a compensation for the insurance company to bear a risk exposure for the unexpected raise of the undiversifiable longevity risk component. Following the conventional Intertemporal CAPM (ICAPM) interpretation (Merton (1973)) expected utility maximizers try to hedge the stochastic changes of their investment opportunity created by unexpected aggregate longevity shocks. The hedging portfolio is determined by an allocation in traded securities whose return is maximally correlated with the longevity shock qk_t. This portfolio is determined by the constrained minimization problem:

$$\begin{aligned} & \min_{\mathbf{w}} Var_{t-1}\left[R_t^{qk}\left(\mathbf{w}_{t-1}\right) - qk_t\right] \\ s.t. \quad : \quad & R_t^{qk}\left(\mathbf{w}_{t-1}\right) = \mathbf{w}_{t-1} \cdot \mathbf{x}_t + W_{0,t-1} rtb_t \end{aligned}$$

where $\mathbf{x}_t$ is a $n+1$ dimensional vector including the log excess returns of market securities plus the annuity log price growth in excess to the T-bill rate, R_t^{qk} is the return on the replication portfolio and $W_{0,t}$ is the investor's wealth at time t. We assume that the trading strategy is constrained by the condition that, in each period t,

$$w_{xrtb,t} + w_{xr,t} + w_{xb,t} + w_{x\Delta p,t} = W_{0,t},$$

and that the real return of the portfolio at each period is given by

$$\begin{aligned} R_t\left(\mathbf{w}_{t-1}, w_{rtb,t-1}\right) &= w_{xr,t-1} \cdot \left(xr_t + rtb_t\right) + w_{xb,t-1} \cdot \left(xb_t + rtb_t\right) \\ &\quad + w_{x\Delta p,t-1} \cdot \left(x\Delta p + rtb_t\right) + \left(W_{0,t} - \mathbf{w_{t-1}} \cdot \mathbf{1}\right) rtb_t \\ \mathbf{w}_{t-1} &= \left[w_{xr,t-1}, w_{xb,t-1}, w_{x\Delta p,t-1}\right], \\ \mathbf{x}_t &= \left[xr_t, xb_t, x\Delta p_t\right], \\ W_{0,t} &= w_{rtb,t-1} + \mathbf{w_{t-1}} \cdot \mathbf{1}. \end{aligned}$$

Recall that the purchase of an annuity is irreversible and payments are done until the death of a single beneficiary, while $x\Delta p_t$ is the annuity price variation between time $t-1$ and t for the 65 year old male cohort in excess to the T-bill rate. Hence the Aggregate Longevity Hedging Portfolio (hereinafter ALHP) is not tradable unless a new security paying off the return

Δp_t on a yearly basis is available to the investor. This is a benchmark example of the theoretical motivations underpinning the necessity of longevity securitization.

By construction, the ALHP tracks the aggregate longevity shock qk_t and is therefore the best available product to reinsure aggregate longevity risk. Note that the efficiency of the replication increases with the number of investment opportunities exposed to aggregate longevity risk. In practice, the hedging portfolio is determined by performing the minimization over the set of unconstrained allocations $\mathbf{w}_t = [w_{xr,t}, w_{xb,t}, w_{x\Delta p,t}]$, while the position in the short rate is set equal to $w_{rtb,t} = W_0 - \mathbf{w}_t \cdot \mathbf{1}$. Since our VAR model generates a stationary dynamics, the minimum variance replication portfolio corresponds to a time-independent allocation $\mathbf{w}^*_{t-1} = \mathbf{w}$. The first order condition[8] therefore implies the solution

$$\begin{aligned} \mathbf{w}^T &= Var\left[\mathbf{x}_t\right]^{-1} \left\{Cov\left[\mathbf{x}_t, qk_t\right]\right\}, \\ w_{rtb} &= W_0 - \mathbf{w} \cdot \mathbf{1} \end{aligned}$$

We consider three alternative replication portfolios corresponding to zero initial investment ($W_0 = 0$) with an increasing set of restrictions on the allocations. Table 4 reports three hedging portfolios. The first portfolio considers an unrestricted allocation, while the second an allocation where investment in equity is not allowed ($w_{xr} = 0$). In both cases the allocation strategy is a short position in T-bill and equity and a long position in long term bond and in the ALS.

The third hedging portfolio is further restricted by forcing a zero allocation in the T-bill, $w_{rtb} = 0$, thus making the long-term bond the only available financial security available to finance the annuity liability. In all the three cases the volatility induced by the aggregate longevity shock as measured by the volatility of the aggregate longevity replication portfolio is close to 60 basis points in annual terms, and this value remains almost

[8]

$$\begin{aligned} & \min_{\mathbf{w}} Var_{t-1}\left[R^{qk}_t(\mathbf{w}) - qk_t\right] \\ = & \min_{\mathbf{w}} Var_{t-1}\left[qk_t\right] + Var_{t-1}\left[R^{qk}_t(\mathbf{w})\right] - 2Cov_{t-1}\left[qk_t, R^{qk}_t(\mathbf{w}_{t-1})\right] \\ = & \min_{\mathbf{w}} Var_{t-1}\left[R^{qk}_t(\mathbf{w})\right] - 2Cov_{t-1}\left[qk_t, R^{qk}_t(\mathbf{w})\right] \\ = & \min_{\mathbf{w}} Var_{t-1}\left[\mathbf{w} \cdot \mathbf{x}_t + W_0 rtb_t\right] - 2Cov_{t-1}\left[qk_t, \mathbf{w} \cdot \mathbf{x}_t + W_0 rtb_t\right] \\ = & \min_{\mathbf{w}} \mathbf{w} Var_{t-1}\left[\mathbf{x}_t\right] \mathbf{w}^T + \mathbf{w} \cdot \left\{-2Cov_{t-1}\left[\mathbf{x}_t, qk_t\right]\right\} \end{aligned}$$

constant for any holding period with an essentially flat term structure of volatility. This value is slightly bigger but comparable to the 50 basis points which are usually considered as the market standard for longevity risk (see Loeys, Panigirtzoglou, and Ribeiro (2007)).

The above measures of aggregate longevity risk and the values of its replication portfolios can be revised on a yearly basis, at the highest revision frequency of the mortality rates. By making ALHP tradable on a yearly basis, one could sell aggregate longevity protection without incurring in the liquidity problem of long duration securities.

According to Campbell (1996), an intertemporal utility-maximizing agent will optimally demand to invest or sell the hedging portfolio for aggregate longevity, if the state variable qk forecasts changes in financial or human capital. The empirical estimation of the extended VAR shows that the aggregate longevity shock indeed predicts price changes in annuity prices and in long term bonds, and supports the hypothesis of existence of non-zero potential demand for ALS. While a complete discussion of the demand for longevity-linked securities requires a structural equilibrium framework like, for example, the model of Cocco and Gomes (2012), in the next section we estimate the size of the compensation for bearing longevity risk assuming that the set of investment opportunities also includes the ALS.

4.1 Pricing longevity risk

Milevsky, Promislow, and Young (2005) propose to use the notion of Sharpe Ratio as an actuarial measure of systematic longevity risk compensation. While the Sharpe Ratio of an investment is determined by the ratio between the expected return from the investment in excess to a benchmark security (usually the T-bill) and the expected volatility, in actuarial science the Sharpe Ratio determines the excess markup per unit of volatility that an aggregate longevity protection seller would charge to the protection buyer.

The discussion of the previous subsections suggests the possibility to use the information conveyed by the VAR dynamic model to estimate a longevity risk compensation. It is easy to understand that within our framework the Sharpe Ratio of the ALHP is a reliable measure of such compensation. Note that by definition the ALHP is a zero investment portfolio, as

$$\begin{aligned} \mathbf{w}^{ALHP} &= \left[w^{ALHP}_{xrtb,t}, w^{ALHP}_{xr,t}, w^{ALHP}_{xb,t}, w^{ALHP}_{x\Delta p,t}\right], \\ \mathbf{w}^{ALHP} \cdot \boldsymbol{\iota} &= 0, \end{aligned}$$

and that the corresponding return can be split as a the differential between the return of a long position in financial securities and a short position

in the ALS liability. Hence, the ratio between the expected differential return and its risk provides a properly defined Sharpe Ratio. Moreover, differently from previous approaches, our estimation procedure identifies a dynamic compensation component whose evolution is maximally correlated with longevity shocks as opposed to other intermediation margins, which are instead not expected to be correlated with longevity shocks.

Conditional expected values of risk and returns may substantially differ from unconditional ones, and therefore the conditional Sharpe ratios depend on the holding period τ and generate a term structure. This term structure depends on the initial level of the VAR state variables corresponding to the current level of financial returns and the current level of predictors. Over an horizon τ the Sharpe Ratio is the ratio between the τ-period expected excess simple return and the τ-period standard deviation. Hence, recalling that the extended VAR(1) models logarithmic returns we have:

$$\begin{aligned}
SR_{\tau,\mathbf{r}_t} &= \frac{Ex_{\tau,\mathbf{r}_t}}{Std_{\tau,\mathbf{r}_t}}, \\
Std_{\tau,\mathbf{r}_t}(\mathbf{w}) &= \sqrt{Var_{t,\mathbf{r}_t}\left[\sum_{k=1}^{\tau}\mathbf{w}\cdot(\mathbf{r}_{t+k}+r_{0,t+k}\boldsymbol{\iota})\right]/\tau}, \\
Std_{\tau,\mathbf{r}_t}(w_{rtb}) &= \sqrt{Var_{t,\mathbf{r}_t}\left[\sum_{k=1}^{\tau}w_{rtb}r_{0,t+k}\right]/\tau}, \\
Ex_{\tau,\mathbf{r}_t} &= E_{t,\mathbf{r}_t}\left[\frac{1}{\tau}\sum_{k=1}^{\tau}\mathbf{w}\cdot\mathbf{r}_{t+k}\right]+\frac{Std^2_{\tau,\mathbf{r}_t}(\mathbf{w})}{2\tau}.
\end{aligned}$$

Figure 9 reports the term structure of equity Sharpe Ratios obtained by setting the initial condition of the state variables to their long-term expected values.

Insert Figure 9 about here

The above conditional model shows that the relative compensation of different securities as measured by the Sharpe Ratios depends on the horizon and on the state of the economy. When the holding period goes to infinity ($\tau \to \infty$) the Sharpe Ratio $SR_{\tau,\mathbf{r}_t}$ for a generic portfolio w converges to a

limit $SR_{\infty}(w)$[9]:

$$\begin{aligned} SR_{\infty} &= \frac{Ex_{\infty}}{Std_{\infty}}, \\ Ex_{\infty} &= \lim_{\tau \to +\infty} \left\{ E_{t,\mathbf{z}_t} \left[\frac{1}{\tau} \sum_{k=1}^{\tau} \mathbf{w} \cdot \mathbf{r}_{t+k} \right] + \frac{Std^2_{\tau,\mathbf{r}_t}(\mathbf{w})}{2\tau} \right\}, \\ Std_{\infty} &= \lim_{\tau \to +\infty} \sqrt{Var_{t,\mathbf{r}_t} \left[\sum_{k=1}^{\tau} \mathbf{w} \cdot (\mathbf{r}_{t+k} + r_{0,t+k}\boldsymbol{\iota}) \right] / \tau}, \end{aligned}$$

As shown in the Figure 9, convergence of the $SR_{\tau,\mathbf{r}_t}$ to SR_{∞} does occur at very long horizons, but the limiting procedure is necessary in order to produce a bona fide unconditional measure of expected performance consistent with the predictability patterns we documented. Table 5 reports the estimation of long term Sharpe Ratios for all the financial securities included in the extended VAR(1): an equity index, a rolling position in long-term bonds and the ALS. As expected the Sharpe Ratios of the ALS security is negative, as its performance is is lower than that of benchmark risk freE security, the T-Bill. On the other hand, the low period variance implies that the unconditional level of the ALS Sharpe Ratio is as high as $SR_{\infty}^{ALS} = (-)\,0.43$, indicating the potential usefulness of this synthetic security as a "liability" offering a good potential reward to investors seeking new diversification strategies. As this liability is financed by a short term T-Bill, and the maturity mismatch is well known to increase interest rate risk variations, the high value of the Sharpe Ratio can however be misleading.

A similar problem arises when measuring the Sharpe Ratio of the ALHP, the measure of aggregate longevity risk premium. The ALHP can be split in a short position in a ALS security and a long position in a portfolio of traded financial securities, thus making the Sharpe Ratio depend on the composition of the portfolio used to finance the short position.

Table 6 reports the Sharpe measure of aggregate longevity risk compensation for the three alternative allocations defined in Table 4: the unrestricted one $\mathbf{w}^{ALHP\_Unr}$, the one excluding allocation to equity, $\mathbf{w}^{ALHP\_1}$, and the one where the ALS stochastic liability can be hedged using only

[9]These quantities can be easily computed using the following expressions of the long run mean and covariance:

$$\begin{aligned} \mu_{\infty} &= (\mathbb{I} - \Phi_1)^{-1} \Phi_0 \\ \mathbf{\Sigma}_{\infty} &= (\mathbb{I} - \Phi_1)^{-1} \Phi_1 (\mathbb{I} - \Phi_1)^{-T} \end{aligned}$$

long term bonds $\mathbf{w}^{ALHP\_2}$. As the Table shows, Sharpe Ratios decrease with increasing restrictions. Moreover, the difference in the Sharpe Ratios is essentially determined by the different position used to hedge the stochastic liability induced by the short position in the ALS. The Sharpe Ratio is the highest if the investor can use all financial securities, and the lowest if she is allowed to hedge using long term bonds (a security typically held by insurance companies). In fact, the hedging portfolio $\mathbf{w}^{ALHP\_2}$ using only long term bonds is safer (but less profitable) than $\mathbf{w}^{ALHP\_1}$ which also uses T-bills, as its duration matches that of the stochastic liability and thus has identical response to (small) interest rate fluctuations.

In conclusion, the actuarial longevity premium estimate consistent with a prudent hedging policy is given by $SR_{\infty}^{ALHP\_2} = 0.33$. Its value is not far from the conventional level 0.25 used in the actuarial pricing of longevity products as discussed in Loeys, Panigirtzoglou, and Ribeiro (2007). This estimation is expected to overestimate the potential Sharpe Ratio from longevity liability, as shorting costs are not explicitly accounted in this analysis. In addition the adverse selection effect is also expected to play a role here: the mortality rates of annuitants are known to be significantly smaller from those of average population (see Poterba (2001) and Mitchell, Poterba, Warshawsky, and Brown (1999)). Given the scarcity of data on prices of traded longevity-linked securities, the same problems affect virtually any empirical measure of longevity risk compensation.

From a normative point of view these considerations indicate a further indirect motivation to promote the integration between financial and actuarial markets: their development would drive a more transparent and precise assessment of the price for aggregate longevity risk. Among other benefits it is worth mentioning that this assessment can certainly reduce the dangerous unawareness of the public and private costs deriving from generalized longevity increase. Unreported robustness checks show that our results are unaffected if the estimation is based on maximum and minimum annuity premia rather than on the mean ones. Similarly, the estimated values are robust to changes (reductions) of the sample used for the extended VAR(1) estimation.

5 Conclusions

Our analysis shows that a promising direction to improve the efficiency of longevity risk sharing is integration between insurance and financial markets. We believe that our results uncover some critical issues to improve

longevity risk securitization. First, the long term nature of the longevity risk requires an accurate analysis of the term structure of the risk return trade-offs generated by including a longevity-linked security in the set of investments. Second, a potential large number of short term investors would be willing to increase their exposure to longevity risk without increasing their investment horizon. This requires the organization of a maturity transformation activity by financial intermediaries that seems to be a crucial step to increase the interest of the market for longevity-linked securities, as well as their liquidity. Third, an integrated market for insurance and financial contracts with a publicly traded longevity index would also imply a more transparent and efficient pricing of life annuities with a direct benefit for annuity subscribers.

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A Identification and estimation of the Lee-Carter mortality model

The Lee and Carter (1992) model consists of a system of equations for logarithms of mortality rates for age cohort x at time t , $q_{x,t},$, and a time-series equation for an unobservable time-varying mortality index k_t:

$$\begin{aligned} \ln(q_{x,t}) &= a_x + b_x k_t + \epsilon_{x,t} \\ k_t &= c_0 + c_1 k_{t-1} + e_t \\ \epsilon_{x,t} &\sim NID\left(0, \sigma_\epsilon^2\right) \\ e_t &\sim NID\left(0, \sigma_e^2\right) \end{aligned} \tag{8}$$

where a_x and b_x are age-specific constants. The error term $\epsilon_{x,t}$ captures cross-sectional errors in the model based prediction for mortality of different cohorts, while the error term e_t captures random fluctuations in the time series of the common factor k_t driving mortality at all ages.. This common factor, usually known as the unobservable mortality index evolves over time as an autoregressive process and the favorite Carter-Lee specification makes is a unit-root process by setting $c_1 = 1$. Identification is achieved by imposing the restrictions $\sum_t k_t = 0$ and $\sum_x b_x = 1$, so that the unobserved mortality index k_t is estimated through Singular Value Decomposition (SVD). SVD is a technique based on a theorem of linear algebra stating that a $(m \times n)$ rectangular matrix M can be broken down into the product of three matrices - an $(m \times m)$ orthogonal matrix U, a diagonal $(m \times n)$ matrix S, and the

transpose of an orthogonal $(n \times n)$ matrix V. The SVD of the matrix M will be therefore be given by $M = USV'$ where $U'U = I$ and $V'V = I$. The columns of U are orthonormal eigenvectors of AA' , the columns of V are orthonormal eigenvectors of $A'A$, and S is a diagonal matrix whose elements are the square roots of eigenvalues from U or V in descending order. The restriction $\sum_t k_t = 0$ implies that a_x is the average across time of $q_{x,t}$, and Equation 8 can be rewritten in terms of the mean-centered log-mortality rate as

$$q_{x,t} - \overline{q}_{x,t} \equiv \tilde{m}_{x,t} = b_x k_t + \epsilon_{x,t}. \tag{9}$$

Grouping all the $\tilde{m}_{x,t}$ in a unique $(X \times T)$ matrix $\tilde{m}$ (where the columns are mortality rates at time-t ordered by age groups and the rows are mortality rates through time for a specific age-group x), leads naturally to use SVD to obtain estimates of b_x and k_t. In particular, if $\tilde{m}$ can be decomposed as $\tilde{m} = USV'$, $b = [b_0, b_1, \ldots, b_X]$ is represented by the normalized first column of U, $u_1 = [u_{0,1}, u_{1,1}, \ldots, u_{X,1}]$, so that

$$b = \frac{u_1}{\sum_{x=0}^{X} u_{x,1}}.$$

On the other hand the mortality index vector $k = [k_1, k_2, \ldots, k_T]$ is given by

$$k = \lambda_1 (\sum_{x=0}^{X} u_{x,1}) \nu_1$$

where $\nu_1 = [\nu_{1,1}, \nu_{1,2}, \ldots, \nu_{1,T}]'$ is the first column of the V matrix and λ_1 is the highest eigenvalue of the matrix S. The values of mortality rates obtained with this method will not, in general, be equal to the actual number of deaths. In Lee and Carter (1992), the authors hence re-estimate k_t in a second step, taking the values of a_x and b_x as given from the first-step SVD estimate and using actual mortality rates. The new values of k are obtained so that, for each year, the actual death rates are equal to the implied ones. This two-step procedure allows to take into account the population age distribution, providing a very good fit for 13 of the 19 age groups in the authors' sample, where more than 95% of the variance over time is explained. For seven of these, more than 98% of the variance is explained.

B Financial asset returns and their predictors: the basic specification of the VAR model for traditional financial investments

Consider the continuously compounded security market returns from time t to time $t+1$, $\mathbf{r}_{t+1}$. Define μ_t, the conditional expected log return given information up to time t as follows:

$$\mathbf{r}_{t+1} = \mu + \mathbf{u}_{t+1},$$

where $\mathbf{u}_{t+1}$ is the unexpected log return. Define the τ-period cumulative return from period $t+1$ through period $t+\tau$, as

$$\mathbf{r}_{t,t+\tau} = \sum_{i=1}^{\tau} \mathbf{r}_{t+i}.$$

The term structure of risk is defined as the conditional variance of cumulative returns, given the investor's information set, scaled by the investment horizon

$$\Sigma_r(\tau) \equiv \frac{1}{\tau} Var(\mathbf{r}_{t,t+\tau} \mid D_t), \tag{10}$$

where $D_t^{Mkt} \equiv \sigma\{z_\tau^{Mkt} : \tau \leq t\}$ consists of the full histories of returns as well as predictors that investors use in forecasting returns. Following Barberis (2000) and Campbell and Viceira (2002, 2005), we describe asset return dynamics by means of a first-order vector autoregressive or VAR(1) model. We choose a VAR(1) as the inclusion of additional lags, even if easily implemented, would reduce the precision of the estimates:

$$\mathbf{z}_t^{Mkt} = \Phi_0^{Mkt} + \Phi_1^{Mkt} \mathbf{z}_{t-1}^{Mkt} + \nu_t^{Mkt}, \tag{11}$$

where

$$\mathbf{z}_t^{Mkt} = \begin{bmatrix} r_{0t} \\ \mathbf{x}_t^{Mkt} \\ \mathbf{s}_t^{Mkt} \end{bmatrix}$$

is a $(m \times 1)$ vector, with r_{0t} being the log real return on the asset used as a benchmark to compute excess returns on all other asset classes, $\mathbf{x}_t^{Mkt}$ being the $(n \times 1)$ vector of log excess returns on all other asset classes with respects to the benchmark, and $\mathbf{s}_t^{Mkt}$ is the $((m-n-1) \times 1)$ vector of returns predictors. In the VAR(1) specification, Φ_0^{Mkt} is a $(m \times 1)$ vector of intercepts and Φ_1^{Mkt} is a $(m \times m)$ matrix of slopes. Finally, ν_t^{Mkt} is a $(m \times 1)$ vector

of innovations in asset returns and returns' predictors for which standard assumptions apply, i.e.

$$\nu_t^{Mkt} \sim \mathcal{N}(0, \Sigma_\nu^{Mkt}), \tag{12}$$

where Σ_ν^{Mkt} is the $(m \times m)$ variance-covariance matrix. Note that

$$\Sigma_\nu^{Mkt} = \begin{bmatrix} \sigma_0^2 & \sigma_{0x}' & \sigma_{0s}' \\ \sigma_{0x} & \Sigma_{xx} & \Sigma_{xs}' \\ \sigma_{0s} & \Sigma_{xs} & \Sigma_{ss} \end{bmatrix}$$

and the unconditional mean and variances-covariance matrix of $\mathbf{z}_t$, assuming that the VAR is stationary end therefore that this moments are well-defined, can be represented as follows:

$$\begin{aligned} \mu_z^{Mkt} &= \left(I_m - \Phi_1^{Mkt}\right)^{-1} \Phi_0^{Mkt} \\ vec\left(\Sigma_{zz}^{Mkt}\right) &= \left(I_{m^2} - \Phi_1^{Mkt} \otimes \Phi_1^{Mkt}\right)^{-1} vec\left(\Sigma_\nu^{Mkt}\right). \end{aligned}$$

The conditional mean of the cumulative asset returns at different horizons are instead

$$E_t(z_{t+1}^{Mkt} + ... + z_{t+\tau}^{Mkt}) = \left(\sum_{i=0}^{\tau-1} (\tau - i) \left(\Phi_1^{Mkt}\right)^i\right) \Phi_0^{Mkt} + \left(\sum_{j=0}^{\tau} \left(\Phi_1^{Mkt}\right)^j\right) \mathbf{z}_t^{Mkt},$$

and their variance is:

$$\begin{aligned} Var_t(z_{t+1}^{Mkt} + ... + z_{t+\tau}^{Mkt}) &= \Sigma_\nu^{Mkt} + (I + \Phi_1^{Mkt})\Sigma_\nu^{Mkt}(I + \Phi_1^{Mkt})' + \\ &\quad (I + \Phi_1^{Mkt} + \left(\Phi_1^{Mkt}\right)^2)\Sigma_\nu^{Mkt}(I + \Phi_1^{Mkt} + \left(\Phi_1^{Mkt}\right)^2)' + ... \\ &\quad + (I + \Phi_1^{Mkt} + ... + \left(\Phi_1^{Mkt}\right)^{\tau-1})\Sigma_\nu(I + \Phi_1^{Mkt} + ... + \left(\Phi_1^{Mkt}\right)^{\tau-1})'. \end{aligned} \tag{13}$$

Once the conditional moments of excess returns are available the following selector matrix extracts for each period, τ-period conditional moments of log real returns

$$M_r = \begin{bmatrix} 1 & 0_{1\times n} & 0_{1\times(m-n-1)} \\ \iota_{n\times 1} & I_{n\times n} & 0_{n\times(m-n-1)} \end{bmatrix}$$

which implies

$$\frac{1}{\tau} \begin{bmatrix} E_t\left(r_{0,t+1}^\tau\right) \\ E_t\left(\mathbf{r}_{t+1}^\tau\right) \end{bmatrix} = \frac{1}{\tau} M_r E_t(z_{t+1}^{Mkt} + ... + z_{t+\tau}^{Mkt})$$

$$\frac{1}{\tau}\begin{bmatrix} Var_t\left(r^{\tau}_{0,t+1}\right) \\ Var_t\left(\mathbf{r}^{\tau}_{t+1}\right) \end{bmatrix} = \frac{1}{\tau} M_r Var_t(z^{Mkt}_{t+1} + ... + z^{Mkt}_{t+\tau}) M_r'.$$

Therefore after the estimation of the VAR it is possible to derive unconditional and conditional moments for returns and excess returns at all different investment horizons. These moments deliver the dynamics of returns and the risk of different assets across investment horizons. This information forms the input for portfolio allocation. Following Campbell and Viceira (2005), we consider a benchmark portfolio to be obtained by attributing optimal weights to bond, stock and T-bills. Therefore we include in $\mathbf{x}^{Mkt}_t$ excess returns on stocks and bonds, real returns on T-bills, while we include in $\mathbf{s}^{Mkt}_t$ three factors commonly recognized as good predictors of these assets' returns. In particular, the predictors are the nominal short-term interest rate, the dividend price ratio and the yield spread between long-term and short term bonds.

C The dataset

C.1 Annuity prices

We estimate the change in the annuity price included in our VAR(1) model as

$$\Delta p_{65,t+1} = \ln\left(\frac{P^A_{t+1,65}}{P^A_{t,65}}\right),$$

where $P^A_{t+1,65}$ is the annuity price offered on the US market for 1$ monthly life annuities written on 65-year-old males. Consequently Δp^A_{t+1} is the yearly log price change of a standardized 65 year-old annuity between time t and $t+1$. Our annuity prices consist of the sample average premiums for immediate $1 monthly life annuities for 65-year old US males issued during the 1952-2007 period. In order to have the longest time series of prices, we collected premiums from different sources. In order to have the longest time series of prices, we collected premia from different sources. Following Warshawsky (1988) and Friedman and Warshawsky (1988), premia over the 1952-1967 period come from successive annual issues of Spectator's Handy Guide and A.M. Best's Flitcraft Compend , whereas premia over the 1968-1985 years come from the successive annual issues of A.M. Best's Flitcraft Compend. Following Koijen and Yogo (2012), Cox and Lin (2007) and Brown, Mitchell, and Poterba (2002), we compile premia for the 1986-2007 years from the semiannual issues of Annuity Shopper. These data are integrated with those

obtained from the annual issues of the Life/Health editions of Best's Review for the 1995-1998 period.

Given the length of the sample period and the different sources of data we use, our sample premia refer to an unbalanced panel of companies. Although the correct approach should be to use only rates reported by the same companies, this would substantially reduce the number of premia available each year for computing the minimum, the maximum and the average annual annuity premium which, consequently, might not be reflective of the true value of annuity price. The pricing approach adopted in the present paper is essentially based on the assumption that price changes of annuities reflect changes of fundamental risks priced by buyers and sellers. A side product of this estimation analysis is a direct empirical measure of the effective price reaction to changes in systematic mortality trends Clearly, empirical evidence of this connection would support the hypothesis that some form of competition drives the prices in the market for annuities.

C.2 Marketed securities and the state variables

The Campbell and Viceira (2005) model is developed using quarterly data. As adapting the mortality series to this frequency is both hardly feasible (lack of mortality data for frequencies higher than yearly) and less meaningful (for example, some months of the year experience higher mortality rates than others), we focus our analysis on annual data. We download the financial data from Robert Shiller's website[10]for the postwar period 1952-2007 and, following Campbell, Chan, and Viceira (2003) construct the financial time series as:

- Short-term ex-post real rate: return on 6-month commercial paper bought in January and rolled over July, minus the Producer Price Index (PPI).
- Excess return on stocks: log return on the S&P 500 Stocks, from which the short-term interest rate is subtracted.
- Excess return on bonds: returns are obtained using the loglinear approximation described in Campbell, Lo, MacKinlay, and Whitelaw (1998)

$$r_{n,t+1} = D_{n,t} y_{n,t} - (D_{n,t} - 1) y_{n-1,t+1},$$

[10]http://www.econ.yale.edu/~shiller/data.htm

where n is the Bond maturity, the Bond yield is $Y_{n,t}$, the log Bond yield is $y_{n,t} = (1 + Y_{n,t})$ and $D_{n,t}$ is the Bond duration, calculated at time t as

$$D_{n,t} \approx \frac{1 - (1 + Y_{n,t})^{-n}}{1 - (1 + Y_{n,t})^{-1}}$$

with n set to 20 years and $y_{n-1,t+1}$ approximated by $y_{n,t+1}$.

- Excess annuity prices' growth, as described in the previous subsection of this Appendix.

  Finally the following set of state variables are included in the VAR to parametrize the opportunity set faced by the investor

- Nominal T-bill rate: return on 6-month commercial paper bought in January and rolled over July.

- Log dividend-price ratio: natural logarithm of the S&P 500 dividend series minus the logarithm of the S&P 500 price series.

- Yield spread: difference between the log yield of the long Bond and the short yield on the commercial paper.

- Aggregate longevity shocks: qk_t as defined in Section 2, are average differences between predicted and fitted mortality rates for the cohorts underlying life annuities (in our case, 65-110).

D Figures and Tables

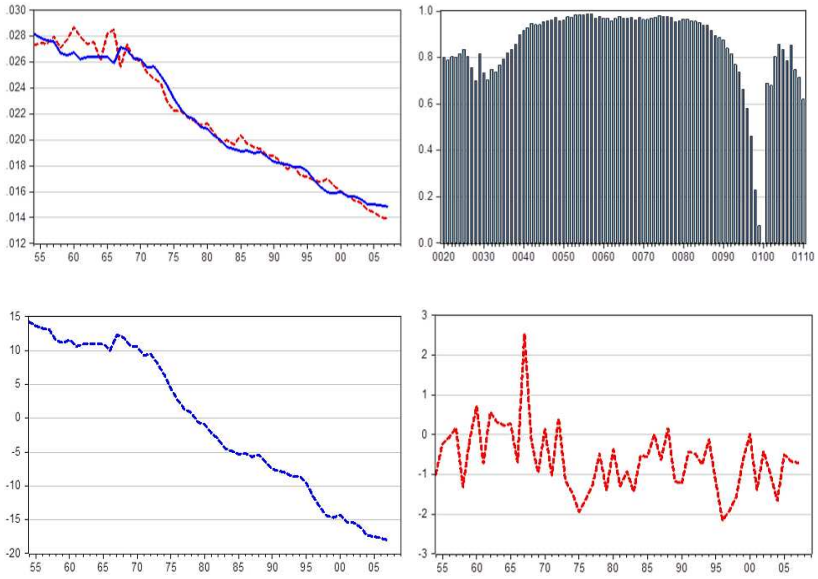

Figure 1: Lee-Carter Fitted mortality and pseudo out-of-sample (1991-2007) projections. Figure 1.1: Fitted mortality rates at 65. Figure 1.2: Cross-Sectional R^2 of Equation (3) Figure 1.3: The unobservable mortality index k_t. Figure 1.4: Innovations in k_t.

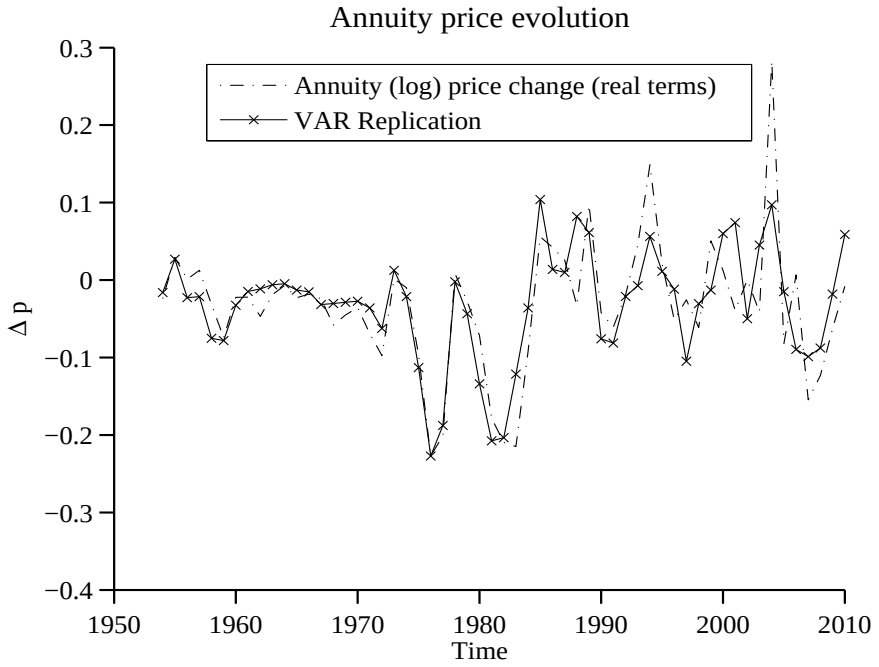

Figure 2: Time series of historical (real) logarithmic price changes (dashed line) vs the replication as operated by the VAR dynamic model.

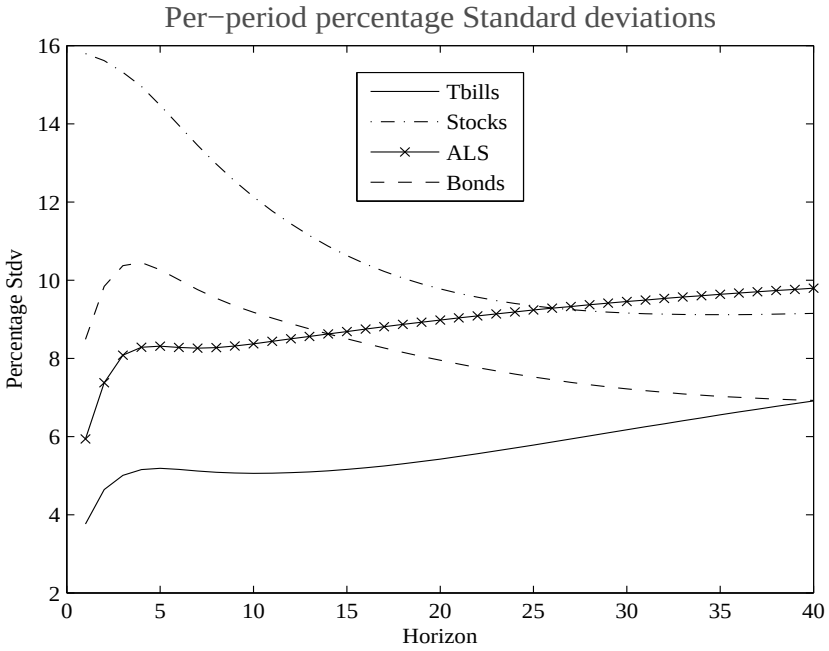

Figure 3: Term structure of risks for the securities included in the Extended VAR model.

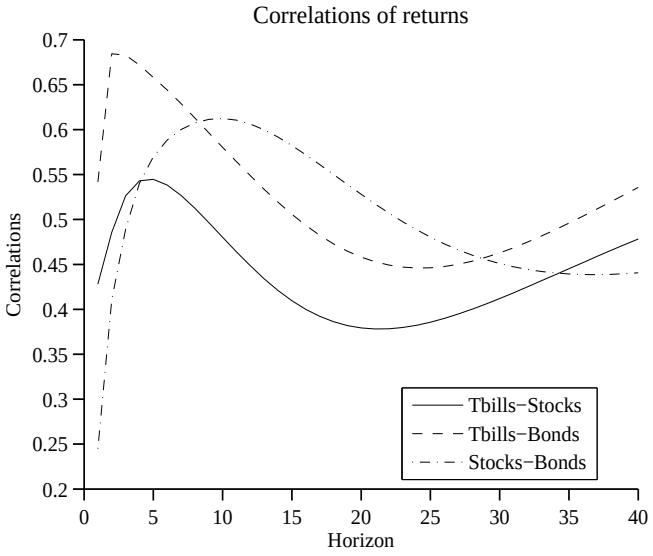

Figure 4: Term Structure of correlations between financial securities included in the Extended VAR model

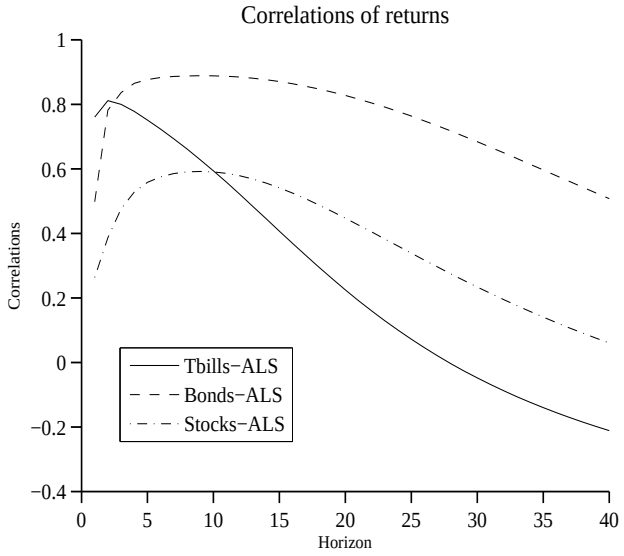

Figure 5: Term Structure of correlations between financial securities and the Annuity-Linked Security.

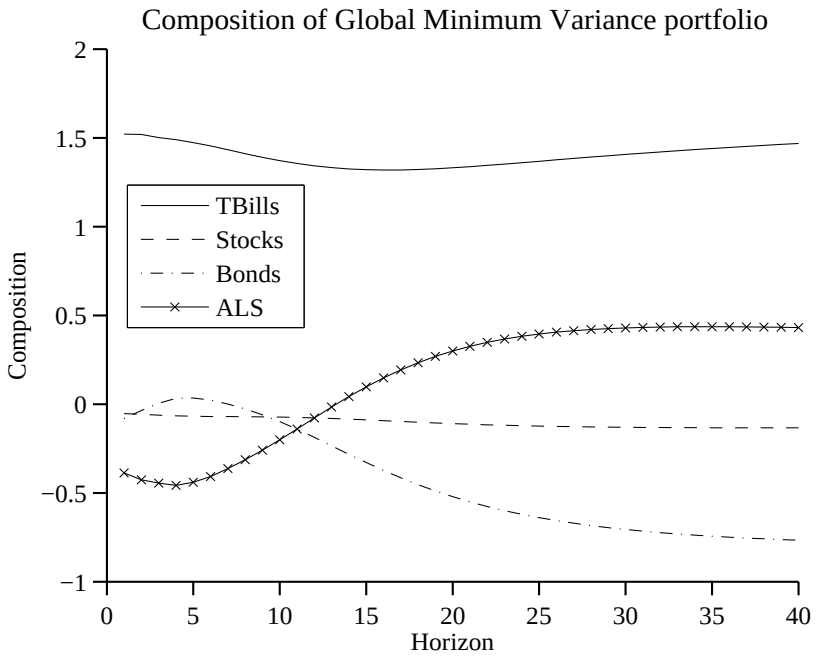

Figure 6: Term structure of allocations forming the GMV portfolio at different horizons.

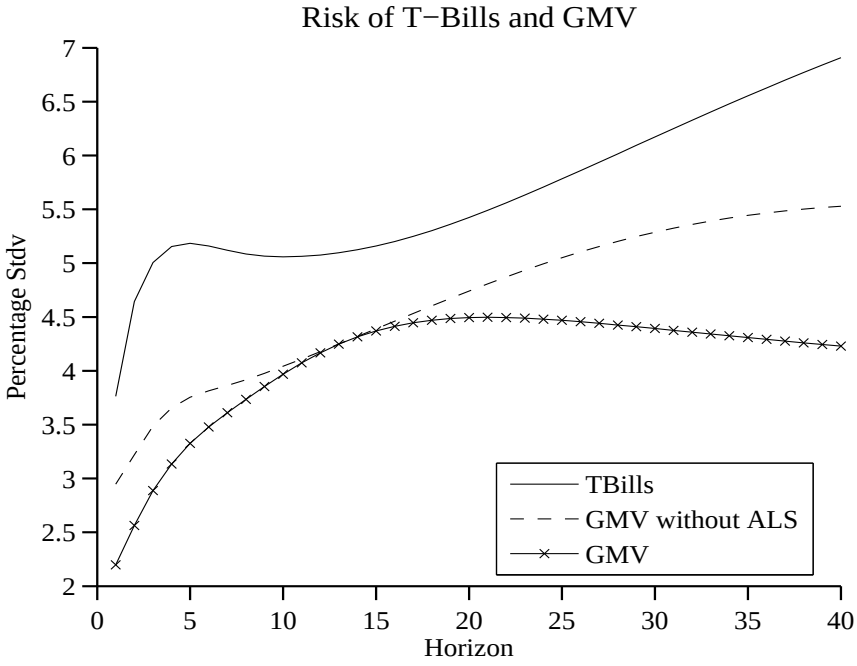

Figure 7: Term structure of risks for an allocation in T-Bill (continuous line), in the GMV portfolio restricted to financial securities (dashed line), in the GMV portfolio including also the Annuity-Linked Security.

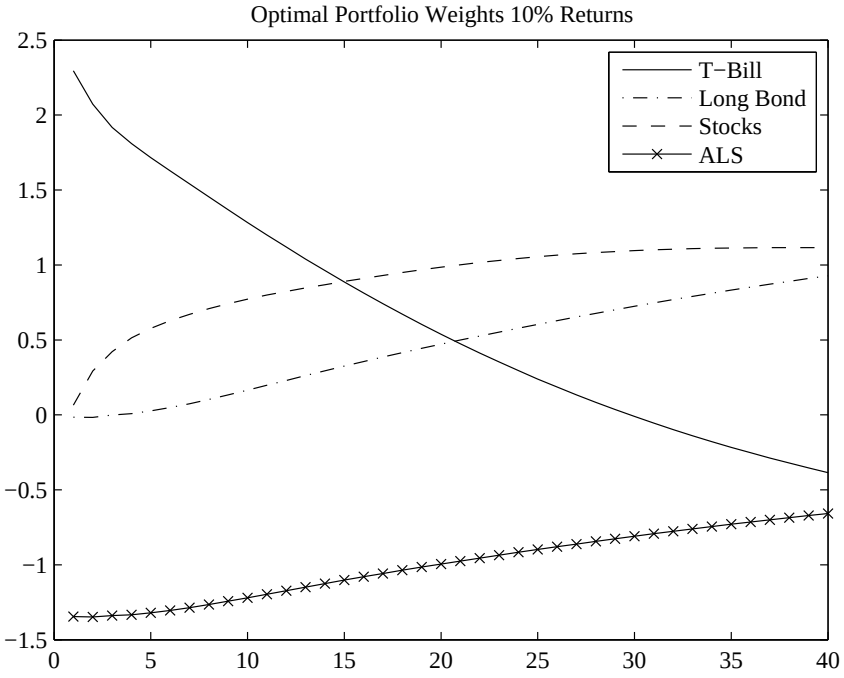

Figure 8: Term structure of the efficient allocation with target expected return of 10%.

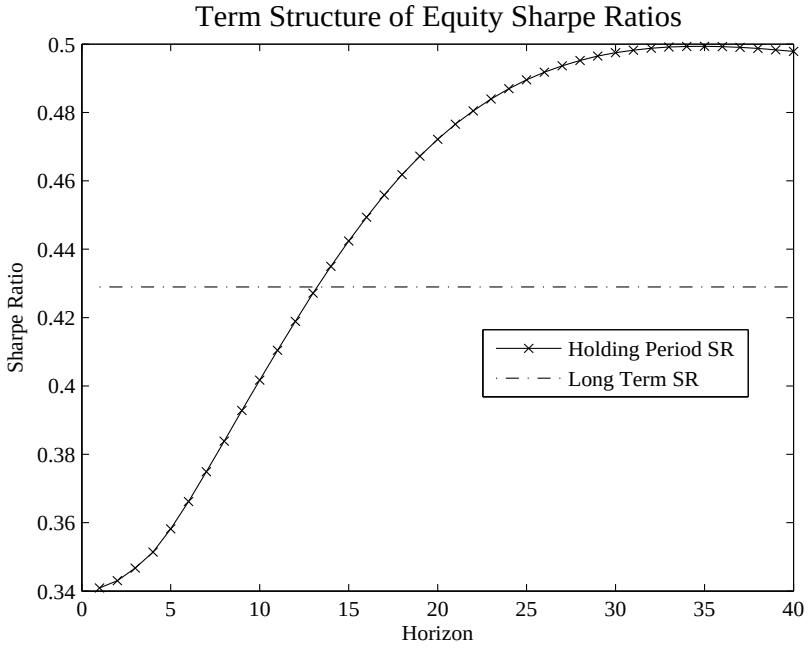

Figure 9: Term Structure of the Equity Sharpe Ratio where the initial level of the state variables is set to their long term expected value.

	rtb	xr	xb	$x\Delta pr$	y	$d-p$	spr	qk
Mean	0.027	0.056	0.0046	−0.027 5	0.056	−3.493	0.004	0.00(4)
Std Dev	0.052	0.160	0.108	0.057	0.031	0.411	0.014	0.041
Sharpe Ratio	-	0.353	0.04 259 3	−0.482 46	-	-	-	-

Table 1: Summary Statistics. Mean returns is computed including the Jensen correction term, thus are computed as $\mu + 0.5\sigma^2$. Sharpe Ratio is computed as the ratio between Mean and Std Dev. **Note**: rtb = ex post real T-bill rate, xr = excess stock return, xb = excess bond return, $(d-p)$ = log dividend-price ratio, y = nominal T-bill yield, spr = yield spread.

VAR(1) - Matrix Φ_1 - Yearly Sample 1953-2007. Original Financial Variables

	rtb_t	xr_t	xb_t	y_t	$(d-p)_t$	spr_t	R^2	$adjR^2$
	(t)	(t)	(t)	(t)	(t)	(t)		
rtb_{t+1}	0.567	-0.027	-0.110	0.414	-0.008	-0.640	0.530	0.471
	(3.825)	(-0.662)	(-1.843)	(2.350)	(-0.462)	(-1.507)		
xr_{t+1}	0.099	0.021	-0.003	-0.935	0.107	0.271	0.087	-0.027
	(0.192)	(0.130)	(-0.013)	(-1.056)	(1.963)	(0.171)		
xb_{t+1}	0.830	-0.087	-0.508	0.246	-0.036	4.684	0.546	0.489
	(3.649)	(-1.286)	(-4.315)	(0.558)	(-1.235)	(5.132)		
y_{t+1}	-0.218	0.036	0.043	1.006	0.005	0.211	0.784	0.757
	(-3.686)	(2.463)	(2.053)	(9.821)	(1.337)	(1.214)		
$(d-p)_{t+1}$	-0.548	0.071	0.049	0.398	0.939	1.414	0.872	0.855
	(-0.993)	(0.402)	(0.237)	(0.417)	(16.822)	(0.787)		
spr_{t+1}	0.126	-0.030	0.009	-0.039	-0.001	0.405	0.455	0.387
	(2.778)	(-2.680)	(0.706)	(-0.532)	(-0.434)	(3.204)		

Cross-Correlations of Residuals

	rtb	xr	xb	y	$(d-p)$	spr
rtb	3.552	0.464	0.070	-0.247	-0.501	0.285
xr	-	13.507	0.170	-0.406	-0.977	0.435
xb	-	-	7.273	-0.637	-0.191	0.160
y	-	-	-	1.403	0.425	-0.845
$(d-p)$	-	-	-	-	14.480	-0.436
spr	-	-	-	-	-	1.066

Table 2: VAR(1) coefficients with relative t-statistics and Cross-Correlations of Residuals. **Note**: rtb_t = ex post real T-bill rate, xr_t = excess stock return, xb_t = excess bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread.

VAR(1) - Matrix Φ_1 - Yearly Sample 1953-2007. Annuities

	rtb_t	xr_t	xb_t	$x\Delta pr_t$	y_t	$(d-p)_t$	spr_t	$(qk)_t$	R^2	$adjR^2$
	(t)	(t)	(t)	(t)	(t)	(t)	(t)	(t)		
rtb_{t+1}	0.477	0.007	-0.106	0.074	0.590	-0.010	-0.680	-0.016	0.475	0.389
	(3.265)	(0.203)	(-1.753)	(0.640)	(2.534)	(-0.568)	(-1.452)	(-0.116)		
xr_{t+1}	0.190	0.048	0.131	0.929	-0.215	0.154	-0.892	-0.236	0.167	0.031
	(0.388)	(0.363)	(0.679)	(1.660)	(-0.225)	(2.423)	(-0.462)	(-0.372)		
xb_{t+1}	0.917	-0.116	-0.502	0.116	0.267	-0.054	4.362	0.588	0.562	0.491
	(4.121)	(-1.972)	(-4.311)	(0.370)	(0.550)	(-1.921)	(4.203)	(2.721)		
$x\Delta pr_{t+1}$	0.531	-0.099	0.057	-0.188	-1.146	-0.034	-0.446	0.271	0.516	0.437
	(4.294)	(-2.228)	(0.970)	(-0.968)	(-3.950)	(-1.871)	(-0.675)	(2.199)		
y_{t+1}	-0.204	0.025	0.038	-0.046	0.940	0.007	0.286	-0.064	0.794	0.761
	(-3.702)	(2.477)	(1.702)	(-0.841)	(9.253)	(1.896)	(1.315)	(-1.728)		
$(d-p)_{t+1}$	-0.812	0.149	-0.081	-0.898	-0.038	0.885	2.475	0.148	0.845	0.819
	(-1.392)	(0.808)	(-0.363)	(-1.504)	(-0.035)	(13.163)	(1.141)	(0.221)		
spr_{t+1}	0.104	-0.017	0.014	0.048	0.036	-0.002	0.339	0.011	0.456	0.367
	(2.452)	(-1.775)	(1.040)	(1.069)	(0.466)	(-0.585)	(2.076)	(0.404)		
$(qk)_{t+1}$	-0.113	-0.051	0.000	-0.059	-0.008	-0.009	0.373	0.747	0.575	0.506
	(-1.341)	(-1.671)	(0.002)	(-0.617)	(-0.034)	(-1.172)	(1.292)	(6.625)		

Cross-Correlations of Residuals

	rtb	xr	xb	Δpr	y	$(d-p)$	spr	$(-qk)$
rtb	3.763	0.206	0.117	0.192	-0.298	-0.234	0.318	0.104
xr	-	14.584	0.040	-0.063	-0.248	-0.970	0.295	0.118
xb	-	-	7.184	0.176	-0.620	-0.089	0.132	-0.014
Δpr	-	-	-	3.927	-0.547	0.022	0.583	-0.044
y	-	-	-	-	1.368	0.262	-0.843	-0.041
$(d-p)$	-	-	-	-	-	15.768	-0.277	-0.098
spr	-	-	-	-	-	-	1.047	0.047
$(-qk)$	-	-	-	-	-	-	-	2.617

Table 3: VAR(1) coefficients with relative t-statistics and Cross-Correlations of Residuals. **Note**: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, xb_t = excess bond return, Δpr_t = log difference on annuities premium $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread, $(-qk)$ = aggregate mortality shock.

$HedgingPortfolio$	w_{xr}	w_{rtb}	w_{rtb}	$w_{x\Delta pr}$
$Unconstrained$	-0.042	-0.014	0.009	0.047
$Constrained\_1\_w_{xr} = 0$	0	-0.062	0.004	0.058
$Constrained\_2\_w_{rtb} = w_{xr} = 0$	0	0	-0.059	0.059

Table 4: Optimal Allocation for Aggregate Longevity Hedging Portfolio under different constraints on the set of financial securities included in the hedging portfolio.

SR_{∞}^{Equity}	$SR_{\infty}^{LongBond}$	SR_{∞}^{ALS}
0.429	0.0323	(-) 0.437

Table 5: Long term Sharpe Ratios for securities included in the Extended VAR. Values are computed as illustrated in eq.8.

SR_{∞}^{Unc}	$SR_{\infty}^{Cons\_1}$	$SR_{\infty}^{Cons\_2}$
(-) 0.504	(-) 0.421	(-) 0.333

Table 6: Long term Sharpe Ratios for Aggregate Longevity Hedging Portfolios. Values are computed as illustrated in eq.8.