# Risk, Uncertainty, Value, and Public Policy

## to honor of Massimo Marinacci

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# Doubts

*Knowledge would be fatal, it is the uncertainty that charms one. A mist makes things beautiful.*

*Oscar Wilde*, The Picture of Dorian Gray, 1891

*Our doubts are traitors,*
*And make us lose the good*
*we oft might win,*
*By fearing to attempt.*

*William Shakespeare*, Measure for Measure, act 1, scene 4

# Uncertainty

- What is it?
- Why do we care?
- How do we represent it?
- How do we measure it?
- Who confronts it?
- How does it affect equilibrium concepts?
- What does it do to
  - quantities?
  - prices?
- How does it affect design of good government policies?

# What is it?

Fear of model misspecification.

# A model = a stochastic process

A *model* is a probability distribution over a sequence.

# Digression on rational expectations

- A rational expectations model is shared by every agent inside a model, by nature, and by the econometrician.
- The 'sharing with nature' part *precludes* concerns about model misspecification (along with belief heterogeneity).

# Why do we care?

Two distinct reasons:

- Ellsberg experiments make Savage axioms dubious.
- It is difficult statistically to distinguish alternative models from samples of the sizes of typical macroeconomic data sets.

I will emphasize the second reason.

# How do we represent it?

As a decision maker who has a set of models.

# How do we manage it?

- Construct bounds on value functions.
- Our tool for constructing bounds on value functions: min-max expected utility.
  - Two-player zero-sum game.
  - A minimizing player helps a maximizing player to compute bounds on value functions and to evaluate fragility of decision rules.

# How do we measure it?

- Relative entropy.
  - An expected log likelihood ratio.
  - A statistical measure of model discrepancy.
  - It tells how difficult it is to distinguish two models statistically.
  - It bounds rates of statistical learning as sample size grows.

# Entropy

*When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: 'Call it entropy. It is already in use under that name and besides, it will give you a great edge in debates because nobody knows what entropy is anyway.' Quoted by Georgii, "Probabilistic Aspects of Entropy," 2003.*

# How do we measure it?

Size of set of decision maker's statistical models, as measured by relative entropy.

# Who confronts it?

- We the model builders do. So do ...
- Agents inside our models.
  - Private citizens.
  - Government policy makers.

# How does it affect equilibrium concepts?

We want:

- An equilibrium concept as close as possible to rational expectations.
- A common approximating model for all agents in model.
- An extension of either
  - A recursive competitive equilibrium.
  - Nash or subgame perfect equilibrium.
  - Self-confirming equilibrium.

# Belief heterogeneity

- Common approximating model for all agents in model, but ...
- Diverse interests and min-max expected utility give rise to *ex post* heterogeneity of (worst-case) models.
- A disciplined model of belief heterogeneity.

# What it does to quantities

- An increase in model uncertainty operates like an increase in the discount factor.
- Observational equivalence results ($\exists$ a $\beta - \theta$ ridge).
- There is a form of precautionary saving differing from the usual kind based on convexity of the marginal utility of consumption.

# What it does to prices

- Makes a potentially volatile 'preference shock' multiply the ordinary stochastic discount factor.
- Portfolio holders' worst-case beliefs affect state contingent prices.
- That gives rise to a 'market price of model uncertainty'.
- Helps attain Hansen-Jagannathan asset pricing bounds by increasing volatility of stochastic discount factor under the common approximating model.

# Does uncertainty aversion resemble risk aversion?

- Yes, in some ways, but ...
- It activates attitudes about the intertemporal distribution of uncertainty that distinguish it from risk aversion.

# Can small amounts of uncertainty aversion substitute for large amounts of risk aversion?

- Pratt experiment for calibrating magnitude of risk aversion.
- Anderson-Hansen-Sargent measures of statistical discrepancies between alternative statistical models for calibrating magnitude of uncertainty.

# How does it affect government policy design problems?

- Portfolio holders' worst-case beliefs show up in state contingent prices.
- This can make a disciplined form of purposeful belief manipulation concern a Ramsey planner.
- Alternative ways to configure model uncertainties. (Massimo and Fabio and Simone and Pieropaolo's new paper on uncertainty and self-confirming equilibria)

# Why not learn your way out?

- Some specifications are statistically difficult to distinguish (e.g., low frequency attributes where laws of large numbers and central limit theorems ask for patience).
- How do you learn when you don't have a single model?
  - A Bayesian knows the correct model from the beginning.
  - Bayesian learns by conditioning in light of a single model (i.e., a probability distribution over a sequence).
  - How do you learn when you don't have a single model?
- Massimo, Simone, Fabio, and Luigi's new paper extending de Finetti's fundamental theorem of statistics.

# What it does to public policy

In a dynamic game or a competitive equilibrium with a government, there are various things to be uncertain about.

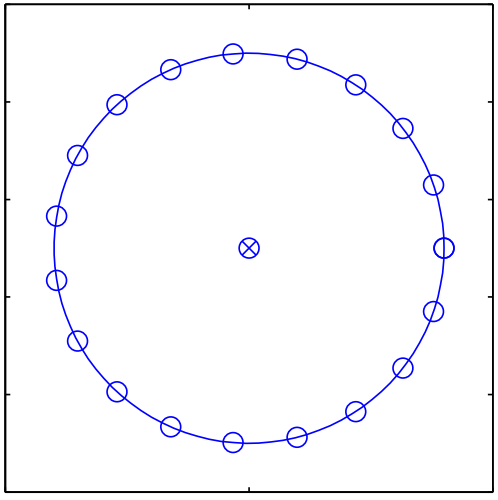
# Multiple uncertainties

Four configurations of uncertainties between a government or Ramsey planner with model(s) x and a private sector with model(s) o.

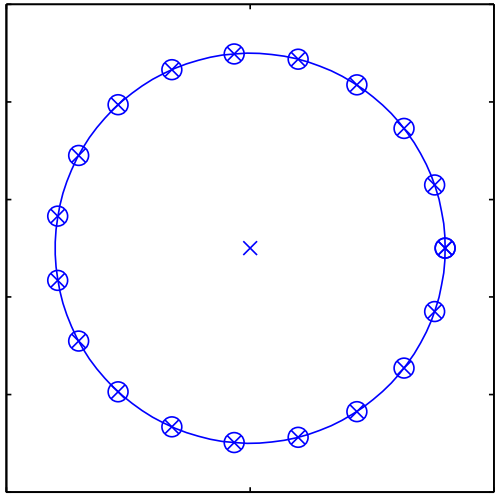
- **Type 0**: Ramsey planner trusts its approximating model (x), knowing private agents (o) don't trust it.
- **Type I**: Ramsey planner has set of models (x) centered on an approximating model, while private sector knows a correct model (o) among Ramsey planner's set of models x.
- **Type II**: Ramsey planner has set of models (x) surrounding its approximating model, which private sector trusts (o).
- **Type III**: Ramsey planner has single model (x) but private sector has another model in an entropy ball around (x).

# Multiple uncertainties

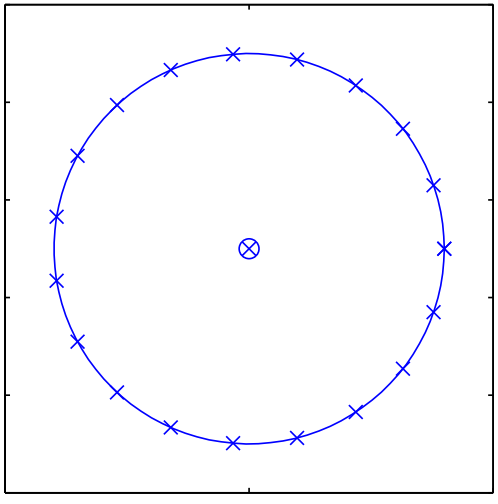
Type 0



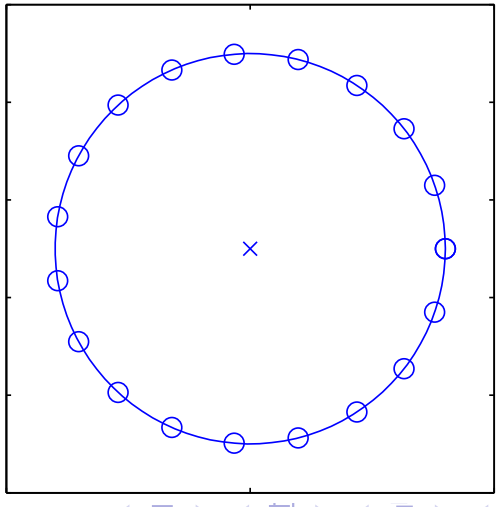
Type I



Type II



Type III



# Densities and ratios

- A random variable $c$
- A probability density $f(c)$
- Another 'nearby' probability density $\tilde{f}(c)$
- A likelihood ratio

$$g(c) = \frac{\tilde{f}(c)}{f(c)}$$

- Evidently

$$\tilde{f}(c) = g(c)f(c)$$

$$Eg(c) = \int g(c)f(c)dc = 1$$

# Entropy

- Entropy is an expected log-likelihood ratio

$$\text{ent} = \int \log(g)\tilde{f}\,dc$$

$$\text{ent} = \int \log(g)gf dc \geq 0$$

# How do we represent it?

Constraint preferences:

$$\min_{g \geq 0} \int U(c)g(c)f(c)dc$$

subject to

$$\begin{aligned} \int gfdc &= 1 \\ \int log(g)gfdc &\leq \eta \end{aligned}$$

# How do we represent it?

Multiplier preferences:

$$\min_{g \geq 0} \int \left[ U(c) + \theta \log(g(c)) \right] g(c) f(c) dc$$

subject to

$$\int gfdc = 1 \tag{1}$$

# How do we represent it?

Minimizing distortion $g$:

$$\hat{g}(c) \propto \exp\left(\frac{U(c)}{-\theta}\right)$$

$$\hat{g}(c) = \frac{\exp\left(\frac{U(c)}{-\theta}\right)}{\int \exp\left(\frac{U(c)}{-\theta}\right) dc}$$

Bucklew (2004, p. 27) calls this a statistical version of *Murphy's law*:

> "The probability of anything happening is in inverse ratio to its desirability."

# How do we represent it?

Risk-sensitive operator:

$$\mathsf{T}U(c) \doteq \min_{g \geq 0} \int \left[ U(c)g(c) + \theta \log(g(c)) \right] g(c) f(c) dc$$

subject to

$$\int gfdc = 1 \tag{2}$$

Indirect utility function:

$$\mathsf{T}(U) = -\theta \log E \exp\left( \frac{U(c)}{-\theta} \right)$$

# Relationship among preferences

- Constraint and multiplier preferences differ.
- Constraint preferences are more natural.
- Multiplier preferences are easier to work with.
- Fortunately, for the purposes of asset pricing, their indifference curves are tangent at a given allocation.

# Indifference curves

- Expected utility:

$$\frac{dc_2}{dc_1} = -\frac{\pi_1}{\pi_2}\frac{u'(c_1)}{u'(c_2)}$$

- Constraint and *ex post* Bayesian preferences:

$$\frac{dc_2}{dc_1} = -\frac{\hat{\pi}_1}{\hat{\pi}_2}\frac{u'(c_1)}{u'(c_2)}$$

  where $\hat{\pi}_1, \hat{\pi}_2$ are the minimizing probabilities computed from the worst-case distortions.

- Multiplier and risk-sensitive preferences:

$$\frac{dc_2}{dc_1} = -\frac{\pi_1}{\pi_2}\frac{\exp(-u(c_1)/\theta)}{\exp(-u(c_2)/\theta)}\frac{u'(c_1)}{u'(c_2)}$$

# State prices

The state prices are

$$q_i = \pi_i \hat{g}_i u'(\bar{c}_i) = \pi_i \left( \frac{\exp(-u(\bar{c}_i)/\theta)}{\sum_j \pi_j \exp(-u(\bar{c}_j)/\theta)} \right) u'(\bar{c}_i).$$

The worst-case likelihood ratio $\hat{g}_i$ operates to increase prices $q_i$ in relatively low utility states $i$.

# Indifference curves

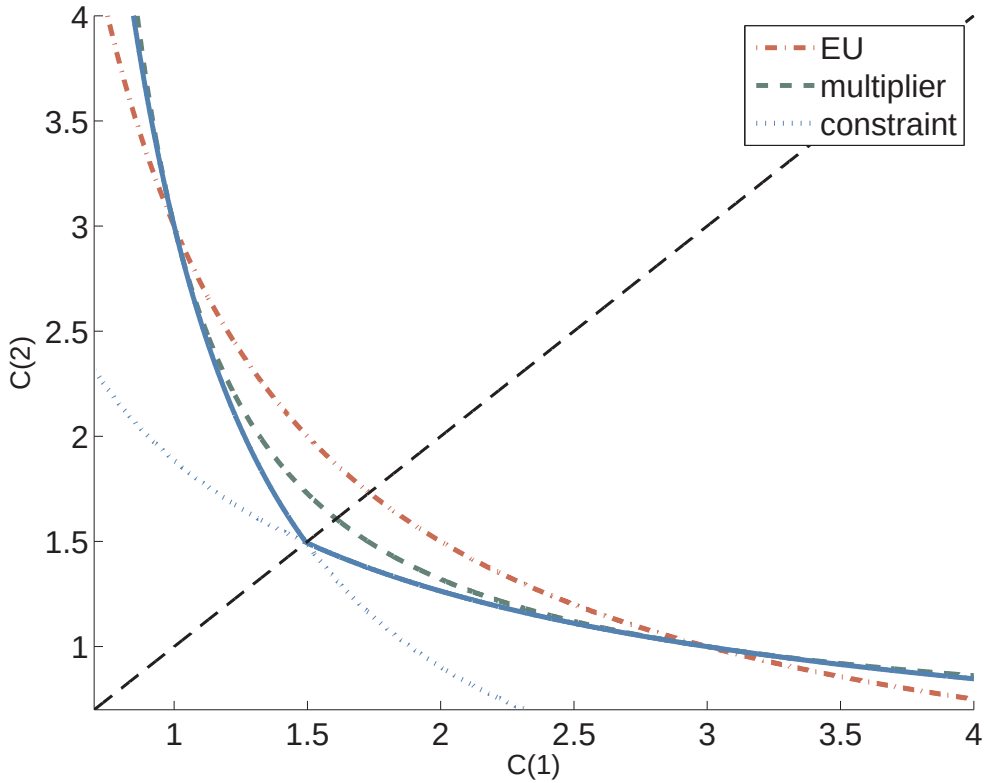



Figure: Indifference curves through point $(c_1, c_2) = (3, 1)$ for expected logarithmic utility (less curved smooth line), multiplier (more curved line), constraint (solid line kinked at 45 degree line), and *ex post* Bayesian (dotted lines) preferences. The worst-case probability $\hat{\pi}_1 < .5$ when $c_1 = 3 > c_2 = 1$ and $\hat{\pi}_1 > .5$ when $c_1 = 1 < c_2 = 3$.

# How do we represent it?

- $E\exp\big(-u(c)/\theta\big) = \int \exp\big(-u(c)/\theta\big) f(c)dc$ is a moment generating function for $u(c)$.
- $h(\theta^{-1}) \doteq \log \int \exp\big(-u(c)/\theta\big) f(c)dc$ is a cumulant generating function.
- $h(\theta^{-1}) = \sum_{j=1}^{\infty} \kappa_j \frac{(-\theta^{-1})^j}{j!}$, where $\kappa_j$ is the $j$th cumulant of $u(c)$.
- Thus, $\mathsf{T}u(c) = -\theta h(\theta^{-1}) = -\theta \sum_{j=1}^{\infty} \kappa_j \frac{(-\theta^{-1})^j}{j!}$.

# How do we represent it?

- When $\theta < +\infty$, $\mathrm{T}u(c)$ depends on cumulants of all orders.
- For the particular case $u(c) \sim \mathcal{N}(\mu_u, \sigma_u^2)$, $\kappa_1 = \mu_u, \kappa_2 = \sigma_u^2$ ,and $\kappa_j = 0 \ \forall j \geq 3$, so that $\mathrm{T}u(c) = \mu_u - \frac{1}{2\theta}\sigma_u^2$, or

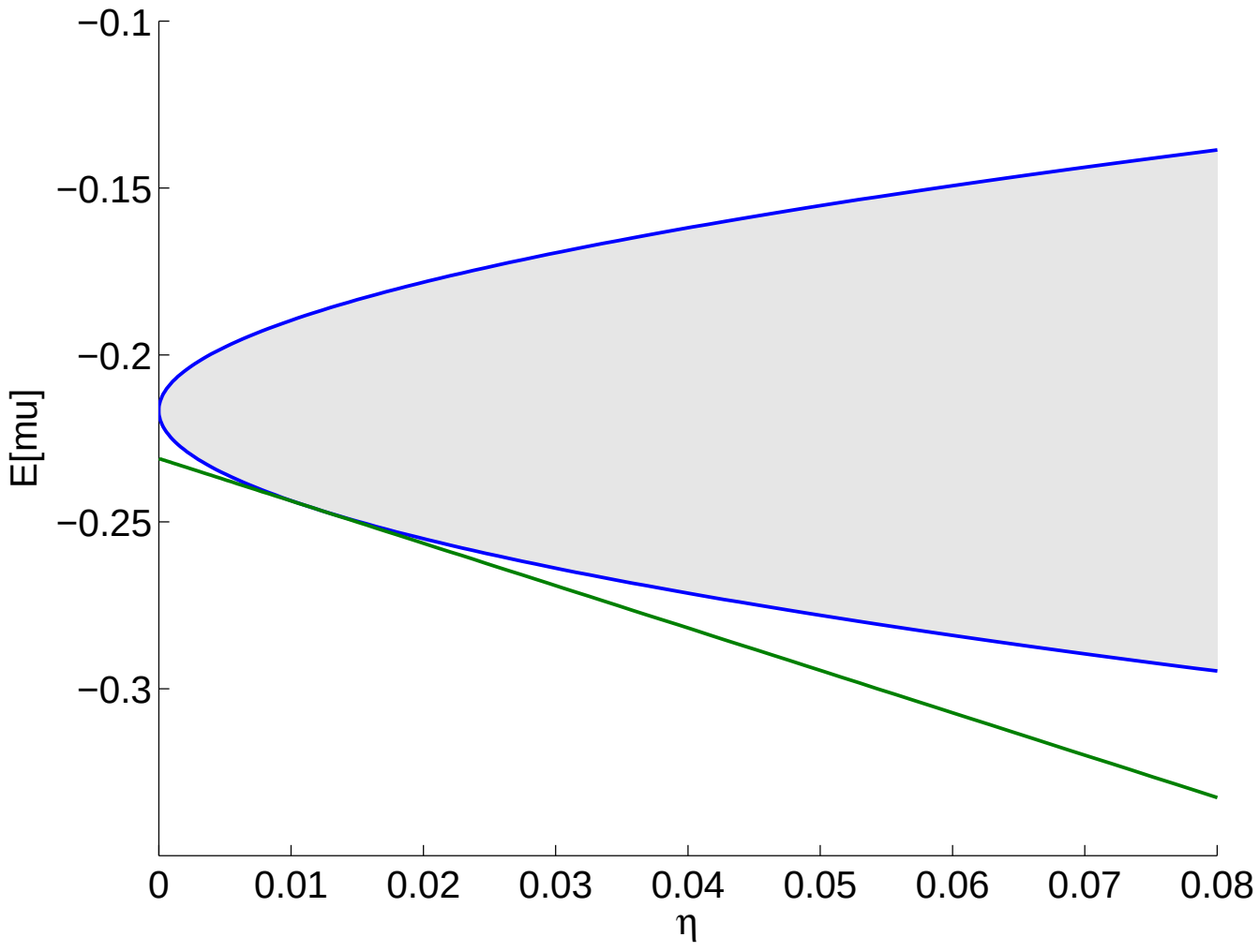
$$\mathrm{T}u(c) = E(u) - \frac{1}{2\theta}\mathrm{var}(u)$$

- $\mathrm{T}u(c)$ becomes expected utility $\mu_u$ when $\theta^{-1} = 0$.
- Duffie and Epstein's stochastic differential utility.

# Robustness bound

$$\int g(c)u(c)f(c)dc \geq \mathsf{T}_\theta u(c) - \theta \int g(c) \log g(c) f(c) dc.$$

# Robustness bound



# Dynamics

$F(x^t)$ joint density over $x^t = (x_t, x_{t-1}, ..., x_0)$. Factor it:

$$F(x^t) = f(x_t|x^{t-1})F(x^{t-1})$$

Distorted joint density

$$\widehat{F}(x^t) = G(x^t)F(x^t)$$

where $G(x^t)$ is a likelihood ratio. Factor it:

$$\begin{aligned} G(x^t) &= g(x_t|x^{t-1})G(x^{t-1}) \\ E\left[g(x_t|x^{t-1})|x^{t-1}\right] &= 1 \Rightarrow \\ EG(x^t)|x^{t-1} &= G(x^{t-1}) \end{aligned}$$

The likelihood ratio $G$ is a martingale under $F$.

# Dynamics

$g(x_t|x^{t-1})$ distorts $f(x_t|x^{t-1})$

$$\widehat{f}(x_t|x^{t-1}) = g(x_t|x^{t-1})f(x_t|x^{t-1})$$

In our applications, worst-case distortion is

$$\check{g}(x_t|x^{t-1}) \propto \exp\left(\frac{-V(x^t)}{\theta}\right)$$

# Bellman equation

Ordinary:

$$U(x^t) = u(x^t) + \beta E_t U(x^{t+1})$$

Multiplier preferences:

$$V(x^t) = u(x^t) + \beta \mathsf{T}_t(V(x^{t+1}))$$

or

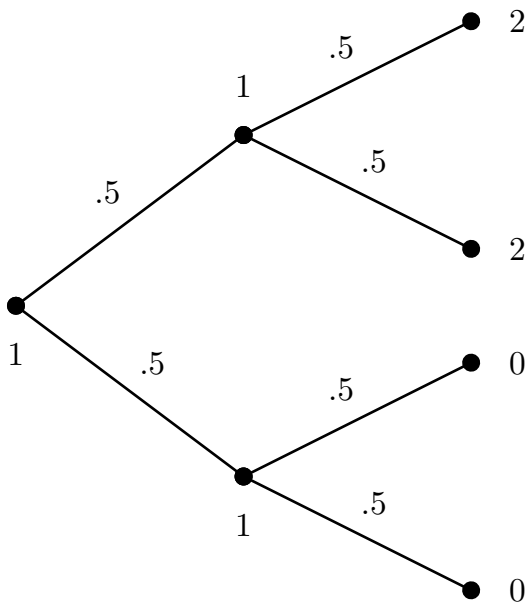
$$V(x^t) = u(x^t) - \beta\theta \log E_t \left[\exp\left(\frac{-V(x^{t+1})}{\theta}\right)\right]$$

# Bellman equation, multiplier preferences

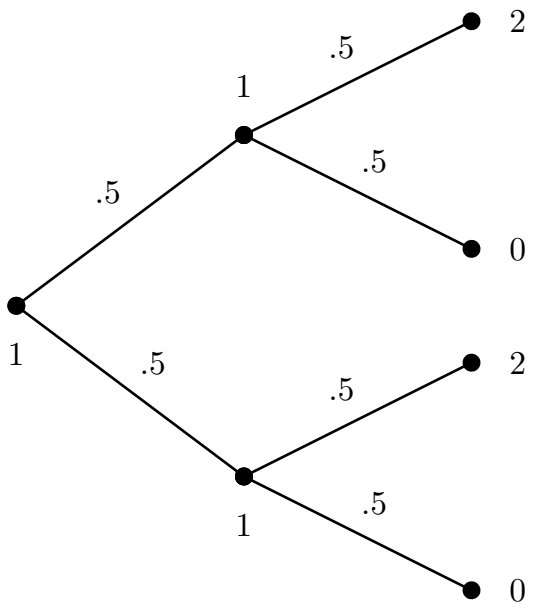
$$
\begin{aligned}
V(x^t) &= u(x^t) + \beta E_t(\check{g}(x_{t+1}|x^t)V(x^{t+1})) \\
&\quad + \beta\theta E_t\left[\log \check{g}(x_{t+1}|x^t)\check{g}(x_{t+1}|x^t)\right]
\end{aligned}
$$

$$
\check{g}(x_{t+1}|x^t) \propto \exp\left(\frac{-V(x^{t+1})}{\theta}\right)
$$

# Attitude toward timing



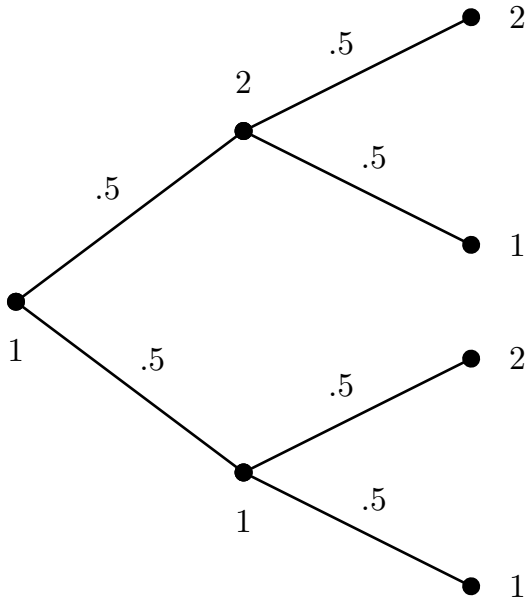

Plan A



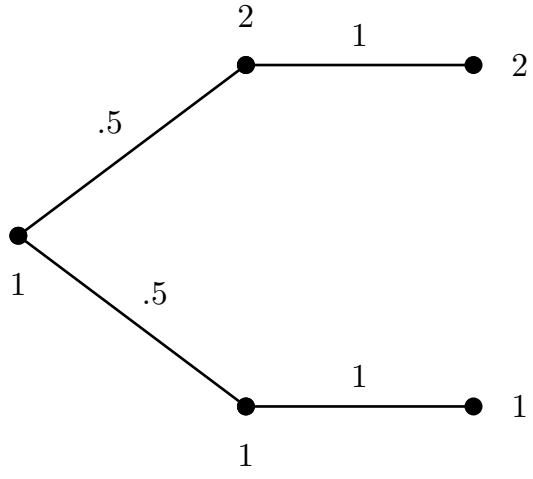

Plan B

Figure: Plan A has early resolution of uncertainty. Plan B has late resolution of uncertainty.

# Attitude toward persistence




Plan C




Plan D

Figure: Plan C has i.i.d. risk. Plan D has persistent risk.

# Punch line

Person with multiplier preferences:

- Likes early resolution of uncertainty.
- Dislikes persistence of uncertainty.

Expected utility person is indifferent to both.

# Optimism and pessimism

With $\theta < +\infty$, aversion to persistence of risk implies worst-case model asserts:

- Good news is temporary.
- Bad news is persistent.

This is a possible definition of a pessimist.

# Optimism and pessimism

- The probability choosing minimizing agent uses his entropy budget wisely by distorting low-frequency, difficult to detect features of the stochastic dynamics.
- This has a beautiful interpretation in terms of a frequency domain representation of quadratic objective functions.

# Disciplining $\theta$

$F(x_t)$ - model A
$\hat{F}(x_t) = G(x^t)F(x^t)$ - model W, depends on $\theta$

Detection error probability:
Probability that a likelihood ratio model selection test gives the wrong answer.
(Likelihood ratio is a random variable.)

# Detection error probabilities

Form the log likelihood ratio $\log(G(x^t)) = \log\left(\frac{\hat{F}(x_t)}{F(x^t)}\right)$

if $F(x^t)$ generated the data, $\log(G(x^t))$ should be negative

if $\hat{F}(x^t)$ generated the data, $\log(G(x^t))$ should be positive

Frequency of mistakes: where $I$ is indicator function,

$E[I(\log(G(x^t) > 0)]$ under model $F$

$E[I(\log(G(x^t) < 0)\, G(x^t)]$ under model $\hat{F}$

Assemble these frequencies to get average detection error probability (under models $F$ and $\hat{F}$).

# Punch line

When it comes to explaining 'risk premium puzzles', can a small or moderate amount of model uncertainty substitute for a large amount of risk aversion?

Yes.

# Cost of aggregate fluctuations

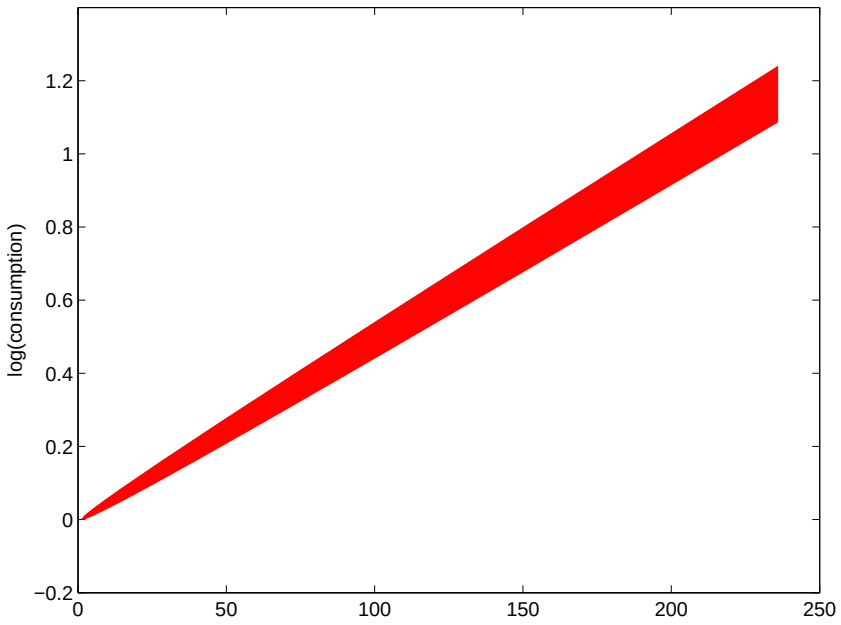



Figure: Lucas's experiment – shut down $\sigma_\varepsilon^2$.

# Cost of aggregate fluctuations

Tallarini's formula:

$$c_0 - c_d = \left(\frac{\beta}{1-\beta}\right)\frac{\gamma\sigma_\varepsilon^2}{2}$$

# Costs of business cycles

*No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it. Robert Lucas, Jr., "Macroeconomic Priorities," 2003.*

# But . . .

See recent empirical work by Levon Barseghyan and co-authors before accepting Lucas's statement about insurance deductibles.

# Uncertainty

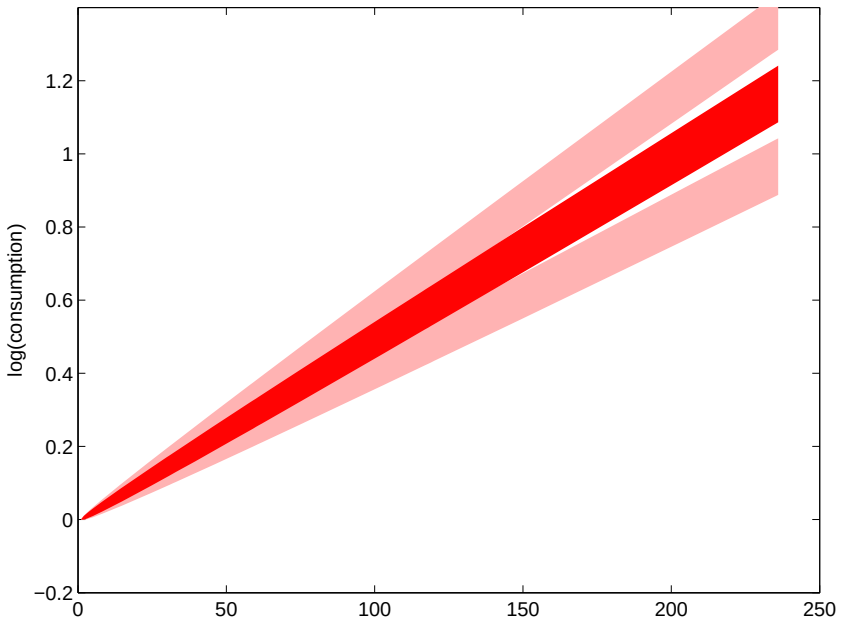



Figure: Model uncertainty.

# Uncertainty elimination

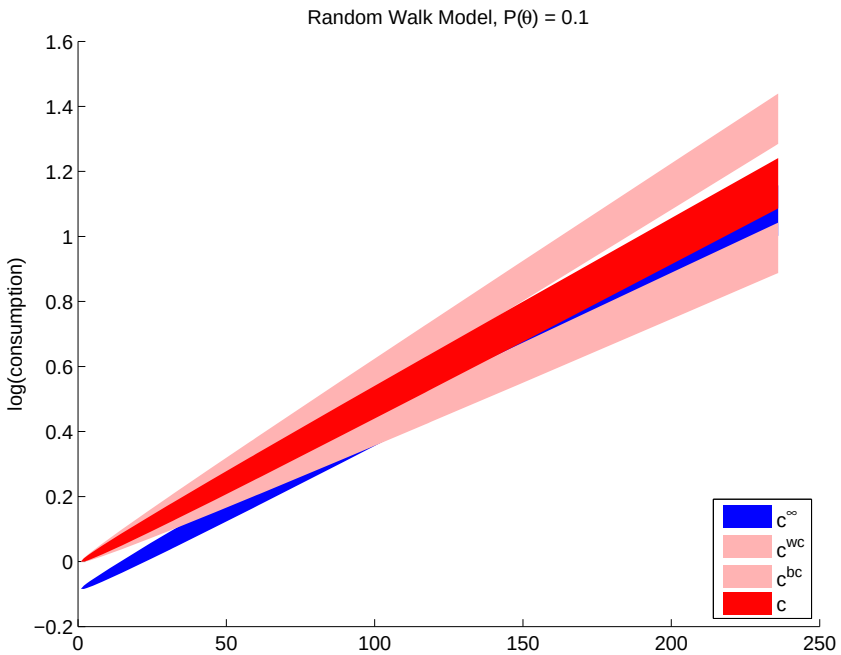



Figure: Elimination of model uncertainty but not risk (reduce $\theta^{-1}$ to zero).

# Costs of uncertainty

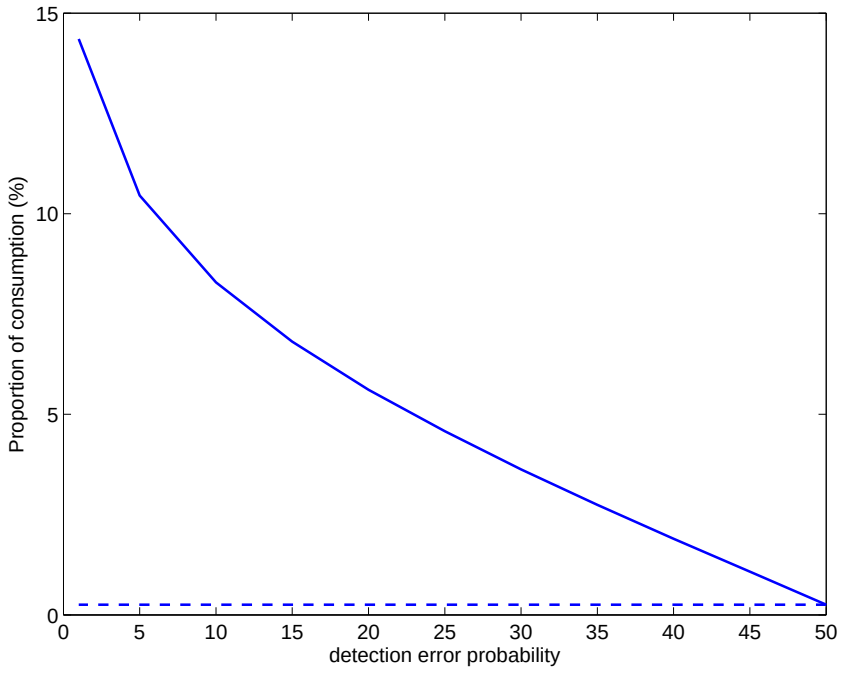



Figure: Proportions $c_0 - c_0^d$ of initial consumption that a representative consumer with model-uncertainty averse (multiplier) preferences would surrender not to confront risk (dotted line) and model uncertainty (solid line) for random-walk model of log consumption growth, plotted as a function of detection error probability.

# Specification differences

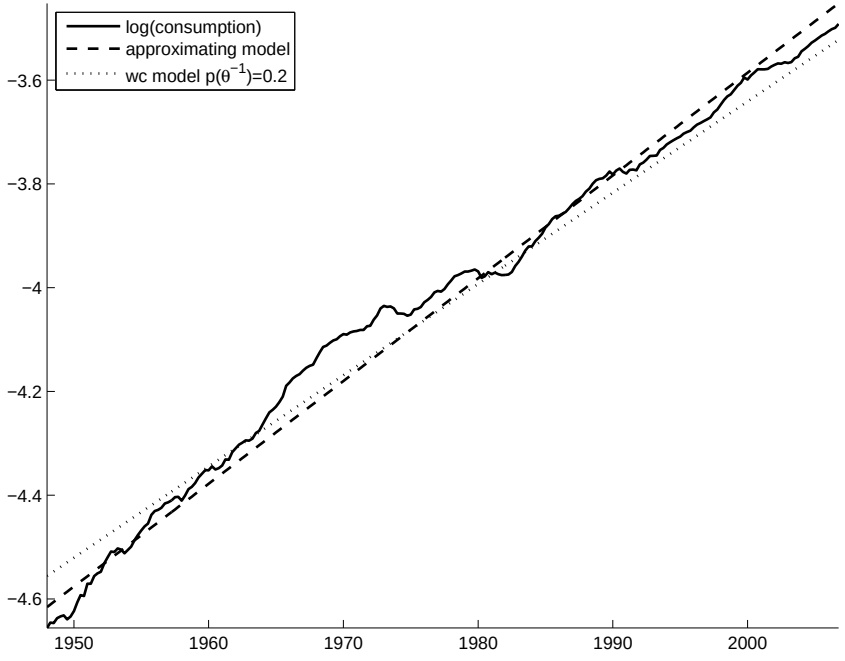



Figure: Log consumption and two lines

# Learning

$F(x^\infty, s^\infty)$ - joint distribution over states and signals
$x^\infty$ - states
$s^\infty$ - signals
Filtering: $F(x^\infty|s^t)$
Robust Filtering - response to not trusting $F(x^\infty|s^t)$

# Markov Setting

$x_t \sim$ hidden state
$\xi$ - sufficient statistics for $f(x_t|s^t)$
hidden Markov model - value function $V(\xi_t)$
evolution of sufficient statistic

$$\xi_{t+1} = v(\xi_t, s_{t+1})$$

# $\mathsf{T}_2$ operator

$V(x_t) \sim$ value function that depends on hidden state replace $EV(x_t) = \int v(x_t) f(x_t|\xi_t) dx_t$ with

$$\mathsf{T}_2 V(x_t) = -\theta \log \left( E \left[ \exp \left( \frac{-V(x_t)}{\theta_2} \right) \right] \right)$$

worst case likelihood ratio

$$h(x_t) \propto \exp \left( \frac{-V(x_t)}{\theta} \right)$$

distorts density of $x_t$ conditional on $s^t$.

# Orientations toward exponential twisting

- Decision making is forward-looking.
- Estimation is backward-looking.

# A frontier: uncertainty and incomplete markets

- General equilibrium theory with state-contingent trading.
- Work by Aloisio Araujo and co-authors. Here model uncertainty can attenuate or shut down some markets.
- Story hinges on *ex post* heterogeneity of beliefs that emerges with multiple priors models. 'There is too much disagreement (or caution) about probability distributions to trade some state-contingent claims.
- A promising model of endogenous incomplete markets.

# Another frontier

- *Self-confirming equilibrium* is an appealing concept for macro policy design questions – SCE are the only possible limit points of adaptive systems.
- There are exciting new ideas in Pierpaolo Battigalli, Simone Cerreia-Vioglio, Fabio Maccheroni, and Massimo Marinacci, 2011, "Self confirming Equilibrium and Uncertainty," Working Papers 428, IGIER (Innocenzo Gasparini Institute for Economic Research), Bocconi University.
- These ideas are even more exciting for macroeconomic applications because here the off-equilibrium-path beliefs of the government are so important in affecting outcomes on an equilibrium path.