# Optimal Control and Filtering in Linear Forward-Looking Economies: A Toolkit*

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**Abstract**

We provide algorithms to solve a linear-quadratic optimal control problem with commitment. By extending to the case of imperfect information a procedure outlined in Ljungqvist and Sargent (2002), we make the results of Svensson and Woodford (2000) easy to implement. We provide a Matlab package that solves this class of models and analyzes their properties using simulations, impulse response functions and other techniques, both with commitment and discretion. A monetary policy application, based on the "new-keynesian" model of Clarida, Gali and Gertler (1999), is used to illustrate how the toolkit can be used.

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# 1. Introduction

This paper presents a toolkit to easily solve for the optimal policy under commitment and for the corresponding rational expectations equilibrium in a dynamic linear-quadratic economy. The general framework adopted here allows for the possibility that some state variables may be imperfectly observed. A closed-form analytical solution for this class of models cannot be obtained in general, and the problem must be solved numerically.

Svensson and Woodford (2000) already addressed this issue and characterized the solutions under two different equilibrium notions: discretion (recursive Markov-perfect equilibrium) and commitment. But, in solving the commitment case, they do not provide explicit algorithms to *compute* the solution.[1] One possible computational strategy, then, is to resort to a standard algorithm used in the optimal control literature which selects the stable (i.e. non-explosive) solution among a class of candidate functions satisfying the first order conditions of the problem (e.g. Söderlind, 1999).

This paper provides a simpler way to solve the commitment case that exploits the recursiveness of the problem. By extending to the case of imperfect information a procedure outlined in Ljungqvist and Sargent (2002), we make the results of Svensson and Woodford (2000) easy to implement. Moreover, this procedure yields a history-dependent *representation* of the optimal commitment policy, allowing the control to be expressed as a function of its own lagged values. For some purposes this may be more revealing than the usual policy representation in terms of the *unobservable* costate variables. Finally, to make these findings operational, we developed an easy-to-use software package written in Matlab (the "Toolkit" in the following) which is distributed with the paper and can be used to solve, simulate and analyze the dynamics of the economy, under both discretion and commitment, using impulse response functions, stochastic simulations and other standard tools.[2]

Our codes complement the set of routines provided by Söderlind in the sense

[1] More specifically, their paper does not provide algorithms to compute the optimal policy and the state space law of motion under commitment (i.e. the matrices $F, \Phi, G, \Gamma, S$ and $\Sigma$ in their equations 3.1-3.3).

[2] The program and the accompanying user's manual can be freely downloaded at `http://francesco-lippi.dadacasa.supereva.it` or obtained from the authors upon request

that we allow for a setup in which information may be imperfect, although we retain the assumption that the private economy and the policy maker share the same information (i.e. information is symmetric). The optimal Kalman filter for this problem is computed and coded (again both for discretion and commitment).

The structure of the paper is the following. The next section describes the setup. In Section 3 we adapt the results of Ljungqvist and Sargent (2002) to the case of imperfect information and derive the algorithms to solve for the commitment case. In Section 4 we present the algorithms to compute several objects of interest in the analysis of the model (impulse response functions, unconditional covariance matrices, value of the intertemporal loss function, etc.). We illustrate the use of these algorithms in Section 5, where we put to work our Matlab package to analyze a model economy by Clarida, Gali and Gertler (1999) as a practical illustration of the simplicity with which a model can be solved and analyzed with the Toolkit. Various appendices provides technical details about the tasks performed by the Toolkit. A final section concludes.

## 2. The economy

This section summarizes the setup and notation (similar to Svensson and Woodford [2000]) used to model a linear-quadratic economy with two agents, a government and an aggregate private sector, which are assumed to have the same information. The economy is described by

$$\begin{bmatrix} X_{t+1} \\ x_{t+1|t} \end{bmatrix} = A^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + A^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + Bi_t + \begin{bmatrix} C_u \\ 0 \end{bmatrix} u_{t+1}, \tag{2.1}$$

where $X_t$ is a vector containing $\mathrm{N_{PD}}$ predetermined variables in period $t$ (also called *natural state* variables in the following), $x_t$ is a vector of $\mathrm{N_{FW}}$ forward-looking variables, $i$ is a vector of $\mathrm{N_{CTL}}$ policy instruments, $u_t$ is a vector of $\mathrm{N_{SK}}$ structural shocks with mean zero and covariance $\Sigma_u^2$ ($\mathrm{N_{SK}}\cdot\mathrm{N_{SK}}$) and $A^1$, $A^2$, $B$, $C_u$ and 0 are matrices of appropriate dimension (the elements of 0 are all zeros). For any variable $z_{t,}$ the notation $z_{t|\tau}$ denotes the expectation $E[z_t \mid I_\tau]$ , i.e. the rational expectation of $z_t$ with respect to the information $I_t$ available in period $t$.

Let $Y_t$ represent the vector of target variables that enter the government criterion function,

$$Y_t = C^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + C^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + C_i i_t, \tag{2.2}$$

where $C^1$, $C^2$ and $C_i$ are matrices of appropriate dimension. Let the quadratic form describing the period loss function be given by

$$L_t \equiv Y_t^{'} W Y_t \tag{2.3}$$

where $W$ is a positive semidefinite matrix of weights. The government actions are aimed at minimizing the intertemporal loss function

$$\Lambda_t = E[\sum_{\tau=0}^{\infty} \delta^{\tau} L_{t+\tau} \mid I_t] \tag{2.4}$$

where $\delta \in (0, 1)$ is the intertemporal discount factor.

Finally, let the vector of $\mathrm{N_Z}$ observable variables $Z_t$ be given by

$$Z_t = D^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + D^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + v_t, \tag{2.5}$$

where the "noise" vector $v_t$ is assumed to be iid with covariance matrix $\Sigma_v^2$ and uncorrelated with $u_t$ at all leads and lags . Information $I_t$ in period $t$ is

$$I_t \equiv \{Z_t, \tau \leq t;\ A^1, A^2, B, C^1, C^2, C_i, C_u, D^1, D^2, W, \delta, \Sigma_u^2, \Sigma_v^2\}.$$

## 3. Model solution with commitment

In the following we first review a conventional method used to characterize the solution under commitment and *complete* information. We then illustrate an alternative method discussed in Ljungqvist and Sargent (2002) that exploits the recursive structure of the problem. Finally, we adapt the latter to the present setting of incomplete information.

Under commitment the policy maker chooses a contingent policy plan at the beginning of times (a sequence of policy functions). Once it is chosen, the plan cannot be abandoned afterwards. With respect to the private sector, the policy maker acts as a Stackelberg leader, taking into account the equations of motion of the forward looking variables $x_t$ as additional constraints in the problem he solves.[3]

### 3.1. The "traditional" solution with complete information

For convenience of notation let us rewrite the transition law of the variables (2.1) under complete information as follows:

$$\Psi_{t+1} = A\Psi_t + Bi_t + w_{t+1} \tag{3.1}$$

where we stack the natural state variables ($X_t$) and the forward looking variables ($x_t$) in the vector $\Psi_t' \equiv [X_t' \; x_t']$ and, being under complete information, we write $A = A^1 + A^2$. Notice that $w_{t+1}' \equiv [(C_u u_{t+1})' \; (x_{t+1} - x_{t+1|t})']$ includes the expectational error incurred in substituting $x_{t+1|t}$ with $x_{t+1}$ in the l.h.s. of (3.1).

In the literature, the usual way to solve for the commitment solution starts by forming the Lagrangian for the problem (see Söderlind, 1999):

$$\mathcal{L}_t = E_t \sum_{\tau=0}^{\infty} \delta^\tau \left\{ Y_{t+\tau}' W Y_{t+\tau} + 2\delta\mu_{t+\tau+1}' [A\Psi_{t+\tau} + Bi_{t+\tau} + w_{t+\tau+1} - \Psi_{t+\tau+1}] \right\} \tag{3.2}$$

The first order conditions are then used to eliminate the control $i_t$ and obtain a difference equation of the form:

$$\mathcal{G} E_t \begin{bmatrix} \Psi_{t+1} \\ \mu_{t+1} \end{bmatrix} = \mathcal{D} \begin{bmatrix} \Psi_t \\ \mu_t \end{bmatrix} \tag{3.3}$$

This difference equation in general admits an infinite number of (explosive) solutions. To ensure that a *stable* solution is selected, i.e. one that satisfies $\sum_{t=0}^{\infty} \delta^t \Psi_t' \Psi_t < +\infty$, it is therefore necessary to impose a transversality condition. The way this is usually done is by computing the real generalized Schur decomposition of the matrices $\mathcal{G}$ and $\mathcal{D}$, which allows the stable solution to be selected as the one associated to the stable generalized eigenvalues (Klein's method). The stable solution is characterized by the feature that the multipliers and the state

[3] These equations are also referred to as "implementability constraint". They typically consist of the euler equations that describe the private sector optimizing behavior.

variables are linked via the relationship:

$$\mu_t = V\Psi_t. \tag{3.4}$$

which we will refer to as the "stabilizing condition".

This "Lagrangian" approach leads naturally to the problem of having to search for stable generalized eigenvalues in systems like (3.3) using a Schur decomposition. Although this is a very powerful technique that can be used to construct solutions of any dynamic system in the form of (3.3), even if not coming directly from an intertemporal optimization problem, it turns out to be not necessary in our case.

### 3.2. An ingenious alternative way

The same solution can indeed be conveniently computed by exploiting the recursive structure of the problem, as suggested in Ljungqvist and Sargent (2002).[4] The first step of this approach involves solving an optimal linear regulator problem (OLRP in the following) disregarding the special nature of the forward looking variables $x_t$ and instead treating them as additional "states". It begins by writing the Bellman equation for the problem as follows:

$$\Psi_t' V^B \Psi_t = \min_{i_t} \left\{ Y_t' W Y_t + \delta E_t \left[ \Psi_{t+1}' V^B \Psi_{t+1} + 2\mu_{t+1}' (A\Psi_t + Bi_t + w_{t+1} - \Psi_{t+1}) \right] \right\} \tag{3.5}$$

where we explicitly keep track of the law of motion (3.1) via the vector of multipliers $\mu_t$. Algebraic manipulations of the first order conditions for this problem yield a standard Riccati equation from which $V^B$ and $F$, the matrix associated with the policy function $i_t = F\Psi_t$, can be computed. The key insight of this method is to show that the matrix $V^B$ of the value function of this problem and the matrix $V$ in (3.4) are actually the same matrix.[5] This insight provides a convenient way to compute a stable solution of (3.2), as $V^B$ can be computed from an ordinary

[4]Early references to this technique can be found in Currie and Levine (1985) and Backus and Driffill (1986).

[5]For a derivation of the algorithms for the computation of $F$ and $V$ see pp.53-54 of Ljungqvist and Sargent (2000,Chapter 4). The same chapter (pages 59-65) also illustrates the equivalence between the value function $V$ associated to the stable solution from the lagrangian formulation (using the eigenvalue decomposition) and the one obtained from the Bellman equation.

OLRP problem without having to compute $\mathcal{G}$ and $\mathcal{D}$ and the associated generalized Schur decomposition. In essence, the Bellman equation implicitly imposes a transversality condition, and therefore immediately yields the stable solution.

Having found $V$, it is easy to characterize the rest of the solution. This is done by partitioning the vector $\mu_t$ and rewriting the stabilizing condition (3.4) as:

$$\mu_t \equiv \begin{bmatrix} \mu_{X,t} \\ \mu_{x,t} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} \tag{3.6}$$

The vector $\mu_t$ contains $\mathrm{N_{PD}}$ lagrange multipliers associated to the natural state variables $X_t$ and $\mathrm{N_{FW}}$ multipliers associated to the implementability constraints (the $x$ variables). Since the $x_t$ can jump at time $t$, the multipliers $\mu_{x,t}$ associated to them can actually be treated as "predetermined" variables, with the interpretation of *shadow prices* measuring the cost in time $t$ loss/utility terms incurred by the optimizing agent in satisfying the implementability constraints. Mechanically, they constitute an additional set of state variables (also called *costate* variables) that can be used to represent the dynamics of the whole system under commitment.

With this interpretation of the $\mu_{x,t}$ in mind, it is immediate to find a closed form solution for the forward looking variables $x_t$ and the policy variable $i_t$ in terms of the predetermined variables of the model. Assuming that $V_{22}$ is invertible, rewrite the lower part of (3.6) to express the forward looking variables $x_t$ as a function of $X_t$, and $\mu_{x,t}$ alone:

$$x_t = -V_{22}^{-1} V_{21} X_t + V_{22}^{-1} \mu_{x,t} \tag{3.7}$$

Next, partition the matrix $F$ of the policy function of the OLRP above to conform to $X_t$ and $x_t$:

$$i_t = [F_1 \ F_2] \begin{bmatrix} X_t \\ x_t \end{bmatrix} \tag{3.8}$$

and substitute expression (3.7) in it to obtain:

$$i_t = F_1^* X_t + F_2^* \mu_{x,t} \tag{3.9}$$

where we defined:

$$\begin{aligned} F_1^* &\equiv F_1 - F_2(V_{22})^{-1}V_{21} \\ F_2^* &\equiv F_2(V_{22})^{-1} \end{aligned}$$

Now (3.7) and (3.9) are indeed the rational expectation solution for the forward looking variables and the commitment solution for the policy variable as a function of the predetermined state ($X_t$) and costate ($\mu_{x,t}$) variables.

### 3.3. Adding incomplete information

In the presence of imperfect information, the problem is only marginally altered. Setting up a lagrangian analogous to (3.2), and following the same steps undertaken to obtain (3.4), one can show that the stable solution of the problem satisfies a relationship of the form:

$$\mu_{t+1|t+1} = V\Psi_{t+1|t+1} \tag{3.10}$$

When instead we follow the alternative route and write the Bellman equation for this problem:

$$\Psi_{t|t}' V^B \Psi_{t|t} = \min_{i_t} E\left\{Y_t^{'} W Y_t + \delta[\Psi_{t+1}' V^B \Psi_{t+1} + 2\mu_{t+1}'(A_t^1\Psi_t + A^2\Psi_{t|t} + Bi_t + w_{t+1} - \Psi_{t+1})] \mid I_t\right\}$$

we obtain the following set of first order necessary conditions:

$$\begin{aligned} C_i' W\left[(C^1 + C^2)\Psi_{t|t} + C_i i_t\right] &= -\delta B'\mu_{t+1|t} \\ \mu_{t+1|t} &= V^B\Psi_{t+1|t} \end{aligned} \tag{3.11}$$

Applying the law of iterated expectations shows that (3.10) implies (3.11). Therefore, the matrices $V$ and $V^B$ appearing in the two expressions must be the same.[6]

The same logic applied in the derivation of the rational expectation solution and the policy function discussed in the previous subsection applies here, except

[6] Note, however, that since (3.11) does not imply (3.10), the two sets of multipliers $\mu_{t|t}$ and $\mu_{t|t-1}$ are actually not necessarily the same. More precisely, the multipliers of the forward-looking variables are the same $\left(\mu_{x,t|t} = \mu_{x,t|t-1}\right)$ while those of the predetermined variables are not (see the next footnote).

that now the predetermined variables are $X_{t|t}$ and $\mu_{x,t}$. In Appendix A we derive the relevant formulas:

$$\begin{aligned} x_t &= G_1(X_t - X_{t|t}) - G_1^* X_{t|t} + G_2^* \mu_{x,t} && (3.12) \\ x_{t|t} &= -G_1^* X_{t|t} + G_2^* \mu_{x,t} && (3.13) \\ i_t &= F_1^* X_{t|t} + F_2^* \mu_{x,t} && (3.14) \end{aligned}$$

together with the equations describing the dynamics of state ($X_t$) and costate ($\mu_{x,t}$) variables in terms of the primitive elements of the problem:[7]

$$\begin{bmatrix} X_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ \mu_{x,t} \end{bmatrix} + \begin{bmatrix} M_{X\,1} & M_{X\,2} \\ M_{\mu\,1} & M_{\mu\,2} \end{bmatrix} \begin{bmatrix} X_{t|t} \\ \mu_{x,t} \end{bmatrix} + \begin{bmatrix} C_u \\ 0 \end{bmatrix} u_{t+1} \quad (3.15)$$

as well as an expression for the measurement equation in terms of state and costate variables:

$$Z_t = LX_t + [N_1 \ N_2] \begin{bmatrix} X_{t|t} \\ \mu_{x,t} \end{bmatrix} + v_t \quad (3.16)$$

To close the model, the value of $X_{t|t}$ must be determined. Appendix B (following Svensson and Woodford, 2000) shows how to compute the optimal linear projection for the natural state variables, $X_{t|t}$, using the Kalman filter. This yields the recursive prediction algorithm:

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t] && (3.17) \\ & \quad \textit{where} \\ K &\equiv PL'(LPL' + \Sigma_v^2)^{-1} && (3.18) \end{aligned}$$

where the matrix $K$ is the steady-state gain matrix associated with the Kalman filter and $P$ is the covariance matrix of the one step ahead forecast error in $X_t$, given by

$$cov(X_t - X_{t|t-1}) \equiv P = H[P - KLP]H' + C_u \Sigma_u^2 C_u' \ . \quad (3.19)$$

It is sometimes convenient to express the optimal projection for $X_{t|t}$ in terms

[7] Note from (3.15) that the costate variables $\mu_{x,t+1}$ do not depend on period $t+1$ innovations, i.e. they are measurable with respect to $I_t$. This observation is useful in the computation of the Kalman filter (see Appendix B).

of the innovations in the observable variable $Z_t$, i.e. using the innovations representation:

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + U\left[Z_t - Z_{t|t-1}\right] \\ & \quad where \\ U &\equiv K(I + N_1 K)^{-1}. \end{aligned} \tag{3.20}$$

# 4. Analyzing the model

Once a model has been solved numerically, the standard way to analyze it typically involves computing impulse response functions, looking at the covariances of the main variables in the system, comparing the value of the intertemporal loss function at different parametrizations and running dynamic simulations. The Matlab Toolkit that comes with the present paper computes all of these objects. In the following subsections we briefly sketch how we set them up, leaving the details of the computations in the Appendices.

## 4.1. Impulse response functions

For the impulse response functions analysis it is convenient to rewrite equations (3.12)-(3.15), the equations that summarize the dynamics of the entire system, in a more parsimonious way as follows:

$$Q_{t+1} = \hat{A}Q_t + \hat{B}\xi_{t+1} \tag{4.1}$$

$$J_t = \hat{C}Q_t + \hat{D}\xi_{t+1} \tag{4.2}$$

$$\text{where} \tag{4.3}$$

$$Q_t \equiv \begin{bmatrix} X_t \\ X_{t|t-1} \\ \mu_{x,t} \end{bmatrix}, \ \xi_t \equiv \begin{bmatrix} u_t \\ v_{t-1} \end{bmatrix}$$

In vector $Q_t$ we stacked the minimum set of variables needed to achieve this representation; $\xi_t$ contains the original structural innovations in the system; $J_t$ instead can potentially contain all the variables of interest in the model. In the Toolkit we decided to set $J_t' = [X_t' \ X_{t|t}' \ x_t' \ x_{t|t}' \ i_t' \ \mu_{xt}']$, which means that the coefficient matrices $\widehat{A}$, $\widehat{B}$, $\hat{C}$ and $\hat{D}$ have the following form:

$$\hat{A} \equiv \begin{bmatrix} (H{+}M_{X\,1}KL) & M_{X\,1}(I-KL) & M_{X\,2} \\ (H{+}M_{X\,1})KL & (H{+}M_{X\,1})(I-KL) & M_{X\,2} \\ M_{X\,2}KL & M_{\mu\,1}(I-KL) & M_{\mu\,2} \end{bmatrix}, \ \hat{B} \equiv \begin{bmatrix} C_u & M_{X\,1}K \\ 0 & (H{+}M_{X\,1})K \\ 0 & M_{\mu\,1}K \end{bmatrix}$$

$$\hat{C} \equiv \begin{bmatrix} I & 0 & 0 \\ KL & I-KL & 0 \\ G_1{-}(G_1{+}G_1^*)KL & -(G_1{+}G_1^*)(I-KL) & G_2^* \\ -G_1^*KL & -G_1^*(I-KL) & G_2^* \\ F_1^*KL & F_1^*(I-KL) & F_2^* \\ 0 & 0 & I \end{bmatrix}, \ \hat{D} \equiv \begin{bmatrix} 0 & 0 \\ 0 & K \\ 0 & -(G_1{+}G_1^*)K \\ 0 & -G_1^*K \\ 0 & F_1^*K \\ 0 & 0 \end{bmatrix}.$$

A side product of writing the system this way is that it becomes very easy to construct "user-defined" variables, i.e. linear combination of the main variables in the model, and to study them with the same tools employed in the analysis of the original state-space variables. In fact, good model-building techniques recommend to write a model in the most parsimonious way but, once the model is solved, it is not uncommon to have to map the impact of the solution on variables not originally included in the system.

The Toolkit does indeed allows the user to construct any number of such variables. As an example, in Appendix C we show how to construct two user-defined variables for the Clarida, Gali and Gertler (1999) model discussed in Section 5.

### 4.2. The value of the intertemporal loss function

To compute the value of the intertemporal loss function $\Lambda_t$ in (2.4) associated with the optimal solution of the model we exploit the recursive nature of the problem and its linear-quadratic structure.

Let us rewrite the optimal value of (2.4) in recursive form:

$$\Lambda_t^* = L_{t|t}^* + \delta \Lambda_{t+1|t}^* \tag{4.4}$$

where the asterisk indicates that the variables are to be evaluated at the solution point. Since the period loss function is quadratic and the constraints are linear, we know that $\Lambda_t^*$ will be a quadratic function of the "initial conditions". But, with forward-looking variables entering the problem, the only relevant initial conditions are those with respect to the variables that cannot jump, i.e. the predetermined variables $X_{t|t}$ (in expected terms) in the discretion case, augmented with the costate variables $\mu_{x,t}$ in the commitment case. For this latter case, then, $\Lambda_t^*$ admits a representation of the form $\Lambda_t^* \equiv \mathcal{S}_{t|t}' V \mathcal{S}_{t|t} + d$, where we stack in the $\mathcal{S}$ vector all the relevant predetermined (state and costate) variables, $\mathcal{S}_t' \equiv [X_t' \ \mu_{xt}']$, and the matrix $V$ and the scalar $d$ are computed in Appendix D.

### 4.3. Computation of unconditional moments

The unconditional covariance matrices for the (estimated and true) state, costate, forward-looking and user-defined variables can be easily computed once the model is solved. Our strategy has been to find $\Sigma^2_{\mathcal{S}_{t|t}}$ first, and then derive all the other covariance matrices as a function of this "fundamental" one.

Let us rewrite the laws of motion for $X_t$ and $\mu_{xt}$ (equations 3.15) in term of the stacked vector $\mathcal{S}$ as follows:

$$\begin{aligned}
\mathcal{S}_{t+1} &= HH \cdot \mathcal{S}_t + MM \cdot \mathcal{S}_{t|t} + uu_{t+1} \\
&\quad where \\
HH &\equiv \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}, \ MM \equiv \begin{bmatrix} M_{X\,1} & M_{X\,2} \\ M_{\mu\,1} & M_{\mu\,2} \end{bmatrix}, uu_{t+1} \equiv \begin{bmatrix} C_u u_{t+1} \\ 0 \end{bmatrix} \\
\Sigma^2_{uu} &\equiv \begin{bmatrix} C_u \Sigma_u^2 C_u' & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{4.5}$$

and the matrices 0, the elements of which are all zeros, are of appropriate dimension.

Similarly, we can write the stacked version of the updating equation as[8]

$$\begin{aligned}
\mathcal{S}_{t+1|t+1} &= \mathcal{S}_{t+1|t} + KK \cdot (\mathcal{S}_{t+1} - \mathcal{S}_{t+1|t}) + vv_{t+1} \\
&\quad where \\
KK &\equiv \begin{bmatrix} KL & 0 \\ 0 & 0 \end{bmatrix}, \ vv_{t+1} \equiv \begin{bmatrix} Kv_{t+1} \\ 0 \end{bmatrix}
\end{aligned} \tag{4.6}$$

[8]To do this we use (3.17) and $\mu_{xt+1|t+1} = \mu_{xt+1|t}$ , which is implied by the fact that $\mu_{x,t}$ is measurable with respect to the information at $t-1$ (this is immediate from 3.15).

$$\Sigma^2_{vv} \equiv \begin{bmatrix} K\Sigma^2_v K' & 0 \\ 0 & 0 \end{bmatrix}$$

Combining the two equations above, we can express $S_{t+1|t+1}$ as a function of $S_{t|t}$ plus some noise and error terms:

$$\mathcal{S}_{t+1|t+1} = (HH + MM)\mathcal{S}_{t|t} + KK \cdot HH(\mathcal{S}_t - \mathcal{S}_{t|t}) + KK \cdot uu_{t+1} + vv_{t+1} \quad (4.7)$$

from which a recursive formula for $\Sigma^2_{\mathcal{S}_{t|t}}$ is easily obtained:

$$\begin{aligned} \Sigma^2_{\mathcal{S}_{t+1|t+1}} &= (HH + MM)\Sigma^2_{\mathcal{S}_{t|t}}(HH + MM)' + \\ &+ (KK \cdot HH)\ cov(\mathcal{S}_t - \mathcal{S}_{t|t})\ (KK \cdot HH)' + KK \cdot \Sigma^2_{uu} \cdot KK' + \Sigma^2_{vv} \end{aligned} \quad (4.8)$$

Notice that the covariance matrix of the contemporaneous prediction error in $\mathcal{S}_t$ that appears in the above formula is given by:

$$cov \begin{bmatrix} (X_t - X_{t|t}) \\ (\mu_{x,t} - \mu_{x,t|t}) \end{bmatrix} \equiv P_{co} = \begin{bmatrix} P_o & 0 \\ 0 & 0 \end{bmatrix} \text{ where } P_o \equiv cov(X_t - X_{t|t}) \quad (4.9)$$

and the matrix $P_o$ is related to the matrix of the one step ahead forecast error $P$ (already computed in (3.19)) by the relationship $P_o = (I - KL)P$. Therefore we can solve (4.8) for the steady state (i.e. unconditional) value of $\Sigma^2_{\mathcal{S}_{t|t}}$ either by iterating to convergence or by direct algebraic manipulation (Hamilton formula).

Appendix E shows how to compute the covariances of the other variables in the model once $\Sigma^2_{\mathcal{S}_{t|t}}$ is known.

## 4.4. A detour: history-dependent representation of optimal policy

It is apparent from equations (3.14) and (3.15) that the optimal commitment policy has an inertial character induced by its dependence on the (history-dependent) costate variables $\mu_x$. This history dependence of policy under commitment can be immediately visualized by a convenient policy representation that expresses the dynamics of the control variables in terms of the observables.

Let $F_2^{*+}$ denote the pseudo-inverse of $F_2^*$. Equation (3.14) for $i_{t-1}$ implies:

$$\mu_{x,t-1} = (F_2^{*+})i_{t-1} - (F_2^{*+}F_1^*)X_{t-1|t-1} \quad (4.10)$$

Using (4.10) and the lower block of (3.15), $\mu_{x,t} = M_{\mu 1} X_{t|t} + M_{\mu 2} \mu_{x,t-1}$, we can thus rewrite the policy function (3.14) as

$$\begin{aligned} i_t &= \Upsilon_1 i_{t-1} + \Upsilon_2 X_{t|t} + \Upsilon_3 X_{t-1|t-1} \\ & \quad where \\ \Upsilon_1 &\equiv F_2^* M_{\mu 2} F_2^{*+} \\ \Upsilon_2 &\equiv F_1^* + F_2^* M_{\mu 1} \\ \Upsilon_3 &\equiv -F_2^* M_{\mu 2} (F_2^{*+} F_1^*) \end{aligned} \tag{4.11}$$

Equation (4.11) offers a convenient history dependent representation of optimal policy in terms of observables, i.e. lagged policy and states. As noted by Woodford (1999), the optimal policy under commitment features "persistence", as captured by the coefficient $\Upsilon_1$. As is well known, such a feature is pervasive in empirical studies of monetary policy, and it is sometimes explained in terms of a postulated costly adjustment of the controls (e.g. interest rates). Under commitment, instead, persistence arises naturally from the dependence of policy on the costate variables.

## 5. Application: a "new synthesis" model

In the following, we use the algorithms derived above to analyze a version of the sticky-price framework developed, among others, by Woodford (2000) and Clarida, Gali and Gertler (1999). In that framework output ($y_t$) and inflation ($\pi_t$) are determined, respectively, by a "dynamic IS" curve and a "Phillips curve", according to:[9]

$$y_t = y_{t+1|t} - \sigma[i_t - \pi_{t+1|t}] + g_t \tag{5.1}$$

$$\pi_t = \delta \pi_{t+1|t} + k(y_t - \bar{y}_t) + u_t \tag{5.2}$$

where $\bar{y}_t$ denotes potential output as of period $t$ (i.e. the output level that would obtain under flexible prices), $i_t$ the nominal interest rate, $g_t$ a demand shock and

[9]These equations are derived from the optimizing behavior of consumers (i.e. an intertemporal Euler equation) and price-setting monopoly firms facing a randomly staggered price adjustment mechanism as in Calvo (1983).

$u_t$ a cost-push shock. The output gap is defined as the difference between actual and potential output: $y_t - \bar{y}_t$.

Following Clarida, Gali and Gertler (1999, CGG henceforth), we assume the economy is subject to three types of shocks: demand ($g_t$) and cost-push shocks ($u_t$) and potential output shocks , $\hat{y}_t$ . They obey the following processes:

$$\bar{y}_t = \gamma \bar{y}_{t-1} + \hat{y}_t \qquad 0 < \gamma < 1 \tag{5.3a}$$

$$g_t = \mu g_{t-1} + \hat{g}_t \qquad 0 < \mu < 1 \tag{5.3b}$$

$$u_t = \rho u_{t-1} + \hat{u}_t \qquad 0 < \rho < 1 \tag{5.3c}$$

where the innovations $\hat{y}_{t+1}$, $\hat{u}_{t+1}$ and $\hat{g}_{t+1}$ are uncorrelated, have zero mean and standard deviation equal to $\sigma_{\hat{y}}$, $\sigma_u$ and $\sigma_g$,respectively. Let us assume the measurable variables are given by:

$$\bar{y}_t^o = \bar{y}_t + \theta_{\bar{y}t} \tag{5.4a}$$

$$y_t^o = y_t + \theta_{yt} \tag{5.4b}$$

$$\pi_t^o = \pi_t + \theta_{\pi t} \tag{5.4c}$$

where the measurement errors $\theta_{jt}$ are iid. Finally, let the central bank period loss function be:

$$L_t \equiv \frac{1}{2}\left[(\pi_t - \pi^*)^2 + \lambda_y(y_t - \bar{y}_t - x^*)^2\right] \tag{5.5}$$

which allows us to encompass some special cases of interest, as done theoretically by Clarida, Gali and Gertler (1999).[10]

[10] Among these is the presence of a systematic inflation bias, $x^* > 0$.

### 5.1. System representation

The CGG model can be represented in terms of the system for the state and forward looking variables (2.1) as:[11]

$$\begin{bmatrix} 1 \\ \bar{y}_{t+1} \\ u_{t+1} \\ g_{t+1} \\ y_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & \frac{-k\sigma}{\delta} & \frac{\sigma}{\delta} & -1 & \frac{k\sigma}{\delta}+1 & \frac{-\sigma}{\delta} \\ 0 & \frac{k}{\delta} & \frac{-1}{\delta} & 0 & \frac{-k}{\delta} & \frac{1}{\delta} \end{bmatrix}}_{A^1} \begin{bmatrix} 1 \\ \bar{y}_t \\ u_t \\ g_t \\ y_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix}}_{B} i_t + \begin{bmatrix} C_u \\ 0_{(\mathrm{N_{FW}},\mathrm{N_{SK}})} \end{bmatrix} \underbrace{\begin{bmatrix} \hat{y}_{t+1} \\ \hat{u}_{t+1} \\ \hat{g}_{t+1} \end{bmatrix}}_{u_{t+1}}$$

$A^2 = 0$ and $C_u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $y_t$ and $\pi_t$ are forward looking variables and the other four variables in the left hand side vector are natural (predetermined) state variables. The objectives of the quadratic period loss function can be written as:

$$Y_t \equiv \begin{bmatrix} \pi_t - \pi^* \\ y_t - \bar{y}_t - x^* \end{bmatrix} = \underbrace{\begin{bmatrix} -\pi^* & 0 & 0 & 0 & 0 & 1 \\ -x^* & -1 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C^1} \begin{bmatrix} 1 \\ \bar{y}_t \\ u_t \\ g_t \\ y_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{C_i} i_t$$

$$W \equiv 1/2 \begin{bmatrix} 1 & 0 \\ 0 & \lambda_y \end{bmatrix}$$

and $C^2 = 0$.

[11]This exact model was coded as a working example in our matlab package. The interested reader can immediately see the mapping between the formulas in this section and the setup file cgg_setup.m.

Table 5.1: Baseline parameter values

| Parameters | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\delta$ | $\gamma$ | $\rho$ | $\mu$ | $k$ | $\sigma$ | $\lambda_y$ | $x^*$ | $\pi^*$ |
| .99 | .7 | .4 | .3 | .05 | 2.0 | .25 | 0.0 | 0.0 |

| Innovations (std) | | | Measurement Errors (std) | | |
|---|---|---|---|---|---|
| $\sigma_{\bar{y}}$ | $\sigma_u$ | $\sigma_g$ | $\sigma_{\theta\bar{y}}$ | $\sigma_{\theta y}$ | $\sigma_{\theta\pi}$ |
| .005 | .015 | .015 | $10^{-8}$ | $10^{-8}$ | $10^{-8}$ |

The setup is closed by the specification of the measurement equation:

$$\begin{bmatrix} \bar{y}_t^o \\ y_t^o \\ \pi_t^o \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D^1} \begin{bmatrix} 1 \\ \bar{y}_t \\ u_t \\ g_t \\ y_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} \theta_{\bar{y}t} \\ \theta_{yt} \\ \theta_{\pi t} \end{bmatrix}}_{v_t}$$

and $D^2 = 0$.

### 5.2. Commitment versus discretion with complete information

Once the model setup is coded and a specific parametrization is chosen, we use our toolkit to solve for the rational expectations equilibrium under commitment and under discretion.[12] The parameters chosen for our benchmark example appear in Table 5.1. We begin the example with the simpler case of complete information.[13] This is done by assuming the measurement error is nil ("very small" numerically).

Under the first parametrization, we use the toolkit to solve the model under both discretion and commitment assuming full information. The code computes the asymptotic losses for the policy maker for each of these cases (0.029 under commitment versus 0.035 under discretion). As is known, commitment yields

[12] See the companion Toolkit Manual and the model files cgg_setup, cgg_param for more details.

[13] Moreover, we assume throughout that the inflation and output-gap targets are zero. Since the CGG model features a long-run non-neutrality, real outcomes in the model are not invariant to these variables. The interested reader can quickly analyze these effects using the toolkit by means of simulations.

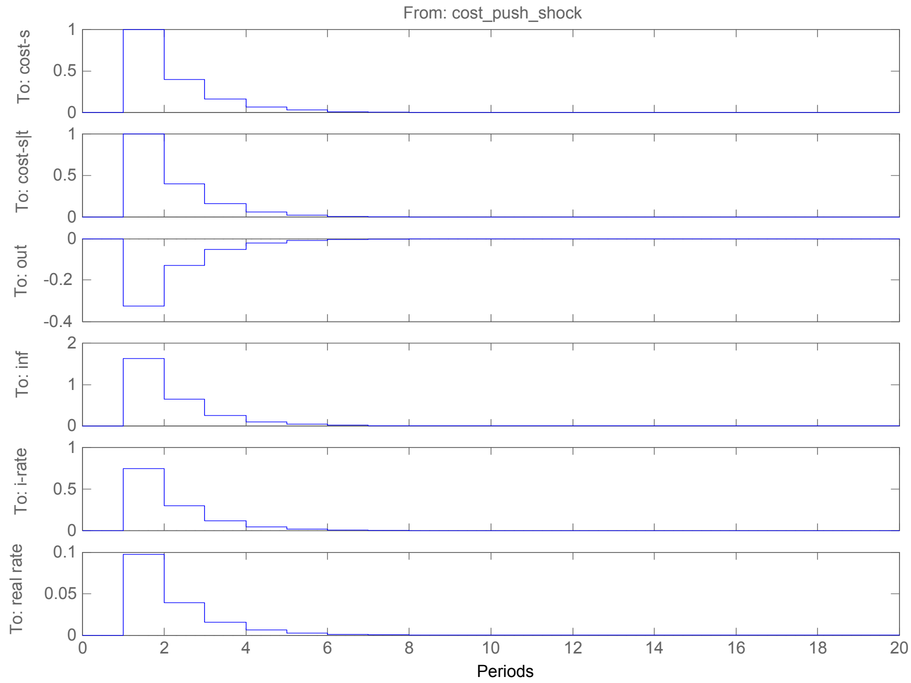


Figure 5.1: Cost-push shock with discretion

a superior welfare outcome even in the absence of a systematic inflation bias because it allows the policy maker to achieve an improved short run tradeoff between inflation and output stabilization (as in Section 4.2 of the CGG paper[14]). The intuitive reason is that under commitment the optimal way to respond to a positive cost-push shock involves reducing inflation expectations (hence current inflation) by committing to tighten policy today *and* tomorrow. This plan is not time consistent (i.e. cannot be implemented under discretion) as the policy maker has an incentive not to adhere to that policy when tomorrow comes.

This key difference between discretion and commitment results in a markedly different dynamic behavior of the economy. As argued by Woodford (1999a), the optimal policy under commitment features history dependence. This feature is apparent from the impulse response functions for the case of a cost-push shock.

[14]Our solution corresponds to what CGG call the "unconstrained optimum" under commitment, as opposed to the "optimum" within a class of policy rules that is constrained to be a linear function of state variables. The key difference is that the "unconstrained" optimum also depends on the costate variables.

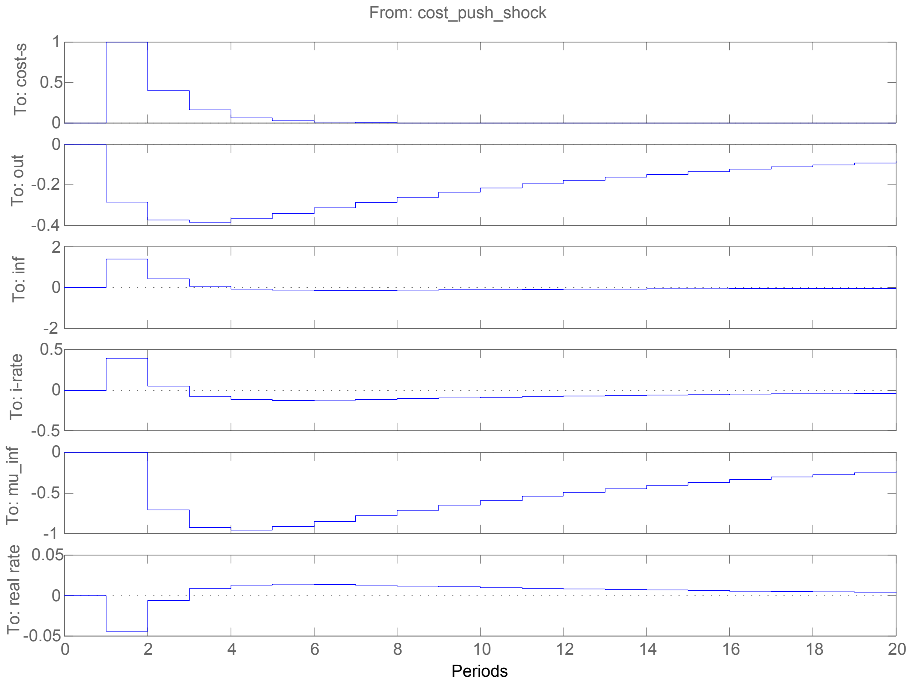


Figure 5.2: Cost-push shock with commitment

Figure 5.1 shows some effects of a (unitary) cost push shock under discretion. The first two boxes describe the dynamic pattern of the true and estimated valued of the shock itself (which has a serial correlation of $\rho = .4$). Obviously in this case of complete information the estimated values coincide with the true ones. The interest rate response to the cost push shock results in a policy tightening (fourth box), and an associated output reduction (third box; since potential output is unaffected by the cost push shock the output dynamics coincide with the output gap dynamics). The bottom box pictures the response of the real interest rate $i_t - \pi_{t+1|t}$ (a user-defined variable), which increase much less than the nominal rate due to the contemporaneous increase in expected inflation. The dynamics of the main macro variables (output, inflation, the interest rate) in this example are inherited directly from the structural persistence of the cost push shock. As the persistence parameter $\rho$ becomes small, the dynamics of the policy response (and all other macro effects) tend to vanish.

Figure 5.2 shows the corresponding effects for the commitment case. The

Table 5.2: Moments of the main variables

| Standard deviation of: | $y_t$ | $y_t - \bar{y}_t$ | $\pi_t$ | $i_t$ | $i_t - \pi_{t+1\|t}$ |
|---|---|---|---|---|---|
| *Discretion* | .010 | .005 | .026 | .015 | .008 |
| *Commitment* | .018 | .016 | .023 | .011 | .008 |

dynamics of the cost shock are unchanged as its motion is exogenous. One major difference with respect to discretion concerns output behavior (second box): under commitment the output response is much more persistent than under discretion. Stabilization of the cost push shock under the optimal policy prescribes a sequence of policy contractions (in terms of the expected real rate $i_t - \pi_{t+1|t}$, see the bottom box in the figure) which bring down inflation expectations. These successive increases in the real rate cause output to remain below trend for many more periods than under discretion.

The fifth box of Figure 5.2 displays the behavior of the lagrange multiplier associated to the inflation equation (a costate variable). Given the way the problem is written in (3.2), the value of this lagrange multiplier indicates the marginal effect on the criterion function of relaxing the constraint. We can think of the value of these multiplier as the utility cost for the policy maker of maintaining a past announcement. It appears that from the second period onwards this variable is non-zero, which indicates that the constraint is binding and, equivalently, that promise-keeping is going to have an effect on policy (see 3.14).[15]

Inspection of the contemporaneous *impact* effect of a cost push shock on inflation is smaller under commitment than under discretion (respectively 1.4 versus 1.6) and the same is true for the impact effect on the output gap (i.e. output since potential output is unchanged; respectively -.28 versus -.33). Moreover, as shown in Table 5.2 inflation volatility is smaller (and the output gap volatility greater) under commitment than under discretion.

[15] A non-zero value of the multiplier also suggests the time-inconsistency of the optimal plan: were the policy maker allowed to "reset" his announcements, it would face an incentive to ignore past promises and choose a different policy.

Table 5.3: Incomplete Information Setup

| Measurement Errors (std) | | |
|---|---|---|
| $\sigma_{\theta\bar{y}}$ | $\sigma_{\theta y}$ | $\sigma_{\theta\pi}$ |
| .01 | .01 | .01 |

### 5.3. The effect of imperfect information

This subsection introduces imperfect information, by adding noise to the measurement block (5.4). This amounts to assuming that potential output, actual output and inflation are subject to the measurement errors reported in Table 5.3.

With imperfect information, the policy maker uses the available information to form an estimate about the true state of the economy (i.e. $X_{t|t}$). Figure 5.3 illustrates this case for a cost push shock under discretion. The first obvious difference with respect to the complete information case of Figure 5.1 is that the true pattern of the shock now differs from the one estimated by the policy maker, as it appears from the two upper boxes in Figure 5.3. The signal extraction problem (solved with the Kalman filter) leads the policy maker to learn only gradually about the realization of the cost push shock: in the current setup, after a unitary cost push shock ($u_t = 1$) occurs, the contemporaneous estimate of the shock by the policy maker is $u_{t|t} = 0.70$. Naturally, the magnitude of the forecast errors induced by imperfect information depends on the assumptions about the properties of the fundamental processes (e.g. the persistence of the various structural shocks $g$, $u$ and $y$ and the signal to noise ratios encoded in $\Sigma_u^2$ and $\Sigma_v^2$). For instance, if we double the amount of noise in the inflation equation (i.e. raise $\sigma_{\theta\pi}$ from 0.1 to 0.2), the estimated value of the shock is much smaller ($u_{t|t} = 0.38$), as one would expect in the presence of more noise in the cost push shock indicator, $\pi_t^o$.[16]

Through its effect on the expectations about the state of the economy (e.g. $X_{t|t}$), imperfect information affects the dynamics of the forward-looking variables. First, the policy response of $i_t$ is less strong than in the full information case, as

[16] Several key objects produced by the filtering problem are computed by our MATLAB code, such as the matrices $P$ and $Po$ corresponding, respectively, to the one-step ahead and contemporaneous forecast errors in $X_t$.

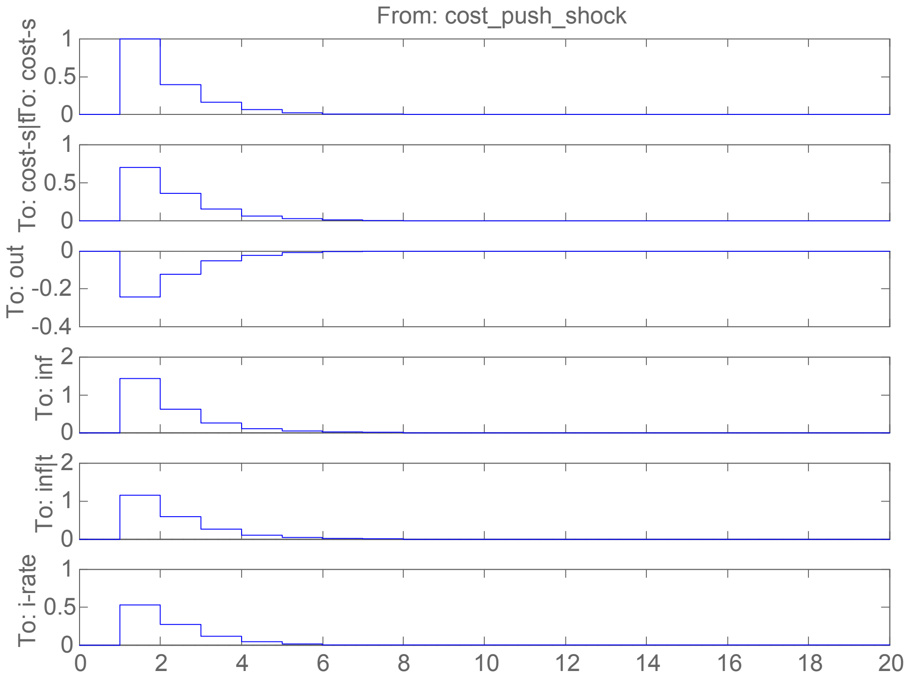


Figure 5.3: Cost-push shock with discretion and imperfect information

the perceived size of the cost push shock is smaller (compare the bottom box in Figure 5.1 and 5.3).[17] The response of output and of inflation is also muted in comparison to the complete information case: output fall by 0.24 (versus 0.32) while inflation increases by 1.4 (versus 1.6). This is due both to the policy response and to the fact that the future expected values of the cost push shock are smaller under incomplete information than under complete information, thus inducing the private economy to expect a different pattern about future shocks and policy.

### 5.4. The macroeconomic consequences of unobservable potential output

We conclude this application by analyzing the effects of imperfect information about potential output in the CGG model (under discretionary policy). Several contributions of Orphanides (e.g. 2000, 2001) offer compelling evidence that potential output estimates are very imprecise in real time. It is argued that basing policy on the estimates of such an unobservable (and noisy) variable may be at the root of important differences between policy based on real-time information and the optimal policy under complete information. It is hypothesized that the large negative shocks to potential output which occurred in the seventies were not recognized in real time by policy makers, who instead perceived a negative output gap and reacted to it by lowering rates. To formalize this argument within the CGG model we compute the effects of a potential output shock in the presence of, respectively, full and incomplete information. The difference in the dynamics of the endogenous variables between these two settings measures the effect of imperfect information.

Figure (5.4) shows the effect of a potential output shock with full information. The interest rate adjusts in such a way that the dynamics of actual output optimally replicate those of potential output (compare the fist two boxes in the figure), e.g. the "output gap" is nil. This policy poses no tradeoff between the objectives of the policy maker, therefore inflation remains constant at its steady state level (zero).

The same potential output shocks leads to different consequences under imperfect information, as shown in Figure (5.5). The first two boxes reveal that

[17] Due to the certainty equivalence feature of our problem, policy differences stemming from imperfect information arise entirely from the estimates of the states as the coefficient $F$ in the the optimal control function ($i_t = FX_{t|t}$) do *not* depend on the uncertainty.

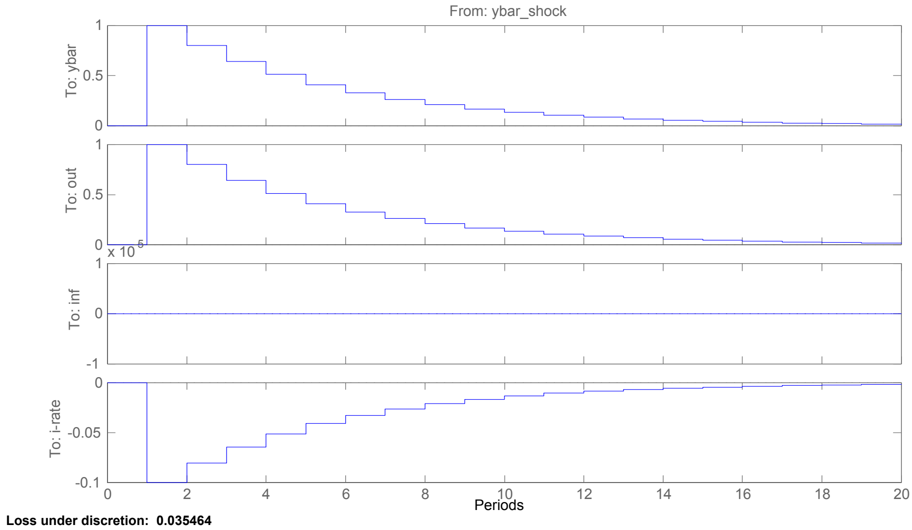


Figure 5.4: Potential output shock with full information and discretion

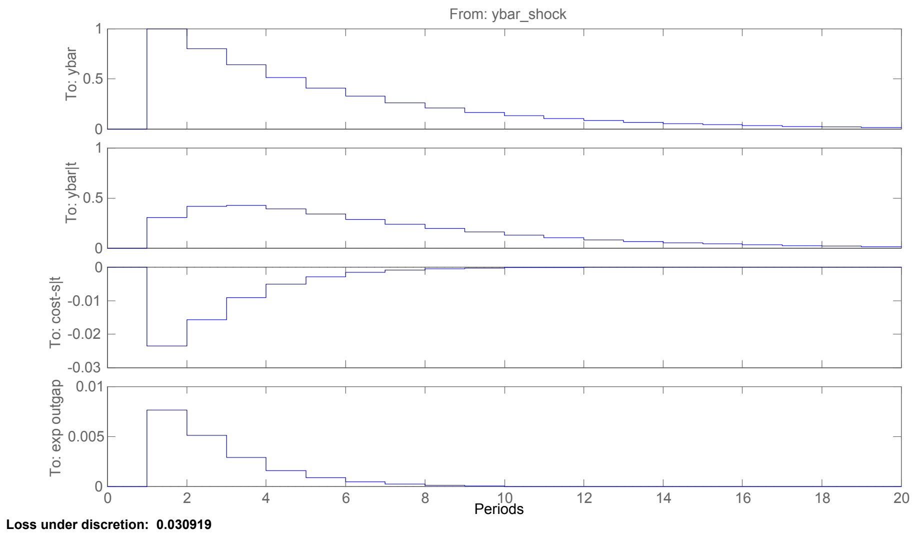


Figure 5.5: PO shock with imperfect information and discretion

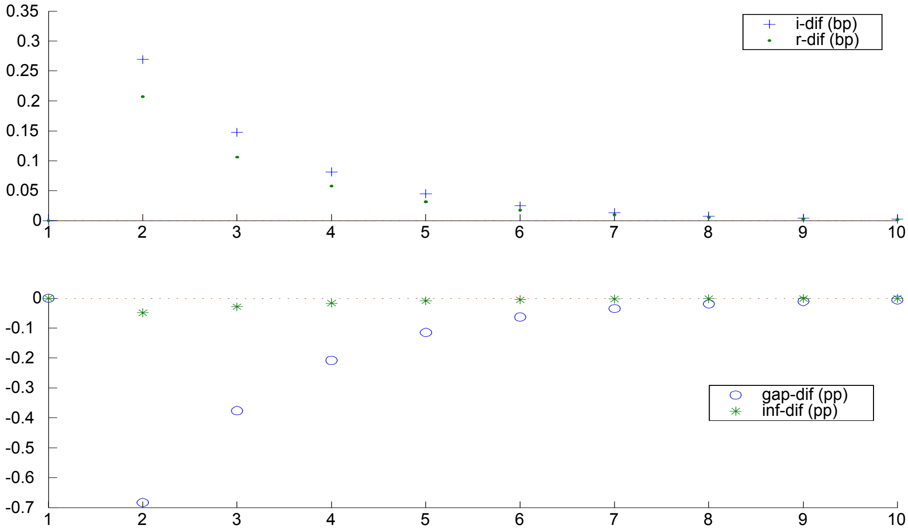


Figure 5.6: Macro effects of imperfect information

the true shock is only partially identified by the policy maker in real time. Note, moreover, that a negative cost push shock (third box) and a positive output gap (fourth box) are perceived by the policy maker.

Figure (5.6) compares the dynamic response of interest rates output and inflation under incomplete versus complete information. It appears that the interest rate (both nominal and real) is relatively loose (i.e. is reduced by a smaller amount) under incomplete information. This occurs because as potential output is underestimated (with incomplete information) the policy maker's perception of how much the interest rate needs to be lowered is smaller than under complete information (recall that the interest rate is proportional to the expected output growth - see equation 5.1). Therefore, the interest rate under incomplete information is tight in comparison to the full information benchmark (of about 25 basis points in our example).[18] As a consequence of different policy and expectations, the dynamics of inflation and the output gap are also affected. The lower panel in Figure (5.6) shows that, following a positive potential output shock, both inflation

[18]There is a second effect which goes in the opposite direction but is dominated under most plausible parameter values. It arises because the perceived negative cost push (under incomplete information) leads the policy maker to lower the interest rate (no effect under full information since there is no cost push shock).

and the output gap are lower than their full information counterpart (respectively of about 0.1 and 0.6 percentage points).

## 6. Concluding remarks

Imperfect information, dynamics and expectations are key ingredients in several problems faced by economic agents. Monetary policy is a classic example: decisions are taken in the presence of considerable uncertainty about the true state of the economy, facing a dynamic economy and a rational forward-looking public.[19] Methods to compute a rational expectations equilibrium with commitment for a linear quadratic problem involving these features are known at least since Backus and Driffill (1986). The technicalities involved, however, might discourage a more widespread application of such methods to the analysis of relevant economic phenomena. This paper tried to bridge this difficulty by providing a set of algorithms, and an associated implementation program, which allow rather involved optimal control problems with filtering to be handled easily. A monetary policy application drawn from the recent "new-keynesian" literature was used to illustrate how a researcher can use the algorithms of this paper to analyze a model by means of stochastic simulations and impulse responses.

[19] Other applications arise naturally in the field of industrial organization. Ljungqvist and Sargent (2002) discuss the case of an oligopolist facing a fringe of small competitive firms, to which imperfect information could be meaningfully added.

# A. Appendix: System dynamics under commitment

This appendix derives the representation of the predetermined state and costate variables in eqns. (3.12) - (3.16).

To derive a solution for the forward looking variables, we start by considering the partition of the stabilizing condition (3.6) in the main text; assuming $V_{22}$ is non-singular, the lower block of that partition implies the following relation:

$$\begin{aligned} x_{t|t} &\equiv G_2^* \mu_{x,t} - G_1^* X_{t|t} \\ &\quad \textit{where} \\ G_1^* &\equiv (V_{22})^{-1} V_{21} \\ G_2^* &\equiv (V_{22})^{-1} \end{aligned} \tag{A.1}$$

Moreover, the lower block of (2.1) yields:

$$A_{21}^1 (X_t - X_{t|t}) + A_{22}^1 (x_t - x_{t|t}) = 0.$$

Using (A.1) and assuming invertibility of $A_{22}^1$ yields the closed form solution (3.12) for the forward-looking variables $x_t$:

$$\begin{aligned} x_t &= G_1 (X_t - X_{t|t}) - G_1^* X_{t|t} + G_2^* \mu_{x,t} \\ &\quad \textit{where} \\ G_1 &\equiv -(A_{22}^1)^{-1} A_{21}^1. \end{aligned} \tag{A.2}$$

Using (A.1) in (3.8) yields (3.14) with

$$\begin{aligned} F_1^* &\equiv F_1 - F_2 (V_{22})^{-1} V_{21} \\ F_2^* &\equiv F_2 (V_{22})^{-1} \end{aligned}$$

The upper partition of the stabilizing condition (3.6) yields:

$$\begin{aligned} \mu_{X,t} &= L_1^* X_{t|t} + L_2^* \mu_{x,t} \\ &\quad \textit{where} \\ L_1^* &\equiv V_{11} - V_{12} G_1^* \\ L_2^* &\equiv V_{12} G_2^*. \end{aligned} \tag{A.3}$$

The motion of the natural state variables can then be computed from the upper block of (2.1) substituting for $i_t, x_t, x_{t|t}$ using (3.14), (A.2) and (A.1) and grouping terms. This yields

$$X_{t+1} = H X_t + M_{X\,1} X_{t|t} + M_{X\,2} \mu_{x,t} + u_{t+1} \tag{A.4}$$

$$
\begin{aligned}
& \qquad where \\
H &\equiv A_{11}^{1}+A_{12}^{1}G_{1} \\
M_{X\,1} &\equiv -A_{12}^{1}(G_{1}+G_{1}^{*})+A_{11}^{2}-A_{12}^{2}G_{1}^{*}+B_{1}F_{1}^{*} \\
M_{X\,2} &\equiv (A_{12}^{1}+A_{12}^{2})G_{2}^{*}+B_{1}F_{2}^{*}
\end{aligned}
$$

Computing the motion of the costate variables $\mu_x$ is slightly more involved. The relevant equations can be derived from the first order conditions with respect to $S_t$ and $S_{t|t}$ of the Lagrangian formulation of the problem (the incomplete information analogous of 3.2). They imply:

$$
\begin{aligned}
\mu_t &= \delta A'\mu_{t+1|t}+C'W'Y_{t|t} \qquad \text{(A.5)} \\
& \quad where \\
A &\equiv A^{1}+A^{2} \\
C &\equiv C^{1}+C^{2}
\end{aligned}
$$

Let us define the matrices

$$
\begin{aligned}
C_{WC} &= C'W'C \equiv \begin{bmatrix} C_{WC}^{X} \\ C_{WC}^{x} \end{bmatrix} \\
C_{WC_i} &= C'W'C_i \equiv \begin{bmatrix} C_{WC_i}^{X} \\ C_{WC_i}^{x} \end{bmatrix} \\
A &\equiv \begin{bmatrix} A^{X} \\ A^{x} \end{bmatrix}
\end{aligned}
$$

which are partitioned in row-blocks that conform to $X$ and $x$ (indexed, respectively, by the superscripts $X$ and $x$). Recalling that $\mu' = [\mu'_X \ \mu'_x]$, and using the lower block of (A.5) and (2.2) yields

$$
\mu_{xt}=\delta\left[A'_{12}\mu_{X,t+1|t}+A'_{22}\mu_{x,t+1}\right]+C_{WC}^{X}S_{t|t}+C_{WC_i}^{x}i_t \qquad \text{(A.6)}
$$

Equation (A.3) and (2.1) imply

$$
\mu_{X,t+1|t}=L_1^{*}[A^{X}S_{t|t}+B_1 i_t]+L_2^{*}\mu_{x,t+1}. \qquad \text{(A.7)}
$$

Plugging (A.7) in (A.6), collecting terms and rearranging yields

$$
\begin{aligned}
\mu_{x,t+1} &= \Sigma\mu_{x,t}+\Gamma S_{t|t}=\Sigma\mu_{x,t}+\Gamma_1 X_{t|t}+\Gamma_2 x_{t|t} \\
& \quad where \\
\Sigma &\equiv [\delta A'_{12}L_2^{*}+A'_{22}]^{-1} \\
\Gamma &\equiv -\Sigma[\left(\delta A'_{12}L_1^{*}A^{X}+C_{WC}^{x}\right)+\left(\delta A'_{12}L_1^{*}B_1+C_{WC_i}^{x}\right)F^{*}]
\end{aligned}
$$

where $\Gamma$ is partitioned in column blocks ($\Gamma = [\Gamma_1\ \Gamma_2]$) conform to $X$ and $x$. Using (A.1) to substitute for $x_{t|t}$ yields the law of motion for $\mu_x$

$$\begin{aligned} \mu_{x,t+1} &= M_{\mu 1} X_{t|t} + M_{\mu 2} \mu_{x,t} \\ & \quad where \\ M_{\mu 1} &\equiv \Gamma_1 - \Gamma_2 G_1^* \\ M_{\mu 2} &\equiv \Sigma + \Gamma_2 G_2^* \end{aligned} \tag{A.8}$$

Equations (A.8) and (A.4) provide the representation of the system (3.15) used in the main text.

Finally, using (A.2), (A.1), (A.2) in (2.5) to write the measurement equations in terms of $X_t$, $X_{t|t}$ and $\mu_{x,t}$ yields:

$$Z_t = LX_t + [N_1\ N_2] \begin{bmatrix} X_{t|t} \\ \mu_{x,t} \end{bmatrix} + v_t$$

where

$$\begin{aligned} L &\equiv D_1^1 + D_2^1 G_1 \\ N_1 &\equiv D_1^2 - D_2^1(G_1 + G_1^*) - D_2^2 G_1^* \\ N_2 &\equiv (D_2^1 + D_2^2) G_2^* \end{aligned}$$

which is equation (3.16) in the main text.

# B. Appendix: The Kalman filter

Below we report the basic steps in the derivation of Kalman filter (3.18)-(3.19). Define the one step ahead forecast errors in $X_t$ and $Z_t$:

$$\begin{aligned} \tilde{X}_t &\equiv X_t - X_{t|t-1} \\ \tilde{Z}_t &\equiv Z_t - Z_{t|t-1} \end{aligned}$$

Let us express the optimal linear forecast of $X_t$ as a function of the new information at time $t$ (since $\tilde{X}_t$ is by definition orthogonal to the information at $t-1$, we know that the this is the form of an optimal linear predictor), as follows

$$X_{t|t} = X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t] \tag{B.1}$$

where the matrix $K_{(\mathrm{N_{PD}},\mathrm{N_Z})}$ has to be determined. Note that (3.15) and (3.16) allow us to write

$$\begin{aligned}\tilde{Z}_t &= Z_t - (L + N_1)X_{t|t-1} - N_2\mu_{x,t} \\ &= L\tilde{X}_t + N_1 K(L\tilde{X}_t + v_t) + v_t\end{aligned}$$

where we used that $\mu_{x,t}$ is measurable with respect to the information at $t-1$ (this is immediate from 3.15). The above defines a measurement equation:

$$\begin{aligned}\tilde{Z}_t &= N\tilde{X}_t + \tau_t \\ & \quad where \\ N &\equiv (I + N_1 K)L \\ \tau_t &\equiv (I + N_1 K)v_t\end{aligned} \tag{B.2}$$

in terms of the innovation as of time $t$ in both the observables and the state variables. Algebraic manipulation of (3.15) after substituting equation (B.1) for $X_{t|t}$ yields

$$\begin{aligned}\tilde{X}_{t+1} &= T\tilde{X}_t + \omega_{t+1} \\ & \quad where \\ T &\equiv H(I - KL) \\ \omega_{t+1} &\equiv C_u u_{t+1} - HKv_t\end{aligned} \tag{B.3}$$

The equation for the states (B.3) and the measurement equation (B.2) allow us to derive an optimal predictor $\tilde{X}_t$ using the Kalman filter. By the Kalman filter the optimal (contemporaneous) projection of $\tilde{X}_t$ on $\tilde{Z}_t$, denoted $\tilde{X}_{t|t}$ is given by (See Ljungqvist and Sargent, 2000, Chapter 21)

$$\begin{aligned}\tilde{X}_{t|t} &= \tilde{K}\tilde{Z}_t \\ & \quad where \\ \tilde{K} &\equiv PN'(NPN' + \Sigma_\tau^2)^{-1}\end{aligned} \tag{B.4}$$

where we used the covariance matrices of $\omega_{t+1}$, $\tau_t$,

$$\begin{aligned}\Sigma_\omega^2 &\equiv C_u\Sigma_u^2 C_u' + (HK)\,\Sigma_v^2(HK)' \\ \Sigma_\tau^2 &\equiv (I + N_1 K)\Sigma_v^2(I + N_1 K)'.\end{aligned}$$

The matrix $P$ is the covariance matrix of the one step ahead forecast error in

$X_{t+1}$ which, from (B.3), is given by

$$cov(X_{t+1} - X_{t+1|t}) \equiv P = TPT' + \Sigma_\omega^2 \tag{B.5}$$

We have to find an expression to link $K$ and $\tilde{K}$. Algebraic manipulation of the state motion for $X_t$ and $Z_t$ implies:

$$\tilde{K} = K(I + N_1 K)^{-1} \tag{B.6}$$

Substituting the expression for $\Sigma_\tau^2, T$ and $N$ in (B.4) and plugging the resulting expression in (B.6) yields (3.18) in the main text. Substituting for $\Sigma_\omega^2, T$ and $K$ into (B.5) yields (3.19) in the main text.

# C. Appendix: User-defined variables in the CGG model

In the following, we show how to construct a vector of "user defined" variables in the Toolkit. Once such a vector is defined, the user can compute impulse response functions, simulations, as well as the moments of these variables, which are often of interest in the analysis of a model.

We set up the Toolkit so that a user defined variable, $y_t$, can be expressed as a linear function of the variables in the vector $J_t' \equiv [X_t' \; X_{t|t}' \; x_t' \; x_{t|t}' \; i_t' \; \mu_{x,t}']$ and/or of their one-step ahead projection $J_{e,t}' \equiv [X_{t+1|t}' \; x_{t+1|t}' \; i_{t+1|t}' \; \mu_{x,t+1}']$. Therefore:

$$y_t \equiv \hat{F} \; J_t + \hat{F}_e \; J_{e,t} \tag{C.1}$$

Hence, the definition of a user defined variable only requires the specification of the row vectors $\hat{F}$ and $\hat{F}_e$ which select the appropriate components of $y_t$ from $J_t$ and $J_{e,t}$. Using (3.15) and (3.12) it is then simple to express the vector $y_t$ as a function of $J_t$ only, by noting that

$$\begin{aligned} J_{e,t} &= \hat{E} J_t \\ & \quad where \\ \hat{E} &= \begin{bmatrix} 0 & H+M_{X\,1} & 0 \;\; 0 \;\; 0 & M_{X\,2} \\ 0 & -G_1^*(H+M_{X\,1}) + G_2^* M_{\mu\,1} & 0 \;\; 0 \;\; 0 & -G_1^* M_{X\,2} + G_2^* M_{\mu\,2} \\ 0 & F_1^*(H+M_{X\,1}) + F_2^* M_{\mu\,1} & 0 \;\; 0 \;\; 0 & F_1^* M_{X\,2} + F_2^* M_{\mu\,2} \\ \underbrace{0}_{X_t} & M_{X\,2} & \underbrace{0 \;\; 0 \;\; 0}_{x_t \; x_{t|t} \; i_t} & M_{\mu\,2} \end{bmatrix} \end{aligned}$$

where the column-blocks of zeros conform to the size of the $J_t$ elements indicated below the curly bracket. Impulse response analysis for the variables in this vector are then readily computable via (C.1) and (4.2), i.e. from[20]

$$\begin{aligned} y_t &= (\hat{F} + \hat{F}_e\hat{E})J_t = \hat{H}\cdot(\hat{C}Q_t + \hat{D}\xi_{t+1}) \\ &\quad where \\ \hat{H} &\equiv \hat{F} + \hat{F}_e\hat{E} \end{aligned} \tag{C.2}$$

As an example of two user defined variables, consider the definition of the real rate and the output gap in the CGG model discussed in Section 5. To write the output gap $(y_t - \bar{y}_t)$ as the first variable in the column vector of user defined variables we shall define:

$$\begin{aligned} \hat{F} &\equiv \left[ \underbrace{0\ \text{-}1\ 0\ 0}_{X_t}\ \ \underbrace{0_{(1,\mathrm{N_{PD}})}}_{X_{t|t}}\ \ \underbrace{1\ 0}_{x_t}\ \ \underbrace{0_{(1,\mathrm{N_{FW}}+\mathrm{N_{CTL}}+\mathrm{N_{FW}})}}_{x_{t|t}\ i_t\ \mu_{xt}} \right] \\ \hat{F}_e &\equiv \left[ \underbrace{0_{(1,\mathrm{N_{PD}}+\mathrm{N_{FW}}+\mathrm{N_{CTL}}+\mathrm{N_{FW}})}}_{X_{t+1|t}\ x_{t+1|t}\ i_{t+1|t}\ \mu_{xt+1}} \right] \end{aligned}$$

where the labels below the vector elements indicate to which elements of $J_t$ these elements conform ($\mathrm{N_{PD}}$, $\mathrm{N_{FW}}$ and $\mathrm{N_{CTL}}$ indicate, respectively, the number of predetermined, forward looking and control variables).[21] Since no forward looking terms enter the definition of the output gap the $\hat{F}_e$ vector is just a zero (of appropriate dimension). Forward looking terms instead enter the ex-ante real rate $(i_t - \pi_{t+1|t})$ definition through the expected inflation term. To write the real rate for the CGG example we shall define

$$\begin{aligned} \hat{F} &\equiv \left[ \underbrace{0_{(1,2*\mathrm{N_{PD}}+2*\mathrm{N_{FW}})}}_{X_t\ X_{t|t}\ x_t\ x_{t|t}}\ \ \underbrace{1}_{i_t}\ \ \underbrace{0_{(1,\mathrm{N_{FW}})}}_{\mu_{xt}} \right] \\ \hat{F}_e &\equiv \left[ \underbrace{0_{(1,2*\mathrm{N_{PD}})}}_{X_{t+1|t}}\ \ \underbrace{0\ \text{-}1}_{x_{t+1|t}}\ \ \underbrace{0_{(1,\mathrm{N_{FW}}+\mathrm{N_{CTL}}+\mathrm{N_{FW}})}}_{i_{t+1|t}\ \mu_{xt+1}} \right]. \end{aligned}$$

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[20] In the numerical implementation of this feature, the Toolkit computes the $\hat{H}$ matrix once $\hat{F}$ and $\hat{F}_e$ are defined by a user. The $y_t$ variables are then stacked as the last elements in the (new) column vector of responses $\tilde{J}_t' \equiv [J_t'\ y_t']$ (i.e. the usual $J_t$ variables plus the new user defined variables). The Toolkit then computes the impulse responses using the augmented system $\tilde{J}_t = \tilde{C}Q_t + \tilde{D}\,\xi_{t+1}$, where the matrices $\tilde{C} \equiv \begin{bmatrix} \hat{C} \\ \hat{H}\hat{C} \end{bmatrix}$ and $\tilde{D} \equiv \begin{bmatrix} \hat{D} \\ \hat{H}\hat{D} \end{bmatrix}$ (as from C.2).

[21] To compute the *expected* output gap, $y_{t|t} - \bar{y}_{t|t}$, one should position the ones so that they pick the corresponding *expected* elements in the $J_t$ vector.

## D. Appendix: The value of the intertemporal loss function

Below it is shown how to compute the matrix $V$ and the scalar $d$ that characterize the quadratic expression of the intertemporal loss function derived in subsection 4.2.

Under commitment, the optimal value of (2.4) admits a representation of the form $\Lambda_t^* \equiv \mathcal{S}'_{t|t} V_c \mathcal{S}_{t|t} + d_c$, where we stack in the $\mathcal{S}$ vector all the relevant state and costate variables, $\mathcal{S}'_t \equiv [X'_t \; \mu'_{x,t}]$,and the matrix $V_c$ and the scalar $d_c$ are to be determined.[22] Substituting this expression in (4.4) we obtain:

$$\mathcal{S}'_{t|t} V_c \mathcal{S}_{t|t} + d_c = L^*_{t|t} + \delta E_t \left\{ \mathcal{S}'_{t+1|t+1} V_c \mathcal{S}_{t+1|t+1} + d_c \right\} \tag{D.1}$$

Our strategy to identify $V_c$ and $d_c$ is to expand the right hand side of (D.1) using the formulas that define the solution under commitment. In order to do that, we first need to express targets $Y$ and period losses in terms of $\mathcal{S}_t$ and $\mathcal{S}_{t|t}$. Simple algebra (using (3.14), (A.1) and (A.2) into (2.2)) yields

$$\begin{aligned} Y_t &= \bar{C}_1 \mathcal{S}_{t|t} + \bar{C}_2 (\mathcal{S}_t - \mathcal{S}_{t|t}) \\ & \quad where \\ \bar{C}_1 &\equiv \left[ \begin{array}{cc} (C_1^1 + C_1^2) - (C_2^1 + C_2^2) G_1^* + C_i F_1^* & (C_2^1 + C_2^2) G_2^* + C_i F_2^* \end{array} \right] \\ \bar{C}_2 &\equiv \left[ \begin{array}{cc} C_1^1 + C_2^1 G_1 & 0 \end{array} \right] \end{aligned} \tag{D.2}$$

and the matrix 0 has zero elements and is conformable to $Y$ and $\mathcal{S}$. This allows the period losses $L^*_{t|t}$ to be rewritten as

$$L^*_{t|t} = \mathcal{S}'_{t|t} \bar{C}'_1 W \bar{C}_1 \mathcal{S}_{t|t} + E_t \left\{ (\mathcal{S}_t - \mathcal{S}_{t|t})' \bar{C}'_2 W \bar{C}_2 (\mathcal{S}_t - \mathcal{S}_{t|t}) \right\}. \tag{D.3}$$

Next, let us rewrite the term $E_t \left\{ \mathcal{S}'_{t+1|t+1} V_c \mathcal{S}_{t+1|t+1} + d_c \right\}$in (D.1) in terms of $S_t$and $S_{t|t}$. Substituting (D.3) and (4.7) into the right hand side of (D.1) and collecting terms, we obtain a quadratic expression in $\mathcal{S}_{t|t}$ plus a scalar on both sides of the resulting equation. Therefore, we derive the following two identities from which $V_c$ and $d_c$ can be computed:

$$V_c = \bar{C}'_1 W \bar{C}_1 + \delta (HH + MM)' V_c (HH + MM) \tag{D.4}$$

$$d_c = \frac{1}{1-\delta} \left[ tr(w_c^1 P_{co}) + \delta \; tr(w_c^2 P_{co}) + \delta \; tr(w_c^3 \Sigma_{uu}^2) + \delta \; tr(w_c^4 \Sigma_{vv}^2) \right] \tag{D.5}$$

[22] The computation of losses under discretion is derived in a completely analogous way (basically replacing the equations for $\mathcal{S}$ (state and costate variables) with equations for $X$; details are available from the authors upon request).

where $w_c^1 = \bar{C}_2' W \bar{C}_2$, $w_c^2 = (HH)'(KK)'V_c(KK)(HH)$, $w_c^3 = (KK)'V_c KK$ and $w_c^4 = V_c$. $P_{co}$ is the covariance matrix of the contemporaneous prediction error in $\mathcal{S}_t$ as defined in (4.9).

# E. Appendix: Computation of unconditional moments

The unconditional covariance matrix of $\mathcal{S}_{t|t}$has been computed in the main text. Here we briefly sketch how to compute the other main covariance matrices. To compute the unconditional covariance matrix for the state vector $\mathcal{S}_t$, note that the identity $\mathcal{S}_t = \mathcal{S}_{t|t} + (\mathcal{S}_t - \mathcal{S}_{t|t})$ and the fact that $E(\mathcal{S}_t - \mathcal{S}_{t|t})\mathcal{S}_{t|t}' = 0$ , imply:

$$P_{co} \equiv cov(\mathcal{S}_t - \mathcal{S}_{t|t}) = \Sigma^2_{\mathcal{S}_t} - \Sigma^2_{\mathcal{S}_{t|t}} \tag{E.1}$$

which shows that $\Sigma^2_{\mathcal{S}_t}$ can be immediately computed once $P_{co}$ and $\Sigma^2_{\mathcal{S}_{t|t}}$ are known.

Next, using equations (3.12) and (3.13) the forward looking variables $x_t$ can be compactly expressed in terms of the $\mathcal{S}_t$ variables as:

$$\begin{aligned} x_t &= \bar{G}(\mathcal{S}_t - \mathcal{S}_{t|t}) + \bar{G}^* \mathcal{S}_{t|t} \\ & \quad where \\ \bar{G} &\equiv [G_1 \qquad 0_{nfw,nfw}],\ \bar{G}^* \equiv [-G_1^* \qquad G_2^*] \end{aligned}$$

which allows the covariance matrices of the forward looking variables $x_t$ and $x_{t|t}$ to be computed:

$$\begin{aligned} \Sigma^2_{x_{t|t}} &\equiv E_0(x_{t|t} x_{t|t}') = \bar{G}^* \Sigma^2_{\mathcal{S}_{t|t}} \bar{G}^{*\prime} \\ \Sigma^2_{x_t} &\equiv E_0(x_t x_t') = \bar{G} P_{co} \bar{G}' + \Sigma^2_{x_{t|t}} \end{aligned}$$

## E.1. Moments of the user-defined variables

From equation (C.2) we can easily infer an expression for the unconditional covariance matrix of $y_{t,}$ $cov(y_t) \equiv E(y_t y_t') - E(y_t)E(y_t')$:

$$cov(y_t) = \hat{H}\left[\hat{C}E(Q_t Q_t')\hat{C}' + \hat{D}E(\xi_t \xi_t')\hat{D}' - \hat{C}E(Q_t)E(Q_t')\hat{C}'\right]\hat{H}'$$

where the only unknown is $E(Q_t Q_t')$ which can be computed directly using $Q's$ definition (4.1) and the moments in $\Sigma^2_{\mathcal{S}_t}$.

## References


[1] Backus, David and John Driffill, 1986, "The consistency of optimal policy in stochastic rational expectations models", CEPR Discussion Paper No. 124.

[2] Clarida, Richard, Jordi Galí and Mark Gertler, 1999. "The Science of Monetary Policy: A New-Keynesian Perspective". **Journal of Economic Literature**. Vol. 37, pp.1661-707.

[3] Currie, David and Paul Levine, 1985, "Time inconsistency and optimal policies in deterministic and stochastic worlds", Prism Research Papers 30, Queen Mary College, London.

[4] Ehrmann, Michael and Frank Smets, 2001 "Uncertain potential output: implications for monetary policy", ECB working paper No.59.

[5] Ljungqvist, Lars and Thomas J. Sargent 2002, "Stackelberg Plans", mimeo, Stanford University.

[6] Ljungqvist, Lars and Thomas J. Sargent 2000, *Recursive Macroeconomic Theory*, MIT Press.

[7] Orphanides, A., 2000, "The quest for prosperity without inflation", manuscript, Board of Governors of the Federal Reserve System.

[8] Orphanides, A., 2001, "Monetary Policy Rules based on Real-Time Data", **American Economic Review**, Vol.91(4): 964-985.

[9] Söderlind, Paul (1999) "Solution and estimation of rational expectations models with optimal policy", **European Economic Review**, 43, pp.813-23.

[10] Svensson, Lars O.E. and Michael Woodford, 2000. "Indicator variables for optimal policy". Mimeo, Princeton University, NJ.

[11] Svensson, Lars O.E. and Michael Woodford, 2002. "Optimal policy with partial information in a forward looking model: a certainty-equivalence redux". Mimeo, Princeton University, NJ.

[12] Woodford, Michael, 1999a. "Optimal Monetary Policy Inertia", NBER working paper No. 7261.