A behavioral characterization of the Drift Diffusion Model and its multi-alternative extension for choice under time pressure

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Abstract

In this paper, we provide an axiomatic foundation for the value-based version of the Drift Diffusion Model (DDM) of Ratcliff (1978), a successful model that describes two-alternative speeded decisions between consumer goods.

Our axioms present a test for model misspecification and connect the externally observable properties of choice with an important neurophysiological account of how choice is internally implemented.

We then extend our axiomatic analysis to multi-alternative choice under time pressure. In a nutshell, we show that binary DDM comparisons of the alternatives, paired with Markovian exploration of the consideration set, approximately lead to softmaximization.

A successful model to describe two-alternative speeded decisions between consumer goods is the value-based version of the Drift Diffusion Model (DDM) of Ratcliff [53]. Here we provide an axiomatic foundation for the DDM and a simple way to elicit its parameters from behavioral data.

When eye-tracking data are also available, our characterization allows to test the Metropolis-DDM algorithm, a recent multi-alternative extension of the DDM due to Cerreia-Vioglio et al. [14], and to identify its parameters.

The use of the DDM to describe value-based decisions, pioneered by Busemeyer and Townsend [10] in the case of choice under uncertainty, has been extensively studied by Krajbich et al. [32] and [34] and by Milosavljevic et al. [44] for consumer goods, and has been recently neurophysiologically motivated by Shadlen and Shohamy [63] in terms of sequential sampling from memory. By now, the DDM is a paradigm for choice between pairs $\{a, b\}$ of alternatives. It explains a wide range of behavioral and neurobiological data, it has a compelling neurophysiological interpretation, and it is optimal in terms of sequential sampling.[1] It has also been shown to successfully describe a wide range of purchasing decisions – from snacks to consumer electronics, from household items to mobile apps.[2]

Specifically, given two alternatives, a and b, this neuro-computational algorithm assumes that decisions are made by accumulating noisy information about them over time, until the net evidence

[1] See, e.g., Bogacz et al. [7], Hare et al. [29], Ratcliff et al. [54] and [55], and Fudenberg et al. [26].

[2] See, e.g., Roe et al. [60], Krajbich et al. [32] and [34], Milosavljevic et al. [44], Clithero [17], and Chiong et al. [16].

in favor of one exceeds a prespecified threshold, λ, at which point the favored alternative is selected. The presence of noise in the accumulation of information implies that choice between the same pair of alternatives does not always terminate at the same time and does not always lead to the same outcome. More formally, the DDM describes how linear evidence accumulation with white Gaussian noise generates the random variables *decision time*, $\mathrm{DT}_{a,b}$, and *decision outcome*, $\mathrm{DO}_{a,b}$, for choice in a two-alternative set $\{a, b\}$.

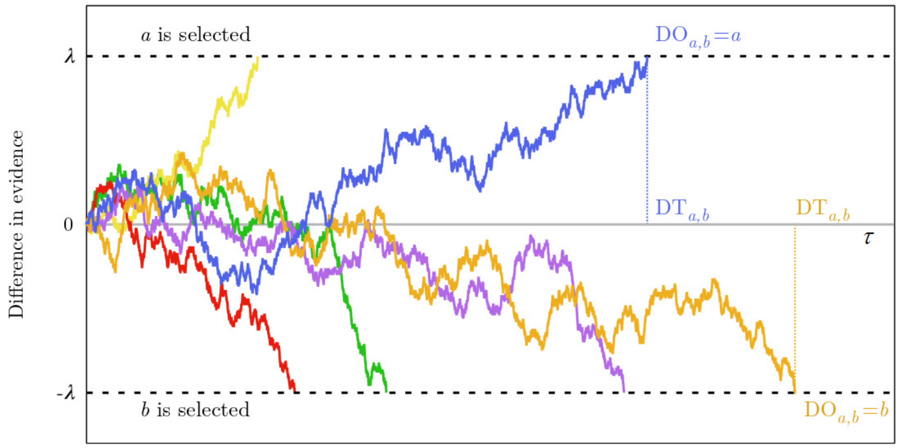

Fig. a: Six realizations of the Drift Diffusion Model.

The DDM naturally captures speed-accuracy tradeoffs: lower thresholds λ produce faster but less accurate responses, whereas higher thresholds λ produce more accurate but slower responses. This feature is particularly relevant for choice under time pressure: empirical evidence confirms the intuition that higher pressure induces lower thresholds.[3]

Our main contribution is providing necessary and sufficient conditions on the observables – that is, on choice frequencies and decision times – that guarantee that the agent behaves "as if" implementing the DDM. In the tradition of psychophysics, these conditions are called "axioms" and can be seen both as an empirical test of the model and as a measurement tool for its parameters.[4] The twist of our approach is combining the choice frequency and decision time components into an axiomatic characterization. Both these observables are at the heart of most psychophysical theories (see, e.g., Luce [36] and [37]). Yet, while the former has been studied in great axiomatic detail under the name of "random choice", the latter has not been analyzed from this perspective, with the exceptions of the recent Echenique and Saito [20] and Alos-Ferrer et al. [2].

Beyond falsifiability of the DDM theory and a better understanding of its behavioral implications, an important experimental advantage of our main representation theorem is that it does not require a parameter fitting routine, but allows to elicit the agent's utility function and decision threshold directly from behavioral data. Also, our framework can be immediately applied to common experimental setups in which each participant contributes only a moderate amount of data and the error rate is low – see Wagenmakers et al. [70], Lerche and Voss [35], and van Ravenzwaaij et al. [56].

Finally, our axiomatization extends beyond two-alternative choice modeling. Indeed, we show that it permits to test the multi-alternative choice procedure under time pressure of Cerreia-Vioglio et al. [14] and to elicit its parameters. In so doing, we also generalize their Metropolis-DDM algorithm to allow for the formation of consideration sets. Because of the importance of these sets

[3]See, e.g., Busemeyer and Townsend [10], Milosavljevic et al. [44], Karsilar et al. [31], and the discussion in Ortega and Stocker [48].

[4]Classical references are Luce [36] and Luce and Suppes [38].

in economics and marketing,[5] this generalization is another relevant contribution of the present paper.

The literature on the DDM is vast but non-axiomatic. We refer readers to the reviews of Fehr and Rangel [25] and Ratcliff et al. [55]. Webb [71] studies the relation between the DDM (and bounded accumulation models, in general) and random utility models. The optimality of the DDM in terms of sequential sampling is analyzed by Gold and Shadlen [28] and Bogacz et al. [7] in the classical case, and by Fudenberg et al. [26] and Epstein and Ji [22] in the Bayesian case. An identification result similar to ours is independently obtained by the recent Chiong et al. [16], who adopt the DDM as a structural model to analyze the effects of advertisement in app purchasing behavior.

The extension of the DDM to menus of $N > 2$ alternatives is a non-trivial issue and different generalizations, with significantly different behavioral and neurobiological properties, have been proposed. See, e.g., Roe et al. [60], Usher and McClelland [68], Anderson et al. [3], McMillen and Holmes [43], Bogacz et al. [8], Ditterich [18], Krajbich and Rangel [33], and Reutskaja et al. [57]. In most of these models, the choice task is assumed to *simultaneously activate N accumulators*, each of which is primarily sensitive to one of the alternatives and integrates evidence relative to that alternative. Choices are then made based on absolute or relative evidence levels. In contrast, the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [14] builds on *sequential activation of 2 accumulators* and Markovian exploration of the menu of alternatives. This feature makes the Metropolis-DDM algorithm more realistic in view of both the available eye-tracking evidence – see, e.g., Russo and Rosen [62] and Russo and Leclerc [61] – and of the known limitations of working memory – see, e.g., Luck and Vogel [39] and Vogel and Machizawa [69]. The same feature allows for model-testing and permits parameters' elicitation by analyzing binary comparisons only. These comparisons are the most studied in many fields of decision theory and their quantitative and experimental analysis is consequently well developed.

The paper is organized as follows: Section 1 is a short and self contained exposition of the value-based Drift Diffusion Model. Section 2 characterizes the DDM via observables; specifically, it provides a "psychometric axioms if and only if DDM" representation theorem and studies its consequences. Section 3 discusses the relation between theoretical choice probabilities and empirical choice frequencies; in particular, it shows that the DDM representation is robust to situations in which data are either scarce or noisy. Section 4 extends the characterization of Section 2 from the DDM to the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [14]; it also generalizes the latter to allow for the formation of consideration sets. Section 5 presents several simulations and discusses the numerical convergence properties of the Metropolis-DDM algorithm. Appendix A contains the proofs of our results and some additional material.

1 The DDM for value-based decisions

Let A be a choice set consisting of at least three distinct alternatives. The DDM is a model of binary comparison between a fixed pair of alternatives a and b in A. According to this model, noisy evidence about the alternatives is accumulated until it reaches some threshold $\lambda > 0$, at which point a decision is taken. Specifically, either alternative is selected as soon as the net evidence in its favor attains level λ. In a neurophysiological perspective, the comparison of a and b is assumed to activate two neuronal populations whose activities (firing rates) provide information

[5] See, e.g., Shocker et al. [64], Roberts and Nedungadi [59], Peter and Olson [50], Eliaz and Spiegler [21], Masatlioglu et al. [42], Hauser [30], Manzini and Mariotti [41], Gaynor et al. [27], and Caplin et al. [11].

for the two alternatives. Denote their mean activities by $u(a)$ and $u(b)$, and assume that each experiences instantaneous independent white noise fluctuations modeled by uncorrelated Wiener processes W_a and W_b. Evidence accumulation in favor of a and b is then represented by the two Brownian motions with drift $V_a(t) \equiv u(a)t + \sigma W_a(t)$ and $V_b(t) \equiv u(b)t + \sigma W_b(t)$, respectively.[6] With this,

- the *net evidence* in favor of a against b is given, at each $t > 0$, by the difference

$$Z_{a,b}(t) \equiv V_a(t) - V_b(t) = [u(a) - u(b)]t + \sigma\sqrt{2}\, W(t)$$

  where W is the Wiener process $(W_a - W_b)/\sqrt{2}$;

- the comparison ends when $Z_{a,b}(t)$ reaches either the barrier λ or $-\lambda$; so, the *decision time* is the random variable

$$\mathrm{DT}_{a,b} \equiv \min\{t : Z_{a,b}(t) = \lambda \text{ or } Z_{a,b}(t) = -\lambda\}$$

- when the comparison ends (at random time $\mathrm{DT}_{a,b}$), the agent selects a if the upper barrier λ has been reached, and selects b otherwise; so, the *decision outcome* is the random variable

$$\mathrm{DO}_{a,b} \equiv \begin{cases} a & \text{if } Z_{a,b}(\mathrm{DT}_{a,b}) = \lambda \\ b & \text{if } Z_{a,b}(\mathrm{DT}_{a,b}) = -\lambda \end{cases}$$

The probability of choosing a from $\{a, b\}$ is thus

$$P_{a,b} \equiv \mathbb{P}[\mathrm{DO}_{a,b} = a]$$

and its explicit logistic formula

$$P_{a,b} = \frac{1}{1 + e^{-\frac{\lambda}{\sigma^2}[u(a)-u(b)]}} \tag{1}$$

can already be found in Ratcliff [53]. Of course, $P_{b,a} = 1 - P_{a,b}$. In particular, the choice of an inferior alternative, a if $u(a) < u(b)$ or b if $u(b) < u(a)$, is called an error. Its probability is the *error rate*

$$\mathrm{ER}_{a,b} \equiv \min\{P_{a,b}, P_{b,a}\} = \frac{1}{1 + e^{\frac{\lambda}{\sigma^2}|u(a)-u(b)|}}$$

and so the *probability of a correct choice* is

$$\mathrm{PC}_{a,b} \equiv 1 - \mathrm{ER}_{a,b} = \max\{P_{a,b}, P_{b,a}\} = \frac{1}{1 + e^{-\frac{\lambda}{\sigma^2}|u(a)-u(b)|}}$$

The explicit formulas of the distribution of $\mathrm{DT}_{a,b}$ and of its moments are also well known (see the appendix). For example, its mean is

$$\overline{\mathrm{DT}}_{a,b} \equiv \mathbb{E}[\mathrm{DT}_{a,b}] = \frac{\lambda}{u(a) - u(b)} \tanh \frac{\lambda[u(a) - u(b)]}{2\sigma^2} = \lambda \frac{2\mathrm{PC}_{a,b} - 1}{|u(a) - u(b)|} \tag{2}$$

(replaced with its limit $\lambda^2/2\sigma^2$ when $u(a) = u(b)$). As intuitive, it increases with the amount of net evidence required to decide, as well as with the payoff proximity of the alternatives.

[6] See, e.g., Bogacz et al. [7] and Fudenberg et al. [26].

In some cases the alternatives a and b may play different roles, say b is the *status quo* or the *incumbent solution* of a decision problem. The amount of net evidence required to maintain b, call it β, may then be different (typically smaller) from the amount of net evidence λ required to switch to a. In these cases, it is necessary to replace $-\lambda$ with $-\beta$ in the expressions of $\mathrm{DT}_{a,b}$ and $\mathrm{DO}_{a,b}$, and formulas (1) and (2) should be modified accordingly. Our multi-alternative generalization of the DDM can be extended to these "asymmetric barriers" cases (see the appendix).

For later reference, we call $P = [P_{a,b}]_{a,b \in A}$ the *stochastic choice matrix* and $\overline{\mathrm{DT}} = \left[\overline{\mathrm{DT}}_{a,b}\right]_{a,b \in A}$ the *decision time matrix* induced by the DDM. More generally, we call (stochastic) choice matrix any quasi-positive matrix $\widetilde{P}$ such that $\widetilde{P}_{a,b} + \widetilde{P}_{b,a} = 1$ for all $a \neq b$ in A and (decision) time matrix any quasi-positive symmetric matrix.

1.1 Normalizations

Notice that the parameters $\{u_a\}_{a \in A}$, σ, and λ are defined up to a common positive scalar multiple. If *all* of them are multiplied by a constant $\alpha > 0$, the predictions of the DDM are unchanged. For instance, choosing $\alpha = 1/\sigma$ amounts to normalize the noise σ of the Brownian motions $\{V_a\}_{a \in A}$ and to replace σ with 1, u with $\hat{u} = u/\sigma$, and λ with $\hat{\lambda} = \lambda/\sigma$.[7]

A different normalization, typical of the mathematical psychology literature, consists in setting $\lambda = 1$, which obviously corresponds to $\alpha = 1/\lambda$.

Finally, observe that u is actually cardinally unique, that is, defined up to positive scalar multiplication and translation by an additive constant. This follows from the fact that the drift of the Brownian motion $Z_{a,b}$ only depends on the difference $u(a) - u(b)$. For example, in a neurophysiological perspective, it may be desirable to normalize the range of u to $[0, 100]$. This leads to the transformation

$$u \mapsto \frac{100}{\max_{a \in A} u(a) - \min_{a \in A} u(a)} \left(u - \min_{a \in A} u(a) \right)$$

In this case, both σ and λ must be multiplied by 100 and divided by $\max_A u - \min_A u$, accordingly.

2 Observability and measurement

In terms of external observability of the DDM, assume that the analyst can observe the agent's choices between pairs (a, b) several times. These observations produce empirical *choice frequencies* $P^o_{a,b} = 1 - P^o_{b,a}$ and empirical *mean decision times* $\overline{\mathrm{DT}}^o_{a,b} = \overline{\mathrm{DT}}^o_{b,a}$ for all $a \neq b$ in A. Formally, we call *observables* any pair $\left(P^o, \overline{\mathrm{DT}}^o\right)$ consisting of a stochastic choice matrix P^o and a decision time matrix $\overline{\mathrm{DT}}^o$.

Example 1 *With three alternatives, the matrices are*

$$\begin{bmatrix} * & P^o_{a,b} & P^o_{a,c} \\ P^o_{b,a} & * & P^o_{b,c} \\ P^o_{c,a} & P^o_{c,b} & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & \overline{\mathrm{DT}}^o_{a,b} & \overline{\mathrm{DT}}^o_{a,c} \\ \overline{\mathrm{DT}}^o_{b,a} & * & \overline{\mathrm{DT}}^o_{b,c} \\ \overline{\mathrm{DT}}^o_{c,a} & \overline{\mathrm{DT}}^o_{c,b} & * \end{bmatrix}$$

The elements on the diagonal, which are conceptually meaningless, can be arbitrarily specified.

[7] This is the normalization that we will adopt, when we do not consider the generic expression. Another normalization of noise, popular in behavioral experiments, corresponds to $\sigma\sqrt{2} = 0.1$ and it determines a choice of $\alpha = \sqrt{2}/20\sigma$ (see Section 5).

Here the superscript "o" stands for "observable" (or "observed"). For instance, $\mathrm{ER}^o_{a,b} \equiv \min\left\{P^o_{a,b}, P^o_{b,a}\right\}$ is the *observed (experimental) error rate*, while $\mathrm{ER}_{a,b} \equiv \min\left\{P_{a,b}, P_{b,a}\right\}$ is the *DDM (theoretical) error rate*. In what follows we use two additional psychological measurements: the *probability of a correct choice*

$$\mathrm{PC}^o_{a,b} \equiv 1 - \mathrm{ER}^o_{a,b} = \max\left\{P^o_{a,b}, P^o_{b,a}\right\}$$

and the *ease of comparison*

$$\mathrm{EC}^o_{a,b} \equiv \log \frac{\mathrm{PC}^o_{a,b}}{\mathrm{ER}^o_{a,b}} = \operatorname{logit} \mathrm{PC}^o_{a,b}$$

that is, the log-odds of a correct choice.[8] The use of these quantities dates back to classical psychometrics (Fechner [24], Thurstone [67], Luce [36], and Rasch [52]) and the relation between them is self explanatory:

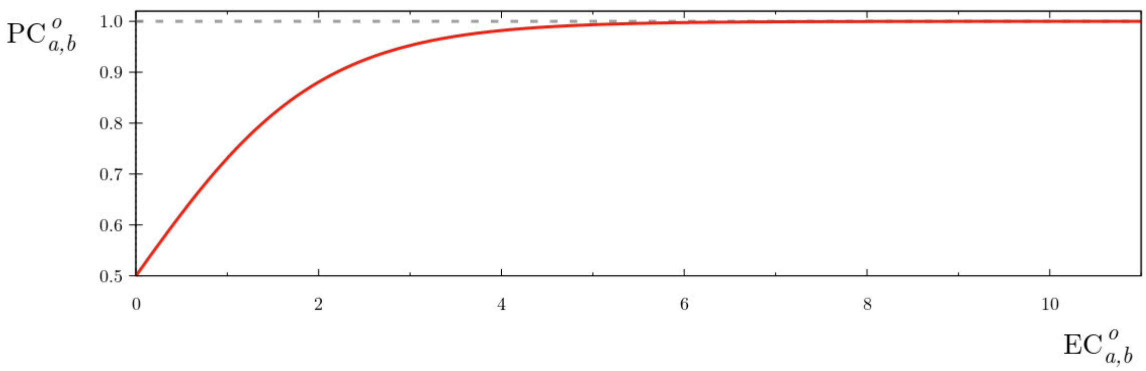

Fig. b: Relation between probability of correct choice $\mathrm{PC}^o_{a,b}$ and ease of comparison $\mathrm{EC}^o_{a,b}$.

In words, *easier choice problems are more likely to elicit correct responses than harder ones.*[9]

In particular, $\mathrm{PC}^o_{a,b}$, measures "how accurate" the comparison is: it ranges between 1/2 (no accuracy, choice is completely random) and 1 (maximum accuracy, choice is deterministic). It is convenient to normalize it to 0 and formally define *accuracy of comparison* as

$$\mathrm{AC}^o_{a,b} \equiv \mathrm{PC}^o_{a,b} - \frac{1}{2}$$

Instead, $\mathrm{EC}^o_{a,b}$ describes ease of comparison as the probabilistic distance $\log \mathrm{PC}^o_{a,b} - \log \mathrm{ER}^o_{a,b}$ between correct responses and errors. Notice that $|\log p - \log q|$ actually defines a distance between probabilities which is characterized by a simple set of properties (see Baucells and Heukamp [6]).[10]

Next we show that the DDM is characterized by simple verifiable conditions on observables, stated as axioms.

Axiom 1 (Positivity) *Choice is a stochastic and time consuming process:*

$$P^o_{a,b} > 0 \qquad \text{and} \qquad \overline{\mathrm{DT}}^o_{a,b} > 0$$

for all distinct a and b in A.

[8] Ease of comparison was originally called "degree of easiness" by Rasch [52]. Also recall that the *log-odds* of an event that occurs with probability p are given by $\operatorname{logit} p = \log p - \log(1-p)$.

[9] See the discussion of Alos-Ferrer et al. [2] on the psychometric function.

[10] See also Proposition 9 in the appendix.

This axiom of Luce [36] is where psychophysics departs from microeconomics, which assumes instantaneous and deterministic choice.

Axiom 2 (Product rule) *Violations of transitivity are only due to noise:*

$$P^o_{a,b}P^o_{b,c}P^o_{c,a} = P^o_{a,c}P^o_{c,b}P^o_{b,a}$$

for all distinct a, b, and c in A.

To interpret this axiom, observe that $P^o_{a,b}P^o_{b,c}P^o_{c,a}$ is the probability of observing the agent choose c from $\{a, c\}$, then b from $\{b, c\}$, and a from $\{a, b\}$, while $P^o_{a,c}P^o_{c,b}P^o_{b,a}$ is the probability of observing the agent choose b from $\{a, b\}$, then c from $\{b, c\}$, and a from $\{a, c\}$. Thus, the product rule asserts that the intransitive cycles

$$a \to c \to b \to a \qquad \text{and} \qquad a \to b \to c \to a$$

must be observed with the same probability. In other words, they are equally likely "mistakes". Indeed, the product rule is equivalent to transitivity in the noiseless case when $P^o_{a,b} = 1$ if a is strictly preferred to b, $P^o_{a,b} = 1/2$ if they are indifferent, and $P^o_{a,b} = 0$ otherwise.[11]

Luce and Suppes [38, p. 341] show that, together with positivity, the product rule characterizes the Luce model of binary choice (see Luce [36, ch. 1-2]).

Axiom 3 (Invariance) *Accuracy is proportional to mean decision time and ease of comparison:*

$$\frac{\overline{\mathrm{DT}}^o_{a,b}\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}} = \frac{\overline{\mathrm{DT}}^o_{a,c}\mathrm{EC}^o_{a,c}}{\mathrm{AC}^o_{a,c}}$$

for all distinct a, b, and c in A.

That is, there exists a constant $\kappa > 0$ such that

$$\overline{\mathrm{DT}}^o_{a,b} = \kappa \frac{\mathrm{AC}^o_{a,b}}{\mathrm{EC}^o_{a,b}} \qquad \text{for all } a \neq b \text{ in } A$$

Thus, the axiom has an (obviously) equivalent interpretation: mean decision time is directly proportional to accuracy and inversely proportional to ease of comparison.[12]

The next theorem, our first main contribution, shows that observables can be explained by the DDM if and only if they satisfy the previous axioms.

Theorem 1 *Let* $\left(P^o, \overline{\mathrm{DT}}^o\right)$ *be the observables. The following are equivalent:*

(i) P^o *and* $\overline{\mathrm{DT}}^o$ *satisfy positivity, the product rule, and invariance;*

(ii) there exist a function $u : A \to \mathbb{R}$ *and two coefficients* $\sigma > 0$ *and* $\lambda > 0$ *such that* $P^o = P$ *and* $\overline{\mathrm{DT}}^o = \overline{\mathrm{DT}}$.

In this case, $\hat{\lambda} = \lambda/\sigma$ *is unique and* $\hat{u} = u/\sigma$ *is unique up to an additive constant. In particular,*

$$\hat{\lambda} = \sqrt{\frac{\overline{\mathrm{DT}}^o_{a,b}}{2}\frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}}} \qquad \text{and} \qquad \hat{u}(a) - \hat{u}(b) = \frac{\operatorname{logit} P^o_{a,b}}{\hat{\lambda}} \tag{3}$$

for all distinct a and b in A.

This theorem has several noteworthy consequences. First, the identification of $\hat{\lambda}$ and $\hat{u}$ allows for inter-agent and intra-agent comparative statics.[13] For instance, one can say that agent 1 is "more reflective" than agent 2 if and only if $\hat{\lambda}_1 > \hat{\lambda}_2$,[14] or that agent 1 is "more risk averse"

[11] Notice that the noiseless choice matrix violates positivity, unless all alternatives are indifferent.

[12] See the discussion of Alos-Ferrer et al. [2] on the chronometric function.

[13] See also Chiong et al. [16, Lemma 1].

[14] Even when they have different utility functions $\hat{u}_1$ and $\hat{u}_2$, but provided they are choosing in the same conditions.

than agent 2 if and only if the certainty equivalents corresponding to $\hat{u}_1$ are smaller than those corresponding to $\hat{u}_2$.[15]

Second, this theorem guarantees that the function u is cardinally unique and that, given u, the coefficients σ and λ are both unique. Beyond its obvious theoretical interest, this fact also provides additional tests for the DDM, when the analyst has a richer database. For example, if instead of empirical mean decision times $\overline{\mathrm{DT}}^o_{a,b}$, empirical distributions $F_{\mathrm{DT}^o_{a,b}}$ of decision times where available, those could be used to compute $\overline{\mathrm{DT}}^o_{a,b}$, for all $a \neq b$ in A. If P^o and $\overline{\mathrm{DT}}^o$ do not satisfy point (i) of Theorem 1, then no DDM can rationalize the data; otherwise, the only possible rationalizing DDM (with $\sigma = 1$) is the one with parameters $\hat{\lambda}$ and $\hat{u}$ given by (3), briefly denoted $\mathrm{DDM}(\hat{u}, \hat{\lambda})$. This does not mean that $\mathrm{DDM}(\hat{u}, \hat{\lambda})$ is the true model, but rather that it is the only DDM candidate. Testing the latter hypothesis is now simple: it amounts to verify whether the observed distributions $F_{\mathrm{DT}^o_{a,b}}$ can be generated by $\mathrm{DDM}(\hat{u}, \hat{\lambda})$, and the theoretical distributions $F_{\mathrm{DT}_{a,b}}$ are known in closed form (see the appendix).

Finally, Theorem 1 shows that utility differences are cardinally measured jointly by choice probabilities and decision times. In this regard, the next proposition shows that either of the two observables is sufficient to ordinally measure such differences. In reading it, recall that $\mathrm{DT}_{a,b}$ stochastically dominates $\mathrm{DT}_{a',b'}$ if and only if

$$\mathbb{P}\left[\mathrm{DT}_{a,b} > t\right] \geq \mathbb{P}\left[\mathrm{DT}_{a',b'} > t\right] \qquad \text{for all } t > 0$$

That is, at each time t, ongoing deliberation between alternatives a and b is more likely than ongoing deliberation between a' and b'. In other words, choice between a and b is more time consuming than choice between a' and b'.

Proposition 2 *Given a function $u : A \to \mathbb{R}$ and two coefficients $\sigma > 0$ and $\lambda > 0$, if $a \neq b$ and $a' \neq b'$ belong to A, then the following conditions are equivalent:*

(i) $\left|u(a) - u(b)\right| \leq \left|u(a') - u(b')\right|$*;*

(ii) $\mathrm{ER}_{a,b} \geq \mathrm{ER}_{a',b'}$*;*

(iii) $\overline{\mathrm{DT}}_{a,b} \geq \overline{\mathrm{DT}}_{a',b'}$*;*

(iv) $\mathrm{DT}_{a,b}$ *stochastically dominates* $\mathrm{DT}_{a',b'}$*.*

The mathematical novelty of this proposition is the equivalence of point (iv) with the remaining points (i), (ii), and (iii). The equivalence of the first three points highlights a significant feature of the value-based DDM: two pairs of alternatives present the same absolute difference in intensity of stimuli if, and only if, they generate the same discrimination error if, and only if, their discrimination time is on average the same. This means that, under the DDM assumptions, the measurement of these differences either by error rates à la Fechner – see, e.g., Luce [36, Ch. 2] and Falmagne [23, Ch. 4] – or by decision times à la Cattel [12] actually coincide. In this way, two of the historically most important hypotheses of classical psychophysics are reconciled.

At the same time, it is easy to see that this result is peculiar to the DDM with symmetric barriers $-\lambda$ and λ. As soon as asymmetric barriers $-\beta$ and λ are considered, the only surviving relation is the trivial: (iv) implies (iii).[16] Another specific feature of the symmetric DDM, with important algorithmic consequences (see Drugowitsch [19]), is the following.

[15] Even when they have different thresholds $\hat{\lambda}_1$ and $\hat{\lambda}_2$, but provided choice between lotteries is observed.

[16] See Echenique and Saito [20] and Alos-Ferrer et al. [2] for a general revealed-preference approach to ordinal measurement of utility differences through response times.

Proposition 3 *Given a function $u : A \to \mathbb{R}$ and two coefficients $\sigma > 0$ and $\lambda > 0$, if $a \neq b$ belong to A, then $\mathrm{DT}_{a,b}$ and $\mathrm{DO}_{a,b}$ are independent random variables.*

3 Robustness of the DDM

Even if data were actually generated by a DDM, the observables $\left(P^o, \overline{\mathrm{DT}}^o\right)$ of the previous section would perfectly match the theoretical DDM predictions $\left(P, \overline{\mathrm{DT}}\right)$ – that is, they would satisfy Axioms 1 to 3 – if and only if empirical frequencies matched limit frequencies. Thus, Theorem 1 describes what we would see if we had abundant and very accurate data. In this section, we investigate what happens when the product rule (Axiom 2) and invariance (Axiom 3) are violated, either because of the quantity/quality of available data or because of model misspecification. Positivity is assumed to hold by restricting the analysis to pairs of alternatives for which choice is not deterministic.[17]

We first investigate violations of the product rule. To simplify the form of the robustness bounds, it is convenient to consider log-odds

$$\ell^o_{a,b} \equiv \log \frac{P^o_{a,b}}{P^o_{b,a}} = \operatorname{logit} P^o_{a,b} \qquad \text{for all } a \neq b \text{ in } A$$

instead of choice frequencies, so that the product rule can be written as

$$\ell^o_{a,b} = \ell^o_{a,c} + \ell^o_{c,b} \tag{4}$$

for all distinct a, b, and c in A.[18] In this perspective, a violation of the product rule amounts to the strict positivity of

$$\max_{a \neq b \neq c} \left| \ell^o_{a,b} - \ell^o_{a,c} - \ell^o_{c,b} \right| = \varepsilon \tag{5}$$

In the next lemma we show that, if the observable choice matrix P^o violates the product rule by ε, there exists an approximating choice matrix $\widetilde{P}$ that satisfies the product rule and is $\varepsilon/4$ close to P^o in the supnorm.

Lemma 4 *Let the observable P^o satisfy positivity. If the product rule is violated by ε in the sense of equation (5), then the choice matrix $\widetilde{P}$ defined by*

$$\widetilde{P}_{a,b} \equiv \frac{1}{1 + e^{-\widetilde{\ell}_{a,b}}} \qquad \text{for all } a \neq b \text{ in } A$$

where $\widetilde{\ell}_{a,b} \equiv |A|^{-1} \sum_{c \in A} \left(\ell^o_{a,c} + \ell^o_{c,b}\right)$, satisfies positivity, the product rule, and is such that

$$\left| P^o_{a,b} - \widetilde{P}_{a,b} \right| \leq \varepsilon/4$$

for all $a \neq b$ in A.

[17] As pointed out by Busemeyer and Townsend [10], extensive evidence shows that individuals often make different choices when confronted with the same set of options repeatedly within the same experimental session and without any outcome feedback. For recent accounts of this evidence, we refer to Agranov and Ortoleva [1], Fudenberg et al. [26], and Alos-Ferrer et al. [2]. For deliberate randomization, see Cerreia-Vioglio et al. [13].
To ease terminology, we say that matrix $\widetilde{P} = \left[\widetilde{P}_{a,b}\right]_{a,b \in A}$ satisfies *positivity* if $\widetilde{P}_{a,b} > 0$ for all $a \neq b$ in A.

[18] The convention $\ell^o_{a,a} = 0$ for all a in A will be adopted, when needed.

We can now define ease of comparison, $\widetilde{\mathrm{EC}}$, and accuracy of comparison, $\widetilde{\mathrm{AC}}$, for the approximating $\widetilde{P}$ as we did for the observable P^o and for the theoretical P. The next lemma helps us to separate violations of the product rule and violations of invariance.

Lemma 5 *Let the observables* $\left(P^o, \overline{\mathrm{DT}}^o\right)$ *satisfy positivity and the product rule. Then, invariance is satisfied if and only if*

$$\overline{\mathrm{DT}}^o_{a,b} \frac{\widetilde{\mathrm{EC}}_{a,b}}{\widetilde{\mathrm{AC}}_{a,b}} = \overline{\mathrm{DT}}^o_{a,c} \frac{\widetilde{\mathrm{EC}}_{a,c}}{\widetilde{\mathrm{AC}}_{a,c}} \tag{6}$$

for all distinct a, b, *and* c *in* A.

Property (6) is then called *quasi-invariance.* In particular, in Theorem 1 we can replace invariance with quasi-invariance. Therefore, a violation of invariance which is not related to a violation of the product rule amounts to strict positivity of

$$\max_{\substack{a \neq b \\ c \neq d}} \left| \overline{\mathrm{DT}}^o_{a,b} \frac{\widetilde{\mathrm{EC}}_{a,b}}{\widetilde{\mathrm{AC}}_{a,b}} - \overline{\mathrm{DT}}^o_{c,d} \frac{\widetilde{\mathrm{EC}}_{c,d}}{\widetilde{\mathrm{AC}}_{c,d}} \right| = \delta \tag{7}$$

In reading the following robustness result, recall that both ε and δ are readily computable from the observables because the matrices ℓ^o, $\tilde{\ell}$, $\widetilde{P}$, $\widetilde{\mathrm{EC}}$, and $\widetilde{\mathrm{AC}}$ are all easily derived from P^o.

Theorem 6 *Let the observables* $\left(P^o, \overline{\mathrm{DT}}^o\right)$ *satisfy positivity. If the product rule and invariance are violated by* ε *and* δ *in the sense of equations (5) and (7), then there exist a function* $u : A \to \mathbb{R}$ *and two coefficients* $\sigma > 0$ *and* $\lambda > 0$ *such that*

$$\left| P^o_{a,b} - P_{a,b} \right| \leq \varepsilon/4 \qquad \text{and} \qquad \left| \overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b} \right| \leq \delta/4$$

for all $a \neq b$ *in* A. *In this case,* $\tilde{\lambda} = \lambda/\sigma$ *and* $\tilde{u} = u/\sigma$ *can be chosen as follows*

$$\tilde{\lambda} = \sqrt{\frac{1}{|A|\left(|A|-1\right)} \sum_{a \neq b} \frac{\overline{\mathrm{DT}}^o_{a,b}}{2} \frac{\widetilde{\mathrm{EC}}_{a,b}}{\widetilde{\mathrm{AC}}_{a,b}}} \qquad \text{and} \qquad \tilde{u}\left(a\right) - \tilde{u}\left(b\right) = \frac{\operatorname{logit} \widetilde{P}_{a,b}}{\tilde{\lambda}} \tag{8}$$

all $a \neq b$ *in* A. *Moreover,* $\widetilde{P} = P$.

In a nutshell, ε and δ violations of the axioms in the data correspond to $\varepsilon/4$ and $\delta/4$ errors in the approximation of the data themselves by a DDM.

Remark 1 *One may wonder how the result changes if* δ *is replaced by the gross measure of violation of invariance given by*

$$\max_{\substack{a \neq b \\ c \neq d}} \left| \overline{\mathrm{DT}}^o_{a,b} \frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}} - \overline{\mathrm{DT}}^o_{c,d} \frac{\mathrm{EC}^o_{c,d}}{\mathrm{AC}^o_{c,d}} \right| = \gamma \tag{9}$$

where observed ease of comparison and accuracy, EC^o *and* AC^o, *instead of their "product-rule-compliant" versions,* $\widetilde{\mathrm{EC}}$ *and* $\widetilde{\mathrm{AC}}$, *are considered. Proposition 8 in the appendix shows that changes are minor, but formulas become less neat because* γ *is affected by both violations of invariance and of the product rule, while* δ *is unaffected by the latter.*

Summing up, small violations of the axioms correspond to *proportionally* small errors in the DDM description of behavior. Moreover, as it happens in the "exact" case, the approximating parameters can be directly derived from behavioral data – see (16) in the appendix.

4 An application: the Metropolis-DDM algorithm

In this section, we present an application of the previous analysis to multi-alternative choice under time pressure. Here A represents the set of **available alternatives** and an **exogenous time limit** t is imposed on agents. For example, they might have to choose one of the following 9 available snacks in 4 seconds:

Fig. c-1: Menu of nine available snacks.

Our analysis of this problem is based on the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [14]. We explain the algorithm in this section, however we refer the reader to [14] for an in-depth discussion of the algorithm and of its relations with the literature. The novel contributions of the present section consist, first, in showing how the axioms we introduced so far allow to study multi-alternative choice environments and, second, in generalizing the original Metropolis-DDM algorithm to allow for the formation of consideration sets. For example, our agent might restrict his attention to the subset C of available sweet snacks:

Fig. c-2: Consideration set of six sweet snacks.

Formally, given a set A of available alternatives, a *consideration set* is a subset C of A consisting of the items among which a consumer actually chooses in a given decision episode. These sets are central in marketing,[19] where their formation is assumed to be the first step in a two-step decision process (the second step being the choice of an alternative from the consideration set). For this reason, as Ringel and Skiera [58] write, they are "the ultimate arbiters of the competition" among brand managers, whose objective is to maximize the chances that their products belong to these sets.[20]

[19] See the review of Shocker et al. [64], Roberts and Nedungadi [59] where an issue of the International Journal of Research in Marketing on this topic is forewarded, and the more recent Hauser [30], or Peter and Olson [50] for a textbook treatment.

[20] More recently, consideration sets have also attracted attention in economics. See, e.g., Eliaz and Spiegler [21], Masatlioglu et al. [42], Manzini and Mariotti [41], Gaynor et al. [27], and Caplin et al. [11].

Before describing the Metropolis-DDM algorithm, we recall some eye-tracking experimental findings (in italics) on multi-alternative choice under time pressure that inspired it,[21] along with (in roman) the corresponding "ingredient" of the algorithm itself.

F1 *Multi-alternative choice procedures are composed primarily of sequential pairwise comparisons, in which actual evaluative processing takes place.*

We describe these pairwise comparisons via the Drift Diffusion Model with utility $u : A \to \mathbb{R}$, threshold λ, and diffusion coefficient 1, briefly denoted $\mathrm{DDM}(u, \lambda)$.

F2 *Increases in time pressure lead to acceleration of information processing, often at the cost of accuracy.*

We allow the threshold λ to depend on the deadline t.

F3 *Search strategies and consideration sets are adapted to time constraints and affected by visual saliency, and agents do not eliminate alternatives after they are rejected in a previous pairwise comparison.*

We describe consideration sets by a partition $\mathcal{C}$ of A; for example, $\mathcal{C} = \{$"sweet snacks", "salty snacks"$\}$. This partition may depend on the time constraint t. Moreover, we denote by $Q(a \mid b)$ the probability of considering a new alternative a for comparison with the temporary solution b. This probability is allowed to vary with t too.

F4 *Agents' exploration of menus is driven by the similarity and proximity of available alternatives, that is, on their perceptual distance.*

This finding suggests a simple parametric form for Q that, although not necessary for our analysis, is intuitive and performs well in simulations:

$$Q(a \mid b) = \begin{cases} \dfrac{1}{|A|-1} \dfrac{1}{d(a,b)^{\rho}} & \text{if } a \neq b \\ 1 - \sum_{c \neq b} \dfrac{1}{|A|-1} \dfrac{1}{d(c,b)^{\rho}} & \text{if } a = b \end{cases}$$

Here d is a perceptual distance (that is, a symmetric function) between alternatives such that $\min_{a \neq b} d(a,b) = 1$ and $\max_{a \neq b} d(a,b) \leq \infty$; it captures both physical proximity and subjective similarity. The exponent $\rho \in (0, \infty)$ is an exploration aversion parameter. When ρ is very large, the agent basically regards as close only the nearest neighbors of the temporary solution b; instead, when ρ is very small, all the considered alternatives are regarded as, essentially, equally distant.

For example, in the case of our 9 snacks, a simple perceptual distance is given by

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \text{ and } b \text{ are adjacent and are either both sweet or both salty} \\ 2 & \text{if } a \text{ and } b \text{ are not adjacent and are either both sweet or both salty} \\ \infty & \text{if one is sweet and the other salty} \end{cases}$$

[21] See Russo and Rosen [62], Russo and Leclerc [61], Nowlis [47], Pieters and Warlop [51], Chandon et al. [15], Krajbich et al. [32], Krajbich and Rangel [33], Reutskaja et al. [57], Milosavljevic et al. [45], and Karsilar et al. [31].

Since this distance takes into account the sweet/salty partition, so does the corresponding transition probability matrix Q. This example also highlights the fact that $d(a,b)$ is necessarily infinite if a belongs to a different consideration set than b.[22]

F5 *The initial fixation is random and independent of value.*[23]

Our final ingredient is thus an initial probability distribution μ on A, which may depend on t as well.

Together, all our ingredients suggest the following decision procedure. When a menu A and a deadline t are given, our agents first select a sub-menu C of A and an initial element b in C according to the consideration sets' partition $\mathcal{C}$ and the initial distribution μ. Then, they consider an alternative solution a in C with probability $Q(a \mid b)$, and compare it to b via $\mathrm{DDM}(u,\lambda)$. If proposal a is judged superior to incumbent b, then a becomes the new incumbent and another proposal c in C is considered and compared to a via $\mathrm{DDM}(u,\lambda)$; otherwise, b maintains its incumbent status and another proposal is considered and compared. This sequential exploration and comparison continues until the time t available to decide expires and the incumbent solution is chosen from the consideration set C.

Before formally describing this decision procedure, a couple of remarks are in order. First, it is important to observe that **the axioms of the previous sections**, together with the eye-tracking detection of binary comparisons, make these assumptions testable and their parameters quantifiable. Therefore, it is the analysis of the first part of this paper that makes **empirically relevant** what we propose here (see also Section 4.1 below).

Second, note that the nature of consideration sets that we propose is both set-theoretic and probabilistic. Intuitively, a partition $\mathcal{C}$ of A consists of consideration sets if, once a set $C \in \mathcal{C}$ is selected by the agent, then:

1. any element of C can be considered (with strictly positive probability),

2. no element outside C can be considered.

Now, if the agent explores alternatives according to transition matrix Q, this means that given any C in $\mathcal{C}$ and any item c in C:

1. starting from c it is possible to reach any element inside C in a finite number of transitions,

2. starting from c it is impossible to reach any element outside C in a finite number of transitions.

In sum, the partition $\mathcal{C}$ must coincide with the partition of communicating classes determined by the exploration matrix Q.[24]

[22] See the discussion below on the identification of the consideration sets in $\mathcal{C}$ and the communicating classes of Q.

[23] But possibly dependent on consideration sets and visual saliency.

[24] Specifically, given any $c \in C$, if $a \notin C$, there is no finite sequence $c = a_0, a_1, ..., a_n = a$ such that $\prod_{k=0}^{n-1} Q(a_{k+1} \mid a_k) > 0$, in particular $Q(a \mid c) = 0$ for all $a \notin C$ In contrast, if $a \in C$, such a finite sequence exists, in particular $a_0, a_1, ..., a_n \in C$. See Norris [46, Ch. 1].

We are now ready to present our multi-alternative choice model, a generalization of the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [14]. In reading it, notice that the partition $\mathcal{C}$ of A into consideration sets does not appear explicitly in the pseudo-code: once an initial element b has been drawn according to μ, the algorithm is constrained to run within the communicating class C of b determined by Q, until it terminates. Nonetheless, in light of the discussion above, one should see μ as the composition of its marginal $\mu^{\mathcal{C}}$ on $\mathcal{C}$ and its conditionals $\{\mu_C\}_{C \in \mathcal{C}}$. With this, the selection of the initial element b synthesizes the selection of a consideration set C according to $\mu^{\mathcal{C}}$ and the subsequent selection of b from C according to μ_C.

Metropolis-DDM Algorithm

Input: Given $t > 0$, set $\mu = \mu_t$, $Q = Q_t$, and $\lambda = \lambda_t$.

Start: Draw a from A according to μ:

- set $s_0 = 0$,
- set $b_0 = a$.

Repeat: Draw a from A according to $Q(\cdot \mid b_n)$ and compare it to b_n via $\mathrm{DDM}(u, \lambda)$:

- set $s_{n+1} = s_n + \mathrm{DT}_{a,b_n}$,
- set $b_{n+1} = \mathrm{DO}_{a,b_n}$,

until $s_{n+1} > t$.

Stop: Set $b^* = b_n$.

Output: Choose b^* from A.

This algorithm can be seen as a parsimonious variation of the standard optimal search algorithm that takes into account the presence of time pressure. In the standard algorithm, agents begin by selecting an initial element b in A, then at each iteration they compare an incumbent and a proposal, and discard permanently the rejected alternative, until the menu is exhausted. Here, the presence of a deadline may lead to the formation of consideration sets, and the possibility of mistakes makes it inadvisable to eliminate proposals that have been rejected in a previous comparison. Nonetheless, the sequential "explore-and-compare" logic of the two procedures is similar.

By implementing the Metropolis-DDM algorithm, the probability of selecting a as a new best candidate given the current incumbent b is

$$M_t(a \mid b) = Q_t(a \mid b) P_{a,b} \qquad \text{for all } a \neq b \text{ in } A$$

The transition probability $M_t(a \mid b)$ combines the stochasticity of the proposal mechanism $Q_t(a \mid b)$ and that of the acceptance/rejection rule $P_{a,b}$ (which also depends on t via λ_t). Therefore, after n iterations of the repeat-until loop, the probability of b being the incumbent is the b-th component of the row vector $\mu_t M_t^n$. The next result shows that the limiting behavior of this sequence turns out to be classical softmaximization, conditional on the communicating classes determined by Q.

Theorem 7 *Let $u : A \to \mathbb{R}$ be a function, $\lambda_t > 0$ a coefficient, and Q_t a symmetric stochastic $A \times A$ matrix. Then, M_t is reversible with respect to the multinomial logit distribution*

$$p_A^{(u,\lambda_t)}(a) = \frac{e^{\lambda_t u(a)}}{\sum_{b \in A} e^{\lambda_t u(b)}} \qquad \text{for all } a \text{ in } A$$

and, given any probability distribution μ_t on A,

$$\lim_{n\to\infty} \mu_t M_t^n = \sum_{C \in \mathcal{C}_t} \mu_t(C)\, p_C^{(u,\lambda_t)}$$

where $\mathcal{C}_t$ is the partition of A into its communicating classes with respect to Q_t.[25]

In particular, if Q_t is irreducible, then M_t is irreducible, aperiodic, and

$$\lim_{n\to\infty} \mu_t M_t^n = p_A^{(u,\lambda_t)}$$

If Q_t is irreducible,[26] the Metropolis-DDM algorithm thus approximates the *multinomial logit* (or *softmax*) distribution $p_A^{(u,\lambda_t)}$, irrespective of the initial distribution μ_t. Otherwise, the algorithm selects a consideration sub-menu C of A and approximates the conditional multinomial logit there. Remarkably, this result continues to hold when asymmetric barriers $-\beta_t$ and λ_t are considered: as shown in the appendix, the lower barrier $-\beta_t$ does not affect the stationary distribution.

Beyond the mathematical novelty, the conceptual innovation of the algorithm presented above relative to the original Metropolis-DDM is allowing the exploration strategy – in particular, the consideration sets' structure – to depend on the time constraint. This is potentially relevant in today's marketplaces, in which consumers have immense choice sets, life styles dramatically reduce deliberation times, choice behavior is easily observable (e.g., through cellphone cameras, even at the eye-tracking level).

Last but not least, the parameters u and λ_t of the limit multi-alternative choice distribution appearing in Theorem 7 are those that govern the pairwise comparisons that lead to it, and are thus identified by Theorems 1 and 6. We discuss next their elicitability from eye-tracking data and the implications of this fact in terms of testability of the Metropolis-DDM algorithm.

4.1 Eye-tracking and axioms to test the algorithm

Eye-tracking data come in the form of fixation sequences, say

$$x_0 - x_1 - x_2 - x_3 - \cdots$$

In the eye-tracking practice,[27] refixation subsequences $x - y - x$ (or $x - y - x - y$, ...) are hypothesized to correspond to pairwise comparisons, while the appearance of a new alternative z, leading to $x - y - x - z$ (or $x - y - x - y - z$, ...), relates to exploration. Therefore, by tracking the choice behavior of an agent and detecting refixation subsequences, it is possible to use the axioms of the previous Sections 2 and 3 to check whether his pairwise comparisons are consistent with the DDM and to elicit the corresponding parameters.

Specifically, the refixations between alternatives a and b present the analyst with the quantities $P_{a,b}^o$ and $\overline{\mathrm{DT}}_{a,b}^o$. At this point, Theorems 1 and 6 allow her to establish if the DDM is a plausible

[25] As usual, p_C is the conditional of p_A given C. That is, $p_C(a) = e^{\lambda_t u(a)} / \sum_{c \in C} e^{\lambda_t u(c)}$ if $a \in C$, and $p_C(a) = 0$ otherwise.

[26] This is the case considered by Cerreia-Vioglio et al. [14].

[27] See the review of Orquin and Mueller Loose [49].

description of the agent's binary choice behavior. If this is the case, the observables $\left(P^o, \overline{\mathrm{DT}}^o\right)$ also identify, by (3) and (8), candidate utility and threshold for binary choice. These utility and threshold can then be compared with the ones that better describe the softmax approximation of the agent's choice distribution from A (also observable) and to test the validity of the Metropolis-DDM algorithm as a description of the choice process.

5 Approximation errors and simulations

Theorem 7 shows that the stationary distribution of the sequence of incumbents generated by the Metropolis-DDM algorithm within each consideration set C is the softmax $p_C = p_C^{(u,\lambda_t)}$. At the same time, the algorithm is stopped at time t after a number of iterations which is random because of the stochastic duration $\mathrm{DT}_{a,b}$ of each iteration (in which b is the incumbent and a is the proposal). Moreover, different iterations have different average duration, as described by (2) and Proposition 2. As a consequence, the softmax approximation of the output of the algorithm is affected by a systematic bias due to the fact that some comparisons take longer than others and, therefore, some alternatives are more likely than others to be the incumbent at any given time.

Heuristically, as the number of iterations increases, the fraction of clock-time in which b is the incumbent and a is the proposal is directly proportional to:

- the probability of b being the incumbent and a the proposal, which is approximately $p_C(b)\, Q_t(a \mid b)$,
- the average clock-time it takes to compare a and b, which is $\overline{\mathrm{DT}}_{a,b}$.

Thus, the long run bias in the softmax approximation is given by

$$\underbrace{p_C(b)}_{\text{softmax}} - \underbrace{\frac{\sum_{a\in C\setminus\{b\}} p_C(b)\, Q_t(a \mid b)\, \overline{\mathrm{DT}}_{a,b}}{\sum_{c\in C}\left(\sum_{a\in C\setminus\{c\}} p_C(c)\, Q_t(a \mid c)\, \overline{\mathrm{DT}}_{a,c}\right)}}_{\text{time-adjusted softmax}}$$

The simulations below show that this bias is in general small, yet present, and that it vanishes as the deadline t becomes less stringent and the evidence threshold λ_t more demanding.[28]

To illustrate, we use the parameters of the DDM elicited by Milosavljevic et al. [44]. In particular, they assume $\sigma = \sqrt{2}/20$, consider 5 equally spaced utility levels and calibrate their values and thresholds under high and low time pressure by studying data from 750 choices made by each subject. They find that utility values are unaffected by time pressure, while thresholds are affected. After renormalizing their data so that $\hat{\sigma} = 1$ (see Section 1.1), we obtain

- a range for $\hat{u} = u/\sigma$ of $[0, 7.071]$,
- values of $\hat{\lambda} = \lambda/\sigma$ of 0.849 and 1.442 under high and low time pressure, respectively.

Next we simulate the behavior of the Metropolis-DDM algorithm based on these parameters with 5 alternatives, corresponding to the 5 equally spaced utility values $0,\ 1.768,\ 3.535,\ 5.303,\ 7.071$.

[28] This informal argument can be made precise for a broader class of algorithms: the technical details on time-correction of stationary distributions of stopped algorithms are formally studied in Baldassi et al. [5].

The first four plots describe the following combinations of thresholds and time pressure levels

		THRESHOLD	
		Small	Large
TIME PRESSURE	High	$t = 4 \quad \hat{\lambda} = 0.849$	$t = 4 \quad \hat{\lambda} = 1.442$
	Low	$t = 12 \quad \hat{\lambda} = 0.849$	$t = 12 \quad \hat{\lambda} = 1.442$

The cases of high time pressure with large threshold ($t = 4$ and $\hat{\lambda} = 1.442$) and low time pressure with small threshold ($t = 12$ and $\hat{\lambda} = 0.849$) are not motivated by the evidence of Milosavljevic et al. [44]. They are computational experiments about an agent who is reflective (no matter the time constraint) and another one who is, instead, impulsive (again, no matter the time constraint). Our simulations show that the Metropolis-DDM algorithm (orange curve) almost perfectly converges to the time-adjusted softmax (green curve) and well approximates softmax (blue curve), except when time pressure is high and the threshold is large.[29] The empirical distribution generated by the Metropolis-DDM algorithm is obtained by running it $10,000$ times for each time pressure/threshold pair.[30]

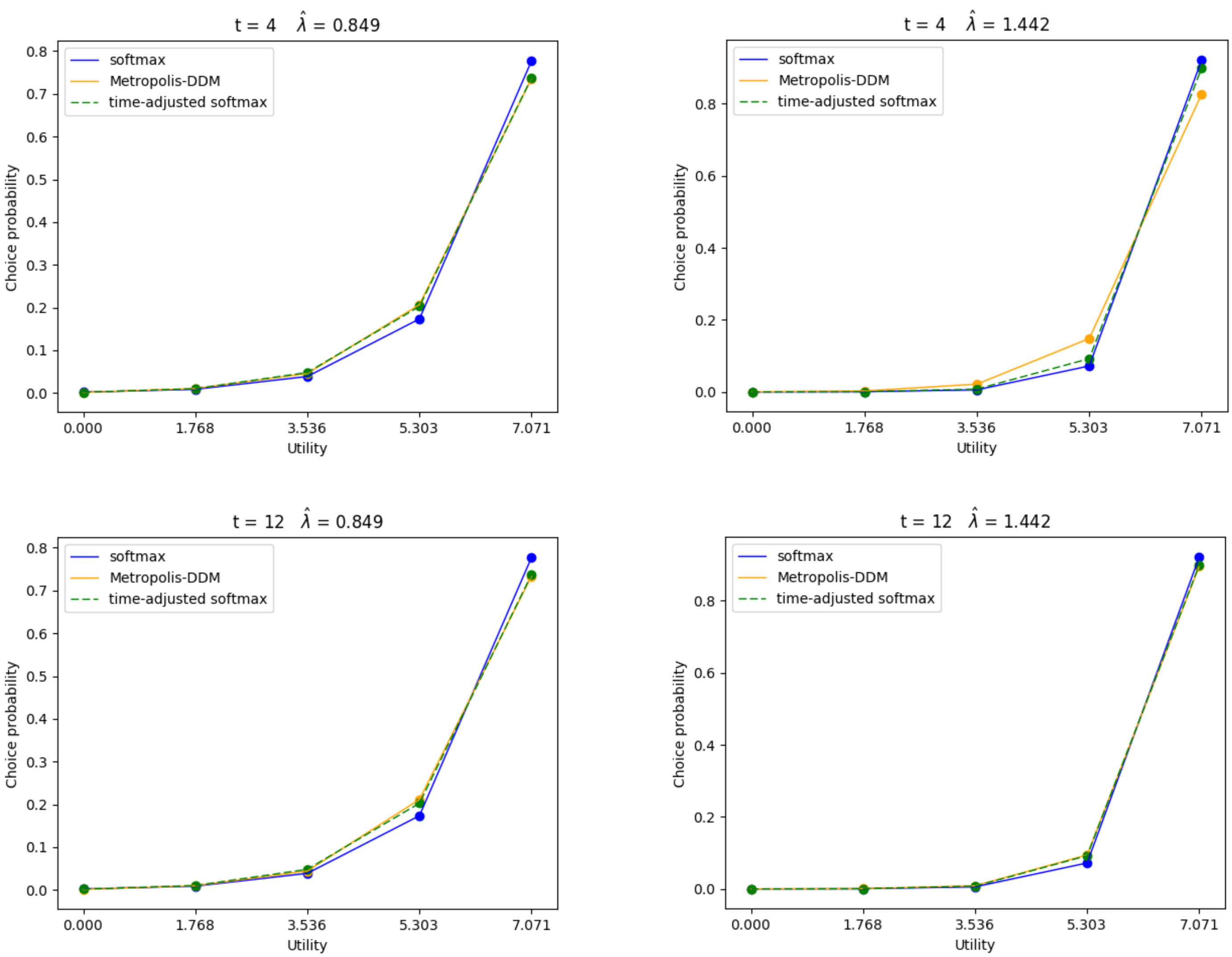

Fig. d: Simulations of Metropolis-DDM for different parameters' settings.

[29] In this case, binary comparisons are too slow for the algorithm to converge, but note that the difference between softmax and time-adjusted softmax remains small.

[30] The code we used is available at https://github.com/carlobaldassi/MetropolisDDM_python.

Our final set of simulations shows that numerical convergence is not seriously affected by the presence of asymmetric barriers. In the plots below, the Metropolis-DDM algorithm is simulated in the cases of high time pressure with small acceptance threshold (left) and low time pressure with large acceptance threshold (right). Varying the rejection threshold β both above and below λ has little-to-no effect on the empirical distribution generated by $10,000$ runs of the algorithm.

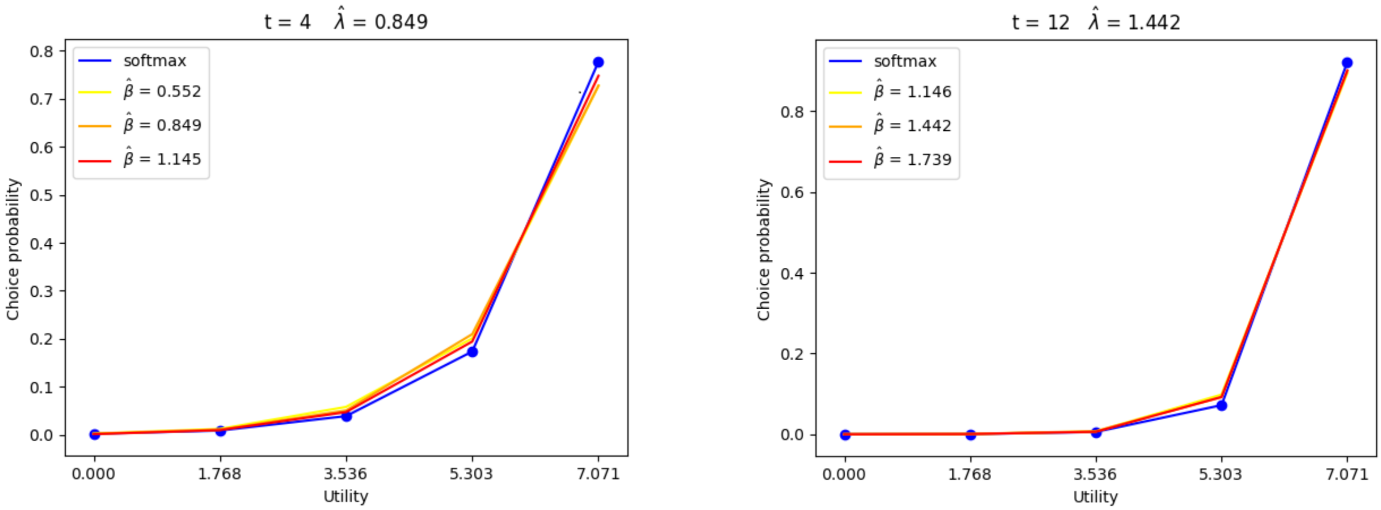

Fig. e: Simulations of Metropolis-DDM with asymmetric barriers.

The standard error bars for each of the simulations in this section turned out to be thinner than the plotted curves themselves, and are therefore omitted.

To sum up, using physiologically calibrated values of the parameters, these simulations show that the algorithm converges to the time-adjusted softmax, relatively fast and irrespective of the symmetry of barriers. Moreover, the latter adjustment is very close to classical softmax.

6 Conclusions

This article is a bridge between decision theory and computational neuroscience, with relevant consumer choice implications.

In the first part of the paper we provide an axiomatic foundation for the value-based Drift Diffusion Model of binary choice and we show how it reconciles the principles of psychophysical discrimination of Fechner and Cattel.

In a nutshell, the behavioral substrate of the DDM consists of three requirements.

Positivity: Choice is a time consuming and noisy process.

Product rule: Violations of transitivity are due to noise.

Invariance: Accuracy is directly proportional to mean decision time and ease of comparison.

Our representation allows to elicit the utilities of alternatives and the other DDM parameters from data, instead of postulating utilities and fitting the remaining parameters of the DDM itself. We also show that such elicitation is robust to the quality of data.

In the second part of the paper we show how our axiomatization of the DDM allows to test an extended version of the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [14] that allows for the formation of consideration sets, and to elicit its parameters.

As remarked by the very recent Stine et al. [65], one of the challenges that the field of cognitive neuroscience faces is the identification of a subject's decision-making strategy from behavioral observations alone. This paper is part of a project aimed at tackling this challenge. The DDM we characterized here is the computational workhorse of the neuroscience of decision making (see Ratcliff et al. [55] and Stine et al. [65]). We are currently exploring the extension of the present analysis to alternative models.

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A Proofs and related material

A.1 Observability and measurement

Proof of Theorem 1 (i) implies (ii). Define the observed odds for a against b as

$$R^o_{a,b} \equiv \frac{P^o_{a,b}}{P^o_{b,a}}$$

for all $a \neq b$ in A. By positivity and the definition of observables, we have that $R^o_{a,b} > 0$, for all $a \neq b$ in A. Arbitrarily choose $c \in A$, set $v(c) \equiv 0$ and

$$v(a) \equiv \log R^o_{a,c} \tag{10}$$

for all $a \neq c$ in A. Then, for all $a \neq b$ in $A \setminus \{c\}$, by the product rule, we have

$$R^o_{a,b} = R^o_{a,c} R^o_{c,b} = \frac{R^o_{a,c}}{R^o_{b,c}} = e^{v(a)-v(b)}$$

and direct application of (10) delivers the same result for $a = c \neq b$ and for $b = c \neq a$. Then $P^o_{a,b} = 1/\left(1 + \left(R^o_{a,b}\right)^{-1}\right)$ implies

$$P^o_{a,b} = \frac{1}{1 + e^{-[v(a)-v(b)]}}$$

for all $a \neq b$ in A.

Tedious verification shows that invariance guarantees that

$$\overline{\mathrm{DT}}^o_{a,b} \frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{\mathrm{EC}^o_{a',b'}}{\mathrm{AC}^o_{a',b'}} \tag{11}$$

for all $a \neq b$ and all $a' \neq b'$ in A. [31]

[31] And not only if $a' = a$ as the axiom requires.

Now arbitrarily choosing $a' \neq b'$ in A, and setting

$$\lambda^2 \equiv \overline{\mathrm{DT}}^o_{a',b'} \frac{\mathrm{EC}^o_{a',b'}}{2\mathrm{AC}^o_{a',b'}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{\log\left(\frac{1-\frac{1}{1+e^{|v(a')-v(b')|}}}{\frac{1}{1+e^{|v(a')-v(b')|}}}\right)}{1-\frac{2}{1+e^{|v(a')-v(b')|}}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{e^{|v(a')-v(b')|}+1}{e^{|v(a')-v(b')|}-1}|v(a')-v(b')| \quad (12)$$

and $u(a) \equiv v(a)/\lambda$ for all a in A, it follows that, for all $a \neq b$ in A,

$$P^o_{a,b} = \frac{1}{1+e^{-[v(a)-v(b)]}} = \frac{1}{1+e^{-\lambda[u(a)-u(b)]}} = P_{a,b}$$

and

$$\begin{aligned}
\overline{\mathrm{DT}}^o_{a,b} &= \lambda^2 \frac{e^{|v(a)-v(b)|}-1}{e^{|v(a)-v(b)|}+1} \frac{1}{|v(a)-v(b)|} = \lambda^2 \frac{e^{\lambda|u(a)-u(b)|}-1}{e^{\lambda|u(a)-u(b)|}+1} \frac{1}{\lambda|u(a)-u(b)|} \\
&= \frac{\lambda}{|u(a)-u(b)|} \frac{e^{\lambda|u(a)-u(b)|}-1}{e^{\lambda|u(a)-u(b)|}+1} = \frac{\lambda}{|u(a)-u(b)|} \tanh\left(\frac{\lambda|u(a)-u(b)|}{2}\right) \\
&= \frac{\lambda}{u(a)-u(b)} \tanh\left(\frac{\lambda[u(a)-u(b)]}{2}\right) = \overline{\mathrm{DT}}_{a,b}
\end{aligned}$$

where the first equality is a consequence of (11) and (12). Thus (ii) holds for the DDM with parameters u, $\sigma = 1$, and λ.

Verifying that (ii) implies (i) is simple and so omitted for brevity.

Finally, if (ii) holds, by (1), we have that, for all $a \neq b$ in A,

$$\begin{aligned}
\operatorname{logit} P^o_{a,b} &= \operatorname{logit} P_{a,b} = \log \frac{P_{a,b}}{P_{b,a}} = \log \frac{\frac{1}{1+e^{-\frac{\lambda}{\sigma^2}[u(a)-u(b)]}}}{\frac{1}{1+e^{-\frac{\lambda}{\sigma^2}[u(b)-u(a)]}}} \\
&= \frac{\lambda}{\sigma^2}[u(a)-u(b)] = \frac{\lambda}{\sigma}\left[\frac{u}{\sigma}(a) - \frac{u}{\sigma}(b)\right]
\end{aligned}$$

and the second part of (3) follows; the first part is a consequence of

$$\begin{aligned}
\overline{\mathrm{DT}}^o_{a,b} &= \overline{\mathrm{DT}}_{a,b} = \frac{\lambda}{u(a)-u(b)} \tanh\left(\frac{\lambda[u(a)-u(b)]}{2\sigma^2}\right) = \frac{\lambda}{|u(a)-u(b)|} \tanh\left(\frac{\lambda|u(a)-u(b)|}{2\sigma^2}\right) \\
&= \frac{\hat\lambda}{|\hat u(a)-\hat u(b)|} \tanh\left(\frac{\hat\lambda|\hat u(a)-\hat u(b)|}{2}\right) = \hat\lambda^2 \frac{1-2\frac{1}{1+e^{\hat\lambda|\hat u(a)-\hat u(b)|}}}{\hat\lambda|\hat u(a)-\hat u(b)|} = \hat\lambda^2 \frac{2\mathrm{AC}^o_{a,b}}{\mathrm{EC}^o_{a,b}}
\end{aligned}$$

as wanted. ■

Proof of Propositions 2 and 3 For notational convenience, we begin with the latter. Arbitrarily choose $u \in \mathbb{R}^A$, $\lambda > 0$ ($\sigma = 1$), and $a, b \in A$. Let $\Delta = u(a) - u(b)$. Since we will repeatedly use the results of the Handbook of Brownian Motion of Borodin and Salminen [9], henceforth HBM, we adopt their notation. Specifically, setting $\mu = \Delta/\sqrt{2}$ and $z = \lambda/\sqrt{2}$,

$$\frac{Z_{a,b}(s)}{\sqrt{2}} = \mu s + W(s) \overset{\mathrm{HBM}}{=} W^{(\mu)}_s$$

$$\mathrm{DT}_{a,b} = \min\{s : |Z_{a,b}(s)| = \lambda\} = \min\left\{s : \left|W^{(\mu)}_s\right| = z\right\} \overset{\mathrm{HBM}}{=} H_{-z,z}$$

$$\mathrm{DO}_{a,b} = \begin{cases} a & \text{if } \dfrac{Z_{a,b}(\mathrm{DT}_{a,b})}{\sqrt{2}} = \dfrac{\lambda}{\sqrt{2}} \\ b & \text{if } \dfrac{Z_{a,b}(\mathrm{DT}_{a,b})}{\sqrt{2}} = -\dfrac{\lambda}{\sqrt{2}} \end{cases} = \begin{cases} a & \text{if } W^{(\mu)}_{H_{-z,z}} = z \\ b & \text{if } W^{(\mu)}_{H_{-z,z}} = -z \end{cases}$$

With this, their Equation 3.0.2 (p. 233) shows that

$$\mathbb{P}\left[H_{-z,z} \in dt\right] = e^{-\frac{\mu^2 t}{2}} \left(e^{-\mu z} + e^{\mu z}\right) \mathrm{ss}_{z,2z}(t)\, dt \tag{13}$$

where $\mathrm{ss}_{z,2z}(t)$ is defined on p. 451 of HBM. Their Equation 3.0.4(b) (p. 233) yields

$$\mathbb{P}\left[W^{(\mu)}_{H_{-z,z}} = z\right] = \frac{e^{\mu z}}{e^{-\mu z} + e^{\mu z}}$$

while Equation 3.0.6(b) (p. 233) gives

$$\mathbb{P}\left[H_{-z,z} \in dt, W^{(\mu)}_{H_{-z,z}} = z\right] = e^{\mu z} e^{-\frac{\mu^2 t}{2}} \mathrm{ss}_{z,2z}(t)\, dt = \mathbb{P}\left[H_{-z,z} \in dt\right] \mathbb{P}\left[W^{(\mu)}_{H_{-z,z}} = z\right]$$

This proves that $\mathrm{DT}_{a,b}$ and $\mathrm{DO}_{a,b}$ are independent random variables, because $\mathrm{DT}_{a,b} = H_{-z,z}$ and $\mathrm{DO}_{a,b}$ only depends on whether $W^{(\mu)}_{H_{-z,z}} = z$ or $W^{(\mu)}_{H_{-z,z}} = -z$.

As to the equivalence between points (i)-(iv) of Proposition 2, by (13), the density of $\mathrm{DT}_{a,b}$ is

$$f_{\mathrm{DT}_{a,b}}(t) = \frac{\lambda e^{-\frac{\Delta^2 t}{4}}}{\sqrt{\pi} t^{3/2}} \cosh\left(\frac{\lambda \Delta}{2}\right) \sum_{k=-\infty}^{\infty} (1+4k)\, e^{-\frac{\lambda^2}{4t}(1+4k)^2} \qquad \forall t \in (0,\infty) \tag{14}$$

but, for all $q \in (0,1)$, $\sum_{k=-\infty}^{\infty} (1+4k)\, q^{\frac{1}{4}(1+4k)^2} = \sqrt[4]{q} \sum_{n=0}^{\infty} (-)^n (2n+1)\, q^{n(n+1)} = \vartheta_1'(0,q)/2$ where ϑ_1 is the first Jacobi theta function. Thus setting $y = |\Delta|$, we have

$$f_{\mathrm{DT}_{a,b}}(t) = f(t,y) = e^{-\frac{y^2 t}{4}} \cosh\left(\frac{\lambda y}{2}\right) \frac{1}{2\sqrt{\pi}} \frac{\lambda}{t^{3/2}} \vartheta_1'\left(0, e^{-\frac{\lambda^2}{t}}\right) \qquad \forall t \in (0,\infty)$$

which is continuous and bounded, as a function of (t,y), on every rectangle $T_x \times Y = (0,x) \times [0, \max_A u - \min_A u]$ with $x \in (0,\infty)$.

Now, for each (fixed) $x \in (0,\infty)$, the distribution function of $\mathrm{DT}_{a,b}$ is

$$F_{\mathrm{DT}_{a,b}}(x) = F(x,y) = \int_0^x f(t,y)\, dt$$

and it is continuous on Y because $f(t,y)$ is continuous and bounded on $T_x \times Y$. Moreover,

$$\frac{\partial f}{\partial y}(t,y) = \frac{y}{2}\left(\frac{\lambda}{y} \tanh\left(\frac{\lambda y}{2}\right) - t\right) f(t,y) \qquad \forall (t,y) \in T_x \times \operatorname{int}(Y)$$

is continuous and bounded too.

Differentiation under the integral sign is then possible, and it shows that, for all $y \in \operatorname{int}(Y)$,

$$\frac{\partial F}{\partial y}(x,y) = \int_0^x \frac{\partial f}{\partial y}(t,y)\, dt = \frac{y}{2} \int_0^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt$$

For $x < \overline{\mathrm{DT}}_{a,b}$ the integrand is positive, and so is $\partial F / \partial y$. While, for $x \geq \overline{\mathrm{DT}}_{a,b}$

$$\begin{aligned}
\int_0^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt &= \int_0^{\overline{\mathrm{DT}}_{a,b}} \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt + \int_{\overline{\mathrm{DT}}_{a,b}}^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt \\
&\geq \int_0^{\overline{\mathrm{DT}}_{a,b}} \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt + \int_{\overline{\mathrm{DT}}_{a,b}}^\infty \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt = 0
\end{aligned}$$

where, in the second line, inequality holds because the integrand of the second summand is negative and the final equality holds because $\int_0^\infty \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\, dt = \mathbb{E}\left[\overline{\mathrm{DT}}_{a,b} - \mathrm{DT}_{a,b}\right]$, and again $\partial F/\partial y$ is positive. Summing up, for each (fixed) $x \in (0,\infty)$, $F(x,y)$ is continuous on $[0, \max_A u - \min_A u]$ and differentiable on $(0, \max_A u - \min_A u)$ with respect to y, and positivity of the derivative yields monotonicity (for fixed x, with respect to $y = |u(a) - u(b)|$).

But this shows that if $|u(a) - u(b)| \leq |u(a') - u(b')|$, then $F_{\mathrm{DT}_{a,b}}(x) \leq F_{\mathrm{DT}_{a',b'}}(x)$ for all $x \in (0,\infty)$, that is, $\mathrm{DT}_{a,b}$ stochastically dominates $\mathrm{DT}_{a',b'}$.

Then (i) implies (iv). On the other hand, if $\mathrm{DT}_{a,b}$ stochastically dominates $\mathrm{DT}_{a',b'}$, then obviously $\overline{\mathrm{DT}}_{a,b} \geq \overline{\mathrm{DT}}_{a',b'}$, so that (iv) implies (iii). Moreover, $\overline{\mathrm{DT}}_{a,b} \geq \overline{\mathrm{DT}}_{a',b'}$ implies

$$\frac{\lambda}{|\Delta|}\tanh\left(\frac{\lambda|\Delta|}{2}\right) = \frac{\lambda}{\Delta}\tanh\left(\frac{\lambda\Delta}{2}\right) \geq \frac{\lambda}{\Delta'}\tanh\left(\frac{\lambda\Delta'}{2}\right) = \frac{\lambda}{|\Delta'|}\tanh\left(\frac{\lambda|\Delta'|}{2}\right)$$

whence $|\Delta| \leq |\Delta'|$ because $(\lambda/y)\tanh(\lambda y/2)$ is strictly decreasing, for fixed $\lambda > 0$, and $y \in [0,\infty)$; but – in turn – this implies

$$\mathrm{ER}_{a,b} = \frac{1}{1+e^{\lambda|\Delta|}} \geq \frac{1}{1+e^{\lambda|\Delta'|}} = \mathrm{ER}_{a',b'}$$

and (iii) implies (ii). Finally, (ii) implies (i) because

$$\frac{1}{1+e^{\lambda|\Delta|}} = \mathrm{ER}_{a,b} \geq \mathrm{ER}_{a',b'} = \frac{1}{1+e^{\lambda|\Delta'|}} \implies |\Delta| \leq |\Delta'|$$

As wanted. ∎

A.2 Robustness of the DDM

Proof of Lemma 4 Recall the convention $\ell^o_{a,a} = 0$ for all $a \in A$. Set

$$v(a) = \frac{1}{|A|}\sum_{c\in A}\ell^o_{ac}$$

for all $a \in A$, and

$$\varepsilon_{ab} = \frac{1}{|A|}\sum_{d\in A}\left(\ell^o_{ab} - \ell^o_{ad} - \ell^o_{db}\right)$$

for all $a \neq b$ in A. Notice that the summands corresponding to $d = a$ and $d = b$ are zero, therefore

$$|\varepsilon_{ab}| = \frac{1}{|A|}\left|\sum_{d\in A}\left(\ell^o_{ab} - \ell^o_{ad} - \ell^o_{db}\right)\right| \leq \frac{|A|-2}{|A|}\varepsilon$$

Moreover

$$\begin{aligned}
(v(a) - v(b)) + \varepsilon_{ab} &= \frac{1}{|A|}\sum_{d\in A}\ell^o_{ad} - \frac{1}{|A|}\sum_{d\in A}\ell^o_{bd} + \frac{1}{|A|}\sum_{d\in A}\left(\ell^o_{ab} - \ell^o_{ad} - \ell^o_{db}\right) \\
(\text{since } \ell^o_{bd} = -\ell^o_{db}) &= \frac{1}{|A|}\sum_{d\in A}\ell^o_{ad} + \frac{1}{|A|}\sum_{d\in A}\ell^o_{db} + \frac{1}{|A|}\sum_{d\in A}\left(\ell^o_{ab} - \ell^o_{ad} - \ell^o_{db}\right) = \ell^o_{ab}
\end{aligned}$$

whence

$$P^o_{a,b} = \frac{1}{1+e^{-\ell^o_{ab}}} = \frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}}$$

Notice that $\widetilde{P}$ in the statement is defined by

$$\widetilde{P}_{a,b} = \frac{1}{1+e^{-|A|^{-1}\sum_{c\in A}\left(\ell^o_{a,c}+\ell^o_{c,b}\right)}} = \frac{1}{1+e^{-(v(a)-v(b))}}$$

thus

$$\left|P^o_{a,b} - \widetilde{P}_{a,b}\right| = \left|\frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}} - \frac{1}{1+e^{-(v(a)-v(b))}}\right|$$

for all $a \neq b$ in A. But the derivative of the logistic function $\left(1+e^{-x}\right)^{-1}$ is always positive and has maximum value 1/4 (at 0). By the Mean Value Theorem,

$$\left|\frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}} - \frac{1}{1+e^{-(v(a)-v(b))}}\right| \leq \frac{1}{4}\left|\varepsilon_{ab}\right| \leq \frac{1}{4}\varepsilon$$

for all $a \neq b$ in A. The rest is trivial. ■

Remark 2 *Lemma 4 shows that, if P^o is a choice matrix that satisfies positivity, and the product rule is violated by ε in the sense of equation (5), then setting*

$$v\left(a\right) = \frac{1}{|A|}\sum_{c \in A} \ell^o_{ac}$$

for all $a \in A$, it follows that

$$\left|P^o_{a,b} - \frac{e^{v(a)}}{e^{v(a)}+e^{v(b)}}\right| \leq \varepsilon/4$$

for all $a \neq b$ in A. This can be seen as a robust version of Theorem 48 of Luce and Suppes [38, p. 350].

Proof of Lemma 5 If the observables $\left(P^o, \overline{\mathrm{DT}}^o\right)$ satisfy positivity and the product rule. Then

$$\widetilde{\ell}_{a,b} = |A|^{-1}\sum_{c \in A}\left(\ell^o_{a,c}+\ell^o_{c,b}\right) = |A|^{-1}\sum_{c \in A}\ell^o_{a,b} = \ell^o_{a,b}$$

therefore

$$\widetilde{P}_{a,b} = \frac{1}{1+e^{-\ell^o_{a,b}}} = P^o_{a,b} \qquad \forall a,b \in A$$

and, a fortiori, $\widetilde{\mathrm{EC}}_{a,b} = \mathrm{EC}^o_{a,b}$ and $\widetilde{\mathrm{AC}}_{a,b} = \mathrm{AC}^o_{a,b}$, for all $a \neq b$ in A. Showing equivalence of quasi-invariance with invariance. ■

Proof of Theorem 6 Choose v and $\widetilde{P}$ like in the proof of Lemma 4. Consider the "fictitious" observables $\left(\widetilde{P}, \overline{\mathrm{DT}}^o\right)$ and notice that, by Lemma 4, $\left(\widetilde{P}, \overline{\mathrm{DT}}^o\right)$ satisfy positivity and the product rule, but they fail invariance by δ. In fact, quasi-invariance would require

$$k\left(a,b\right) = \overline{\mathrm{DT}}^o_{a,b}\frac{\widetilde{\mathrm{EC}}_{a,b}}{\widetilde{\mathrm{AC}}_{a,b}} \qquad \forall\left(a,b\right) \in A^2 : a \neq b \tag{15}$$

to be constant. Notice that $k\left(a,b\right) \geq 0$ by positivity of $\left(\widetilde{P}, \overline{\mathrm{DT}}^o\right)$ and k is not constant because $\delta > 0$ (quasi-invariance is violated by δ). By Ordinary Least Squares, the best constant approximation κ of the function k is the mean

$$\kappa = \frac{1}{|A|\left(|A|-1\right)}\sum_{a \neq b}\overline{\mathrm{DT}}^o_{a,b}\frac{\widetilde{\mathrm{EC}}_{a,b}}{\widetilde{\mathrm{AC}}_{a,b}}$$

Now consider the DDM with

$$\tilde{\lambda} \equiv \sqrt{\kappa/2} \qquad \text{and} \qquad \tilde{u} \equiv v/\tilde{\lambda} \qquad \text{and} \qquad \tilde{\sigma} \equiv 1 \tag{16}$$

The choice probabilities induced by this DDM are

$$P_{a,b} = \frac{1}{1+e^{-\tilde{\lambda}(\tilde{u}(a)-\tilde{u}(b))}} = \frac{1}{1+e^{-(v(a)-v(b))}} = \widetilde{P}_{a,b}$$

and hence, by Lemma 4,

$$\left|P^o_{a,b} - P_{a,b}\right| \leq \varepsilon/4$$

for all $a \neq b$ in A. Moreover, by (15) and (2),

$$\left|\overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b}\right| = \left| k(a,b) \frac{\widetilde{\mathrm{AC}}_{a,b}}{\widetilde{\mathrm{EC}}_{a,b}} - \tilde{\lambda}^2 \frac{1}{v(a)-v(b)} \tanh\left(\frac{v(a)-v(b)}{2}\right)\right|$$

but

$$\frac{\widetilde{\mathrm{AC}}_{a,b}}{\widetilde{\mathrm{EC}}_{a,b}} = \frac{\max\left(\frac{1}{1+e^{-(v(a)-v(b))}}, \frac{1}{1+e^{-(v(b)-v(a))}}\right) - \frac{1}{2}}{\log\left(\frac{\max\left(\frac{1}{1+e^{-(v(a)-v(b))}}, \frac{1}{1+e^{-(v(b)-v(a))}}\right)}{1-\max\left(\frac{1}{1+e^{-(v(a)-v(b))}}, \frac{1}{1+e^{-(v(b)-v(a))}}\right)}\right)} = \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - \frac{1}{2}}{v(a)-v(b)}$$

and

$$\frac{1}{v(a)-v(b)} \tanh\left(\frac{v(a)-v(b)}{2}\right) = 2\frac{\frac{1}{1+e^{-(v(a)-v(b))}} - \frac{1}{2}}{v(a)-v(b)}$$

Therefore

$$\begin{aligned}
\left|\overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b}\right| &= \left|\left(k(a,b) - 2\tilde{\lambda}^2\right) \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - \frac{1}{2}}{v(a)-v(b)}\right| \\
\left(\text{since } 2\tilde{\lambda}^2 = \kappa\right) &\leq \left|\frac{\frac{1}{1+e^{-(v(a)-v(b))}} - \frac{1}{2}}{v(a)-v(b)}\right| \max_{c\neq d} |k(c,d) - \kappa| \\
&\leq \frac{1}{4} \max_{\substack{x\neq y \\ c\neq d}} |k(x,y) - k(c,d)| = \frac{\delta}{4}
\end{aligned}$$

for all $a \neq b$ in A. ■

Proposition 8 *Let the observables* $\left(P^o, \overline{\mathrm{DT}}^o_{a,b}\right)$ *satisfy positivity. If the product rule and invariance are violated by* ε *and* γ *in the sense of equations (5) and (9), respectively, then, setting*

$$\tilde{\lambda} = \sqrt{\frac{1}{|A|(|A|-1)} \sum_{a\neq b} \frac{\overline{\mathrm{DT}}^o_{a,b}}{2} \frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}}} \qquad \text{and} \qquad \tilde{u}(a) - \tilde{u}(b) = \frac{\operatorname{logit} \widetilde{P}_{a,b}}{\tilde{\lambda}} \qquad \text{and} \qquad \tilde{\sigma} = 1$$

and considering the DDM with these parameters, it follows

$$\left|P^o_{a,b} - P_{a,b}\right| \leq \varepsilon/4 \qquad \text{and} \qquad \left|\overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b}\right| \leq \underbrace{\left(2\tilde{\lambda}^2 + \gamma\right)\frac{\varepsilon}{20}}_{\text{violations of the product rule } \varepsilon\neq 0} + \underbrace{\frac{\gamma}{4}}_{\text{violations of invariance}}$$

for all $a \neq b$ *in* A.

Proof Choose v and $\widetilde{P}$ like in the proof of Lemma 4. Consider the "fictitious" observables $\left(\widetilde{P}, \overline{\mathrm{DT}}^o\right)$ and notice that, by Lemma 4, they satisfy positivity and the product rule, but fail invariance by γ. In fact, invariance would require

$$k^o(a,b) = \overline{\mathrm{DT}}^o_{a,b} \frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}} \qquad \forall (a,b) \in A^2 : a \neq b$$

to be constant. Notice that $k^o(a,b) \geq 0$ by positivity of $\left(P^o, \overline{\mathrm{DT}}^o\right)$ and k^o is not constant because $\gamma > 0$ (invariance is violated by γ). By Ordinary Least Squares, the best constant approximation κ^o of the function k^o is the mean

$$\kappa^o = \frac{1}{|A|(|A|-1)} \sum_{a \neq b} \overline{\mathrm{DT}}^o_{a,b} \frac{\mathrm{EC}^o_{a,b}}{\mathrm{AC}^o_{a,b}}$$

Now consider the DDM with

$$\tilde{\lambda} \equiv \sqrt{\kappa^o/2} \qquad \text{and} \qquad \tilde{u}(a) - \tilde{u}(b) \equiv \frac{v(a) - v(b)}{\tilde{\lambda}} \qquad \text{and} \qquad \tilde{\sigma} \equiv 1$$

Clearly,

$$P_{a,b} = \frac{1}{1 + e^{\tilde{\lambda}(\tilde{u}(a) - \tilde{u}(b))}} = \widetilde{P}_{a,b}$$

and hence

$$\left|P^o_{a,b} - P_{a,b}\right| \leq \varepsilon/4$$

for all $a \neq b$ in A. Moreover, by the arguments we used in the proofs of Lemma 4 and Theorem 6,

$$\begin{aligned}
\left|\overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b}\right| &= \left| k^o(a,b) \frac{\frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}} - 1/2}{(v(a)-v(b)) + \varepsilon_{ab}} - 2\tilde{\lambda}^2 \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - 1/2}{(v(a)-v(b))} \right| \\
&\leq \left| k^o(a,b) \frac{\frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}} - 1/2}{(v(a)-v(b)) + \varepsilon_{ab}} - k^o(a,b) \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - 1/2}{(v(a)-v(b))} \right| \\
&\quad + \left| k^o(a,b) \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - 1/2}{(v(a)-v(b))} - \underbrace{2\tilde{\lambda}^2}_{=\kappa^o} \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - 1/2}{(v(a)-v(b))} \right| \\
&\leq \underbrace{|k^o(a,b)| \left| \frac{\frac{1}{1+e^{-(v(a)-v(b))-\varepsilon_{ab}}} - 1/2}{(v(a)-v(b)) + \varepsilon_{ab}} - \frac{\frac{1}{1+e^{-(v(a)-v(b))}} - 1/2}{(v(a)-v(b))} \right|}_{\text{violations of the product rule } \varepsilon_{ab} \neq 0} + \underbrace{\frac{\gamma}{4}}_{\text{violations of invariance}}
\end{aligned}$$

Since the modulus of the first derivative of

$$\frac{\frac{1}{1+e^{-x}} - 1/2}{x}$$

is always smaller than $1/20$, then

$$\left|\overline{\mathrm{DT}}^o_{a,b} - \overline{\mathrm{DT}}_{a,b}\right| \leq \underbrace{\left(2\tilde{\lambda}^2 + \gamma\right) \frac{\varepsilon}{20}}_{\text{violations of the product rule } \varepsilon \neq 0} + \underbrace{\frac{\gamma}{4}}_{\text{violations of invariance}}$$

for all $a \neq b$ in A. ■

A.3 The Metropolis-DDM algorithm

Proof of Theorem 7 In the proof we assume

$$P_{a,b} = \begin{cases} \dfrac{1 - e^{-\beta[u(a)-u(b)]}}{1 - e^{-(\lambda+\beta)[u(a)-u(b)]}} & \text{if } u(a) \neq u(b) \\ \dfrac{\beta}{\lambda + \beta} & \text{if } u(a) = u(b) \end{cases} \in (0,1) \tag{17}$$

thus allowing for asymmetric lower and upper barriers, $-\beta < 0$ and $\lambda > 0$, respectively.

The explicit form of $M = M_t$ (the subscript t will be omitted throughout) is

$$M_{ba} = M(a \mid b) = \begin{cases} Q(a \mid b) P_{a,b} & \text{if } a \neq b \\ 1 - \sum_{c \in A \setminus \{b\}} Q(c \mid b) P_{c,b} & \text{if } a = b \end{cases} \tag{18}$$

and this allows to show that M is a *bona fide* stochastic matrix.

Next we show that M is reversible with respect to $p_A = p_A^{(u,\lambda)}$. Let $a \neq b$ in A.

- If $u(a) - u(b) \neq 0$, then

$$\begin{aligned} M(a \mid b) p_A(b) &= \frac{Q(a \mid b)}{\sum_{x \in A} e^{\lambda u(x)}} \cdot \frac{e^{\lambda u(b)} - e^{-\beta u(a) + \beta u(b) + \lambda u(b)}}{1 - e^{-(\lambda+\beta)[u(a) - u(b)]}} \\ &= \frac{Q(b \mid a)}{\sum_{x \in A} e^{\lambda u(x)}} \cdot \frac{e^{\lambda u(a)} - e^{-\beta u(b) + \beta u(a) + \lambda u(a)}}{1 - e^{-(\lambda+\beta)[u(b) - u(a)]}} = M(b \mid a) p_A(a) \end{aligned}$$

  because Q is symmetric and

$$\frac{e^{\lambda u(b)} - e^{-\beta u(a) + \beta u(b) + \lambda u(b)}}{1 - e^{-(\lambda+\beta)[u(a) - u(b)]}} = \frac{e^{\lambda u(a)} - e^{-\beta u(b) + \beta u(a) + \lambda u(a)}}{1 - e^{-(\lambda+\beta)[u(b) - u(a)]}}$$

- Else $u(a) - u(b) = 0$, that is, $u(a) = u(b)$, then

$$\begin{aligned} M(a \mid b) p_A(b) &= Q(a \mid b) \frac{\beta}{\lambda + \beta} \frac{e^{\lambda u(b)}}{\sum_{x \in A} e^{\lambda u(x)}} \\ &= Q(b \mid a) \frac{\beta}{\lambda + \beta} \frac{e^{\lambda u(a)}}{\sum_{x \in A} e^{\lambda u(x)}} = M(b \mid a) p_A(a) \end{aligned}$$

  because Q is symmetric.

Since $M(a \mid b) p_A(b) = M(b \mid a) p_A(a)$ also if $a = b$, then reversibility holds.

It is then easy to see that, if Q is irreducible, then M is irreducible and aperiodic. In turn this implies that p_A is its stationary distribution and therefore $\mu M^n \to p_A$ as $n \to \infty$ for all $\mu \in \Delta(A)$ (see Madras [40, Ch. 4]).[32]

If instead Q is reducible, since it is symmetric then all communicating classes are closed (see Norris [46, Ch. 1]). In fact, if $Q_{a_1 a_2} Q_{a_2 a_3} \ldots Q_{a_{m-1} a_m} > 0$, then $Q_{a_m a_{m-1}} Q_{a_{m-1} a_{m-2}} \cdots Q_{a_2 a_1} > 0$ and $a_1 \to a_m$ implies $a_m \to a_1$. Rearrange the alternatives so that the communicating classes are

$$A_1 = \{1, ..., |A_1|\}, A_2 = \{|A_1| + 1, ..., |A_1| + |A_2|\}, \ldots, A_K = \{|A| - |A_K| + 1, ..., |A|\}$$

Notice that given any class A_k and any $b \in A_k$, then $Q(a \mid b) = Q_{ba} = 0$ for all $a \notin A_k$,[33] thus for all the rows belonging to A_k the only nonzero elements are in columns belonging to A_k (and also the converse is true by symmetry). That is, $Q = \text{diag}(Q_1, \ldots, Q_K)$ is a block diagonal matrix; moreover, by definition of communicating classes all the Q_k are irreducible (stochastic and symmetric). Now by (18) also $M = \text{diag}(M_1, \ldots, M_K)$ is block diagonal. By the first part

[32] As usual, $\Delta(A)$ is the set of all probability distributions on A.

[33] Otherwise, we would have $b \in A_k$ and $b \to a$, which by closure would imply $a \in A_k$, a contradiction.

of this proof, each of the M_k's is aperiodic, irreducible, with stationary distribution given by the restriction p_k of p_{A_k} to A_k. Then (see again Madras [40, Th. 4.2])

$$M_k^n \to \begin{bmatrix} p_k \\ p_k \\ \vdots \\ p_k \end{bmatrix} \equiv \Pi_k \qquad \forall k = 1, ..., K$$

now let $\mu = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_K \end{bmatrix} \in \Delta(A)$ with $\mu_k \in \mathbb{R}_+^{|A_k|}$ for all $k = 1, ..., K$. Given any $k = 1, ..., K$, since $\mu_k \Pi_k$ is the linear combination of the rows of Π_k with weights given by μ_k,

$$\mu_k M_k^n \to \mu_k \Pi_k = \mu_{k1} p_k + \mu_{k2} p_k + \cdots + \mu_{k|A_k|} p_k = \mu(A_k) p_k$$

therefore, by block-multiplication,

$$\mu M^n = \begin{bmatrix} \mu_1 M_1^n & \mu_2 M_2^n & \cdots & \mu_K M_K^n \end{bmatrix} \to \begin{bmatrix} \mu(A_1) p_1 & \mu(A_2) p_2 & \cdots & \mu(A_K) p_K \end{bmatrix}$$

and so $\mu M^n(a) \to \sum_{k=1}^{K} \mu(A_k) p_{A_k}(a)$ for all a in A. As wanted. ■

A.4 Probabilistic distances

Baucells and Heukamp [6], introduce the idea of risk distance of an event E, that has probability p, from certainty (that may be represented by any event that has probability 1, say Ω). Formally, a *risk distance* is a function

$$\begin{array}{rccc} h: & (0,1] & \to & [0,\infty) \\ & p & \mapsto & h(p) \end{array}$$

such that:

D1 $h(1) = 0$, that is, the distance of certainty from itself is zero;

D2 h is continuous and strictly decreasing, that is, smaller probabilities are farther from certainty;

D3 $h(p_1 p_2) = h(p_1) + h(p_2)$ for all p_1, p_2 in $(0,1]$, that is, if event E is determined by the independent realization of events E_1 (which occurs with probability p_1) and E_2 (which occurs with probability p_2), then its distance from certainty is the sum of the distances.

The resulting Cauchy's logarithmic equation admits solution

$$h(p) = -\kappa \log p = \kappa |\log 1 - \log p| \qquad \forall p \in (0,1]$$

where κ is a strictly positive multiplicative constant.[34] In order to underline the fact that $h(p)$ is a distance between event E and certainty Ω, let us write

$$h(p) = h(p, 1) \qquad \forall p \in (0,1]$$

Now, a reasonable assumption for the distance between event E and a generic event F (which occurs with probability q) is that such distance be symmetric and unaffected by conditioning both events on the realization of a third independent event G (which occurs with probability r). Formally,

[34] See [6, Proposition 2]. Also notice that formally the definition of *risk distance* coincides with that of *uncertainty measure* of event E in Information Theory (see, e.g., Ash [4]), but the conceptual difference is clear.

D4 $h(p,q) = h(q,p)$ for all p, q in $(0,1]$;

D4 $h(p,q) = h(rp, rq)$ for all p, q, r in $(0,1]$.

Proposition 9 *Let* $h : (0,1] \times (0,1] \to [0,\infty)$ *and set* $h(p) = h(p,1)$ *for all* $p \in (0,1]$. *The following conditions are equivalent:*

(i) h *satisfies D1–D5;*

(ii) there exists $\kappa > 0$ *such that* $h(p,q) = \kappa |\log p - \log q|$ *for all* p, q *in* $(0,1]$.

In this case h *is a bona fide distance on* $(0,1]$.

Before entering the proof's details notice that, for $\kappa = 1$, we have

$$h(p,q) = h\left(q\frac{p}{q}, q1\right) = h\left(\frac{p}{q}, 1\right) = h\left(\frac{p}{q}\right) \qquad 0 < p \leq q \leq 1$$

In the perspective of Baucells and Heukamp [6], this has a simple intuition. In fact, if E has probability p, F has probability q, and $E \subseteq F$, the distance of E from certainty conditional on F having occurred is

$$h(\Pr(E \mid F)) = h\left(\frac{\Pr(E)}{\Pr(F)}\right) = h\left(\frac{p}{q}\right) = -\log\frac{p}{q}$$

Proof (i) implies (ii). By [6, Proposition 2], it follows that

$$h(p,1) = -\kappa \log p$$

for some $\kappa > 0$. Now if $0 < p \leq q \leq 1$, by D5,

$$h(p,q) = h\left(q\frac{p}{q}, q1\right) = h\left(\frac{p}{q}, 1\right) = -\kappa \log\frac{p}{q} = \kappa |\log q - \log p|$$

D4 allows to draw the same conclusion if $0 < q \leq p \leq 1$.

Verifying that (ii) implies (i) is simple and so omitted for brevity. ■

Normalize again $\kappa = 1$. Now consider a discrimination task (e.g., the choice between a and b), and denote by S the event "the task is completed successfully" (e.g., the superior alternative is chosen). In this case, the probability of success is $\Pr(S)$ and the error rate is $\Pr(S^c)$. Thus the distance between success and failure is

$$|\log(\Pr(S)) - \log(\Pr(S^c))| = \operatorname{logit}(\Pr(S))$$

that is, the "degree of easiness" of Rasch [52].

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