Multinomial logit processes and preference discovery: inside and outside the black box*

S. Cerreia-Vioglio⋆, F. Maccheroni⋆, M. Marinacci⋆, and A. Rustichini•
⋆Università Bocconi and IGIER and •University of Minnesota

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Abstract

We provide both an axiomatic and a neuropsychological characterization of the dependence of choice probabilities on time in the *softmax* (or *Multinomial Logit Process*) form

$$p_t(a, A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b \in A} e^{\lambda(t)u(b)}} \tag{MLP}$$

where: $p_t(a, A)$ is the probability that alternative a is selected from the set A of feasible alternatives if t is the time available to decide, u is a utility function on the set of all alternatives, and λ is an accuracy parameter on a set of time points.

MLP is the most widely used model of preference discovery in all fields of decision making, from Quantal Response Equilibrium to Discrete Choice Analysis, from Psychophysics and Neuroscience to Combinatorial Optimization. Our axiomatic characterization of softmax permits to empirically test its descriptive validity and to better understand its conceptual underpinnings as a theory of agents' rationality. Our neuropsychological foundation provides a computational model that may explain softmax emergence in human behavior and that naturally extends to multialternative choice the classical Diffusion Model paradigm of binary choice. These complementary approaches provide a complete perspective on softmaximization as a model of preference discovery, both in terms of internal (neuropsychological) causes and external (behavioral) effects.

Keywords: Discrete Choice Analysis, Drift Diffusion Model, Luce Model, Metropolis Algorithm, Multinomial Logit Model, Quantal Response Equilibrium

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1 Introduction

Both human and machine decisions must be often made under external time constraints (deadlines). Introspection and empirical evidence agree that time scarcity leads to mistakes, even when the agent's objectives are clearly defined.

In this paper, we provide both an **axiomatic** and a **neuropsychological** characterization of the dependence of choice probabilities on time in the *softmax* (or *Multinomial Logit Process*) form

$$p_t(a, A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b \in A} e^{\lambda(t)u(b)}} \tag{MLP}$$

where:

- $p_t(a, A)$ is the probability that alternative a is selected from the set A of feasible alternatives if t is the time available to decide,[1]
- $u : X \to \mathbb{R}$ is a utility function on the set X of all alternatives,
- $\lambda : T \to (0, \infty)$ is an accuracy parameter on a set T of time points.

For each fixed time t, our theory coincides with the one of Luce (1959), that is,

$$p_t(a, A) = \frac{v_t(a)}{\sum_{b \in A} v_t(b)}$$

where v_t is a function from X to $(0, \infty)$. Our **axiomatic** contribution consists in characterizing, in terms of observables, the time dynamics of the functions v_t that corresponds to MLP. Specifically, by observing choice frequencies, we are able to establish whether there exist a *time-independent* utility u and a *time-dependent* accuracy parameter λ such that

$$v_t(x) = e^{\lambda(t)u(x)} \qquad \forall (x, t) \in X \times T$$

and, if so, to identify them. Moreover, for $T = (0, \infty)$ we show under which conditions λ is increasing and bijective. In this case, when the agent has no time to decide, all alternatives become equiprobable

$$\lim_{t \to 0} p_t(a, A) = \frac{1}{|A|} \qquad \forall a \in A$$

and choice is completely random. In contrast, longer times allow for greater differences in the selection probability for alternatives that differ in their utilities until, without time pressure, only the utility maximizing ones survive

$$\lim_{t \to \infty} p_t(a, A) = \begin{cases} \dfrac{1}{|\arg\max_A u|} & \text{if } a \in \arg\max_A u \\ 0 & \text{else} \end{cases}$$

and so choice is perfectly rational.

[1] Or the experience level of the agent, see below.

On the other hand, for each pair $\{a, b\}$ of alternatives, our neuro-computational model delivers the same selection probabilities as the (unbiased) Drift Diffusion Model (DDM) of Ratcliff (1978). Indeed,

$$p_t(a, \{a, b\}) = \frac{e^{\lambda(t)u(a)}}{e^{\lambda(t)u(a)} + e^{\lambda(t)u(b)}}$$

is the probability with which the Brownian motion $Z_{a,b}(\tau) = [u(a) - u(b)]\tau + \sqrt{2}W(\tau)$ reaches the decision threshold $\lambda(t)$ before reaching the dual threshold $-\lambda(t)$.[2] Our **neuropsychological** contribution consists in presenting a testable and easy to simulate computational model that extends the DDM to multialternative choice tasks. Our approach builds on the premise that multialternative choice procedures are composed primarily of sequential pairwise comparisons, in which actual evaluative processing takes place, and that agents' exploration strategies are based on the similarity and proximity of available alternatives (Russo and Rosen, 1975, and Reutskaja, Nagel, Camerer, and Rangel, 2011). Specifically, we show that combining Markovian exploration *à la* Metropolis of menu A and DDM pairwise comparison of its alternatives leads to selection probabilities that are described by MLP. Moreover, the fact that $\lambda(t)$ is the decision threshold of the DDM confirms, from a completely different perspective relative to the axiomatic one, its interpretation in terms of accuracy.

As we further discuss below, softmax is the most widely used model of preference discovery in all fields of decision making, from Quantal Response Equilibrium to Discrete Choice Analysis, from Psychophysics and Neuroscience to Combinatorial Optimization. Our axiomatic characterization of MLP permits to empirically test its descriptive validity and to better understand its conceptual underpinnings as a theory of agents' rationality. Our neuropsychological foundation provides a computational model that may explain softmax emergence in human behavior and that naturally extends to multialternative choice the classical Diffusion Model paradigm of binary choice. These complementary approaches provide a complete perspective on softmaximization as a model of preference discovery, both in terms of internal (neuropsychological) causes and external (behavioral) effects.

1.1 More on softmax and on our contribution

1.1.1 Discovered Preference Hypothesis and Quantal Response Equilibrium

On the theoretical side, softmax can be regarded as a formalization of the Discovered Preference Hypothesis (DPH) outlined in Plott (1996). According to the DPH, agents acquire an understanding of how their basic needs are satisfied by the different alternatives in the choice environment through a process of reflection and practice. A process which, in the long run, leads to optimizing behavior. In our model, both reflection and practice are synthesized by the passage of time. This is best seen by considering the two aspects separately:

- If t represents the time limit within which an alternative a must be chosen from a feasible set A, then discovery occurs by reflection only.

[2] In the DDM, $Z_{a,b}(\tau)$ represents, at each τ in $(0, \infty)$, the net accumulated evidence in support of alternative a against alternative b, $-Z_{a,b}(\tau) = Z_{b,a}(\tau)$ the net accumulated evidence in support of b against a, and the first alternative supported by at least $\lambda(t)$ evidence is chosen. See, e.g., Bogacz et al. (2006).

- If t represents the number of times the agent has been facing choice problem A and a is an action to be taken, then discovery mainly occurs by practice. In this case, the fact that λ is increasing captures the dynamics of exploration in earlier decision stages and exploitation in later ones.

MLP is the form that the DPH takes in the logit specification of Quantal Response Equilibrium (McKelvey and Palfrey, 1995). In this case, t is the experience level of the player, that is, the number of times he played the game,[3] $u(a)$ is the expected payoff of action a, and λ captures the player's degree of rationality. From the original data analysis of McKelvey and Palfrey (1995),[4] to the recent Agranov, Caplin, and Tergiman (2015), evidence seems to suggest that, for sophisticated players, λ increases as time passes and the decision making environment is better understood.[5]

Our axiomatic and neuropsychological characterizations of softmax can thus be seen as two alternative foundations of Quantal Response Equilibrium theory.

1.1.2 Discrete Choice Analysis and Multinomial Logit Model

On the empirical side,[6] the most widely used model of Discrete Choice Analysis is the Multinomial Logit Model

$$\mathcal{M} = \left\{ \frac{e^{\lambda u(a)}}{\sum_{b \in A} e^{\lambda u(b)}} : a \in A \right\}_{\lambda \in (0,\infty)} \tag{MNL}$$

This model is popular because for each $\lambda \in (0, \infty)$, called scale parameter, we have

$$\frac{e^{\lambda u(a)}}{\sum_{b \in A} e^{\lambda u(b)}} = \Pr \left\{ u(a) + \frac{\epsilon_a}{\lambda} > u(b) + \frac{\epsilon_b}{\lambda} \quad \text{for all } b \in A \setminus \{a\} \right\} \tag{MLF}$$

where $\{\epsilon_x\}_{x \in X}$ is any random field of independent and identically distributed errors with type I extreme value distribution.[7] That is, MLF describes the probability of choosing a from A by an agent who is maximizing the random utility

$$\tilde{u}_\lambda(x) = u(x) + \frac{\epsilon_x}{\lambda}$$

with systematic component $u(x)$ and disturbance ϵ_x/λ. In this case, the variance of disturbance is independent of the alternative x and inversely proportional to λ^2, so that it vanishes as λ diverges.[8] In discrete choice experiments comprising a panel of repeated choice tasks, the estimation of different values of λ in early, middle, and late phases of the experiment is used to reveal preference learning (decreasing variance – increasing λ) and fatigue (increasing variance – decreasing λ).[9] In the deadlined-choice perspective of this paper, MLF shows that

[3] Our use of the pronoun "he" is completely nature-neutral, our agents might be algorithms.

[4] See also the discussion of Plott (1996) of the Centipede Game experiment of McKelvey and Palfrey (1992).

[5] Interestingly, in Agranov, Caplin, and Tergiman (2015), t is not the experience level but the time the player had to contemplate the altenatives in A before choosing.

[6] See Ben-Akiva and Lerman (1985), Louviere, Hensher, and Swait (2000), and Train (2009).

[7] $\Pr\{\epsilon_x \leq \varepsilon\} = \exp(-e^{-\varepsilon-\gamma})$ for all $x \in X$, where γ is the Euler-Mascheroni constant (see McFadden, 1973).

[8] Specifically, $\operatorname{Var}[\tilde{u}_\lambda(x)] = \operatorname{Var}[\lambda^{-1}\epsilon_x] = \pi^2/6\lambda^2$, also notice that $u(x) = \operatorname{E}[\tilde{u}_\lambda(x)]$, for all $x \in X$.

[9] See Savage and Waldman (2008) and Campbell, Boeri, Doherty, and Hutchinson (2015).

a Multinomial Logit Process describes an agent who is trying to maximize u over A, but – because of time scarcity – makes mistakes in evaluating the utility of the various alternatives, with standard deviation which is inversely proportional to $\lambda(t)$.

Our axiomatic characterization of the Multinomial Logit Model allows to test for model misspecification. Moreover, we also provide simple techniques to directly obtain parameters from data (see Proposition 9).

1.1.3 Psychophysics and Neuroscience

In discrimination tasks where pairs $A = \{a, b\}$ of alternatives are compared, MNL is called "logistic psychometric function family" and is the most used parameterization of response probabilities.[10] In this case, λ is the slope of the psychometric function and it measures accuracy of discrimination or – in the neuroscientific study of choice behavior – sensitivity to utility differences.[11]

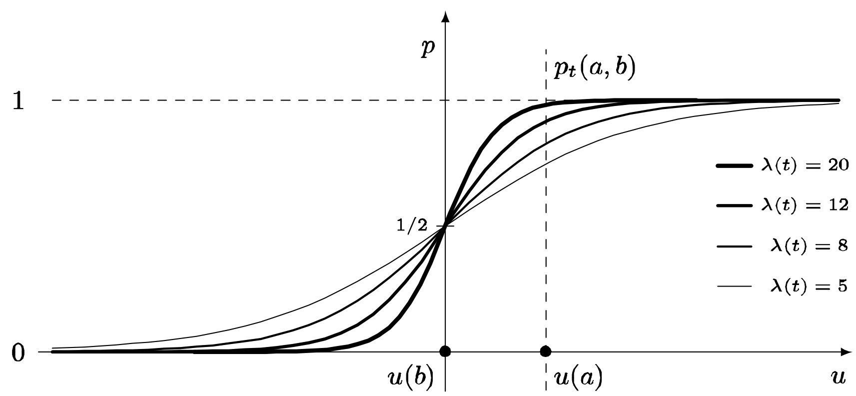


Figure 1: $p_t(a,b) = p_t(a,\{a,b\})$ plotted as a function of $u = u(a)$, for fixed $u(b)$.[12]

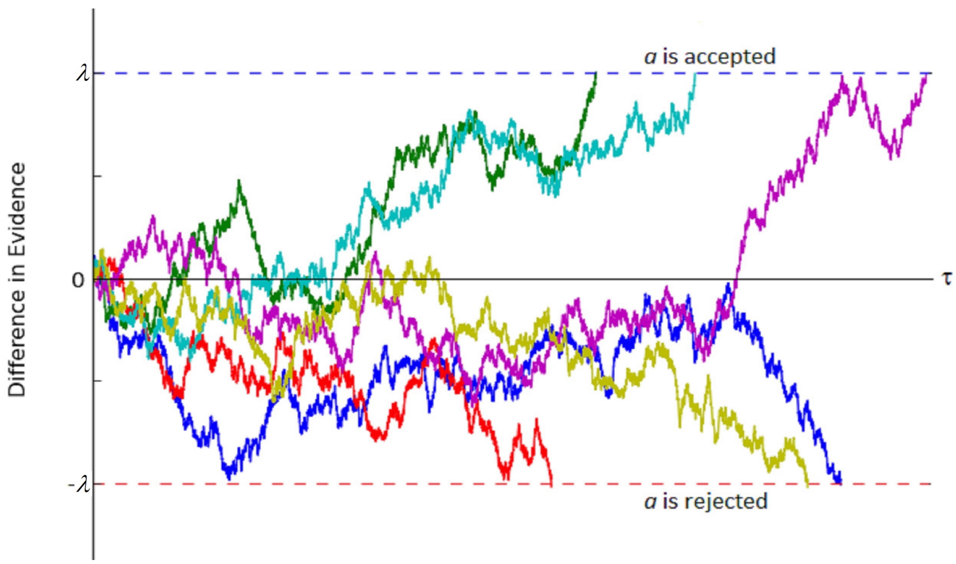
The neuropsychological foundation for the use of this family is the Drift Diffusion Model of Ratcliff (1978), according to which decisions are made by accumulating stochastic information about the alternatives a and b until the net evidence in favor of one, call it "the winner," exceeds threshold λ, at which point the winner is chosen.[13] In a DPH perspective, it is important to observe that the DDM does not assume $u(a)$ and $u(b)$ to be known to the agent. Indeed, the sign of the utility difference $u(a) - u(b)$ is discovered by accumulating (noisy) evidence until the threshold λ is reached.

[10] See Falmagne (1985) and Iverson and Luce (1998).

[11] Rangel (2009), Dovencioglu, Ban, Schofield, and Welchman (2013), and Tsunada, Liu, Gold, and Cohen (2016) are recent neuroscience works that use this specification of the psychometric function and adopt the corresponding interpretation of the parameter λ.

[12] Geometrically, $\lambda(t)/4$ is the slope of the tangent to the graph at $(u(b), 1/2)$.

[13] See Bogacz et al. (2006), Webb (2017), and Section 4 of this paper.



Milosavljevic et al. (2010) are the first to analyze the DDM in the realm of value based choice of consumption. They find that choices are sped up under time pressure by decreasing the threshold λ and making it easier to reach it. More in general, from the pioneering experiments of Cattel (1902) on perceived intensity of light to the recent study of ALQahtani et al. (2016) on the effect of time pressure on diagnostic decisions, evidence has been systematically confirming that accuracy is an increasing function of the time available to discriminate. In particular, Ortega and Stocker (2016) calibrate softmax and find that λ is approximately linear in log-time, a special case that we are able to characterize as a corollary of our main result (see equation LSP).

Our neuropsychological foundation extends, in a novel way based on sequential pairwise comparisons, the scope of the DDM to multialternative choice tasks, while maintaining its discovered preference interpretation. Our axiomatic characterization allows to externally validate the proposed computational model.

1.1.4 Combinatorial Optimization

The Simulated Annealing algorithm of the seminal Kirkpatrick, Gelatt, and Vecchi (1983) builds on the limit properties of softmax.[14] The main idea behind this algorithm is to construct a machine that randomly explores the set A, sequentially selecting alternatives and evaluating their utilities so that, at predetermined periods $\{t_0, t_1, ...\}$, the selection frequencies are approximately given by MLP. Through this procedure, global optima are discovered with increasing probability as the sequence $\{\lambda(t_0), \lambda(t_1), ...\}$ diverges. In other words, for $T = \{t_n\}_{n\in\mathbb{N}}$ and $\lambda = \{\lambda(t_n)\}_{n\in\mathbb{N}}$, a Multinomial Logit Process can be regarded as describing the ideal behavior of a Simulated Annealing machine.

Our computational model can be seen as the script according to which this machine explores the environment and compares the alternatives on the basis of acquired evidence. Our axioms then describe the observable choice behavior of the machine itself.

[14] Uniformity (i.e., exploration) for small $\lambda(t)$ and optimality (i.e., exploitation) for large $\lambda(t)$.

1.2 Related literature

An important distinction in the axiomatic modelling of stochastic choice processes is whether the final decision time t is exogenous (the time limit set by the experimenter or by the environment) or endogenous – the actual response time of a subject who chooses when to make his decision without external time pressure. This paper considers exogenous deadlines and the present discussion of the literature focuses on models with this feature. In Economics, models where decision time is endogenously – say, optimally – chosen are the subject of an active literature and we refer the reader to Woodford (2014), Steiner, Stewart, and Matejka (2017), and Fudenberg, Strack, and Strzalecki (2017) for updated perspectives. In Psychology, these models have a long history, arguably beginning with Laming (1968). We refer to Bogacz et al. (2006) for more recent developments.

Natenzon (2017) proposes a Multinomial Bayesian Probit model with the aim of jointly accommodating similarity effects, attraction effects, and compromise effects in a preference learning perspective. According to Natenzon's model, when facing a menu of alternatives, the decision maker (who has a priori i.i.d. standard normally distributed beliefs the on the possible utilities of alternatives) receives a random vector of (joint normally distributed) signals that represents how much he is able to learn about the ranking of alternatives before making a choice (say within time t). The decision maker updates the prior according to Bayes' rule and chooses the option with the highest posterior mean utility. Conceptually, both Natenzon's model and ours can be seen as formal embodiments of the Discovered Preference Hypothesis. Natenzon's Multinomial Bayesian Probit predates ours and gains descriptive power by abandoning the random utility paradigm and augmenting the number of parameters. At the same time, our model is easier to falsify thanks to its axiomatic nature and because of the abundance of routine techniques to estimate it. From a neuropsychological viewpoint, both approaches belong to the diffusion models' tradition that started with the seminal works of Ratcliff, Busemeyer, and coauthors (see the reviews of Rieskamp, Busemeyer, and Mellers, 2006, Fehr and Rangel, 2011, Ratcliff, Smith, Brown, and McKoon, 2016).

In a general Random Expected Utility perspective, Lu (2016) captures preference learning through increasingly informative priors on the set of probabilistic beliefs of the decision maker, while in Caplin and Dean (2011) preference learning occurs through sequential search.

Fudenberg and Strzalecki (2015) axiomatize a discounted adjusted logit model. Differently from the present work and the ones discussed above, their paper studies stochastic choice in a dynamic setting where choices made today can influence the possible choices available tomorrow, and consumption may occur in multiple periods. Frick, Iijima, and Strzalecki (2017) characterize its general random utility counterpart. Finally, Caplin and Dean (2013) and Matejka and McKay (2015) identify the Multinomial Logit Model in terms of optimal information acquisition, while Saito (2017) obtains several characterizations of the Mixed Logit Model.

As to the neuropsychological modelling of choice tasks with $n > 2$ alternatives, the vast majority of extensions of the DDM considers *simultaneous evidence accumulation* for all the n alternatives in the menu. In these models, the choice task is assumed to simultaneously activate n accumulators, each of which is preferentially sensitive to one of the alternatives and integrates evidence relative to that alternative; choices are then made based on absolute or relative evidence levels. See, e.g., Roe, Busemeyer, and Townsend (2001), Anderson, Goeree, and Holt (2004), McMillen and Holmes (2006), Bogacz, Usher, Zhang, and McClelland (2007), Ditterich (2010), and Krajbich and Rangel (2011).

Alternatively, Reutskaja, Nagel, Camerer, and Rangel (2011) propose three two-stage models in which subjects randomly search through the feasible set during an initial search phase, and when this phase is concluded they select the best item that was encountered during the search, up to some noise. This approach can be called *quasi-exhaustive search* in that the presence of a deadline may terminate the search phase before all alternatives have been evaluated and introduces an error probability.

Here, instead, we focus on *sequential pairwise comparison* as advocated by Russo and Rosen (1975) in a seminal eye fixation study (see especially their concluding section). The computational model we adopt is the basic version of the Metropolis-DDM decision procedures studied in Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2017). Although different from the models considered by Reutskaja, Nagel, Camerer, and Rangel (2011), our model is consistent with some of their experimental findings about the menu-exploration process and shares the classical choice theory approach according to which multialternative choice relies on binary comparison and elimination.

1.3 Paper outline

Section 2 of this paper presents the mathematical setup and extends the Luce Model by removing the assumption of full support. This is technically important for the proofs of our representation results and it contributes to the theoretical literature on static random choice (see Echenique and Saito, 2017, on this specific topic). In Section 3, we axiomatically characterize Multinomial Logit Processes as well as several special cases (starting with the classical Multinomial Logit Model with affine u). Section 4 shows how softmax distributions emerge from a neuropsychologically inspired decision procedure that combines Markovian search of the menu (*à la* Metropolis et al., 1953) and DDM binary comparisons of its alternatives. The final Section 5 concludes and proofs are relegated to the appendix.

2 Random choice rules

Let $\mathcal{A}$ be the collection of all non-empty finite subsets A of a universal set X of possible alternatives. The elements of $\mathcal{A}$ are called choice sets (or menus, or choice problems). We denote by $\Delta(X)$ the set of all finitely supported probability measures on X and, for each $A \subseteq X$, by $\Delta(A)$ the subset of $\Delta(X)$ consisting of the measures assigning mass 1 to A.

Definition 1 *A* random choice rule *is a function*

$$\begin{array}{rccc} p: & \mathcal{A} & \rightarrow & \Delta(X) \\ & A & \mapsto & p_A \end{array}$$

such that $p_A \in \Delta(A)$ for all $A \in \mathcal{A}$.

Given any alternative $a \in A$, we interpret $p_A(\{a\})$, also denoted by $p(a, A)$, as the probability that an agent chooses a when the set of available alternatives is A. More generally, if B is a subset of A, we denote by $p_A(B)$ or $p(B, A)$ the probability that the selected element lies in B.[15] This probability can be viewed as the frequency with which an element in B is chosen. As usual, given any a and b in X, we set

$$p(a,b) = p(a,\{a,b\}), \quad r(a,b) = \frac{p(a,b)}{p(b,a)}, \quad \ell(a,b) = \ln r(a,b)$$

So, $r(a,b)$ denotes the *odds* for a against b and $\ell(a,b)$ the *log-odds*.[16]

2.1 Luce's Model

The classical assumptions of Luce (1959) on p are:

Positivity $p(a,b) > 0$ *for all* $a, b \in X$.

Choice Axiom $p(a, A) = p(a, B)\, p(B, A)$ *for all* $B \subseteq A$ *in* $\mathcal{A}$ *and all* $a \in B$.

The latter axiom says that the probability of choosing an alternative a from menu A is the probability of first selecting B from A, then choosing a from B (provided a belongs to B). As observed by Luce, formally this assumption corresponds to the fact that $\{p_A : A \in \mathcal{A}\}$ is a conditional probability system in the sense of Renyi (1956).[17] Remarkably, Luce's Choice Axiom is also equivalent to:

Independence from Irrelevant Alternatives

$$\frac{p(a,b)}{p(b,a)} = \frac{p(a,A)}{p(b,A)} \tag{IIA}$$

for all $A \in \mathcal{A}$ *and all* $a, b \in A$ *such that* $p(a,A)/p(b,A)$ *is well defined.*[18]

This axiom says that the odds for a against b are independent of the available alternatives that are different from a and b themselves.

Theorem 1 (Luce) *A random choice rule* $p : \mathcal{A} \to \Delta(X)$ *satisfies Positivity and the Choice Axiom if and only if there exists* $v : X \to (0, \infty)$ *such that*

$$p(a, A) = \frac{v(a)}{\sum_{b \in A} v(b)} \tag{LM}$$

for all $A \in \mathcal{A}$ *and all* $a \in A$.

In this case, v *is unique up to a strictly positive multiplicative constant.*

[15] Formally, $x \mapsto p(x, A)$ for all $x \in X$ is the discrete density of p_A, but notation will be abused and $p_A(\cdot)$ identified with $p(\cdot, A)$.

[16] The advantage of using log-odds for a against b is that they are strictly positive if and only if odds are favorable to a. Indeed, $p(a,b) > p(b,a) \iff r(a,b) > 1 \iff \ell(a,b) > 0$.

[17] See Lemma 2 of Luce (1959) and Lemma 12 in the appendix.

[18] That is, different from 0/0. See Lemma 3 of Luce (1959) for the case in which Positivity holds and our Lemma 12 in the appendix for the general case.

This fundamental result in random choice theory also shows that, under the Choice Axiom, Positivity is equivalent to the stronger assumption that p_A has full support for all $A \in \mathcal{A}$.[19]

Full Support $p(a, A) > 0$ *for all* $A \in \mathcal{A}$ *and all* $a \in A$.

2.2 Luce's Choice Axiom without Positivity

The next result generalizes Theorem 1 by maintaining the behavioral assumption of Independence from Irrelevant Alternatives while removing the technical assumption of Full Support. In our softmax analysis it will allow us to distill the utility function u starting from choice frequencies. In reading it, recall that a *choice correspondence* is a map $\Gamma : \mathcal{A} \to \mathcal{A}$ such that $\Gamma(A) \subseteq A$ for all $A \in \mathcal{A}$. It is *rational* when satisfies the *Weak Axiom of Revealed Preference* of Arrow (1959), that is,

$$B \subseteq A \in \mathcal{A} \text{ and } \Gamma(A) \cap B \neq \varnothing \text{ imply } \Gamma(B) = \Gamma(A) \cap B \qquad \text{(WARP)}$$

Theorem 2 *A random choice rule* $p : \mathcal{A} \to \Delta(X)$ *satisfies the Choice Axiom if and only if there exist a function* $v : X \to (0, \infty)$ *and a rational choice correspondence* $\Gamma : \mathcal{A} \to \mathcal{A}$ *such that*

$$p(a, A) = \begin{cases} \dfrac{v(a)}{\sum_{b \in \Gamma(A)} v(b)} & \textit{if } a \in \Gamma(A) \\ 0 & \textit{else} \end{cases} \qquad \text{(GLM)}$$

for all $A \in \mathcal{A}$ *and all* $a \in A$.

In this case, Γ *is unique and* $\Gamma(A) = \operatorname{supp} p_A$ *for all* $A \in \mathcal{A}$.

The relation $\succsim$ generated by a rational choice correspondence Γ is defined by

$$a \succsim b \iff a \in \Gamma(\{a, b\})$$

It is a weak order such that $\Gamma(A) = \{a \in A : a \succsim b \text{ for all } b \in A\}$. A two-stage decision process thus appears in formula GLM: first rational selection via Γ from menu A, then Lucean randomization to choose among the selected alternatives. In other words, Theorem 2 allows us to regard randomization as a tie-breaking mechanism that takes place after a first stage of optimization has selected a subset $\Gamma(A)$ among the available alternatives.

In this two-stage perspective, the simplest tie-breaking rule is the one in which all selected alternatives have the same probability of being chosen, corresponding to a constant v in GLM. Formally, a random choice rule is *uniform* if and only if

$$p(a, A) = \begin{cases} \dfrac{1}{|\operatorname{supp} p_A|} & \text{if } a \in \operatorname{supp} p_A \\ 0 & \text{else} \end{cases}$$

for all $A \in \mathcal{A}$ and all $a \in A$.

[19]The *support* of p_A is defined by $\operatorname{supp} p_A = \{a \in X : p(a, A) > 0\}$ for all $A \in \mathcal{A}$. It coincides with $\{a \in A : p(a, A) > 0\}$ because $p_A(A) = 1$.

Corollary 3 *A uniform random choice rule $p : \mathcal{A} \to \Delta(X)$ satisfies the Choice Axiom if and only if* $\operatorname{supp} p : \mathcal{A} \to \mathcal{A}$ *is a rational choice correspondence.*

In this case, the relation $\succsim$ generated by $\operatorname{supp} p$, *is characterized by*

$$a \succsim b \iff p(a, b) > 0 \iff p(a, b) \geq p(b, a)$$

These results show how Luce's Choice Axiom seamlessly extends Arrow's traditional postulate of deterministic rationality (WARP) to random choice behavior, thus paving the way to an economic theory based on stochastic rationality.[20]

Finally, our Theorem 2 shows that the random choice rules satisfying Independence from Irrelevant Alternatives (but not necessarily Full Support) are a special case of the rules characterized by Echenique and Saito (2017). These rules take the form GLM where $v : X \to (0, \infty)$, but the choice correspondence Γ does not necessarily satisfy WARP. Echenique and Saito achieve such a general characterization with four axioms that, in our special case, can be replaced by Luce's one.

3 Random choice processes

Let $T \subseteq (0, \infty)$ be a – discrete or continuous – set of points of time.

Definition 2 *A* random choice process *is a collection $\{p_t\}_{t \in T}$ of random choice rules.*

For each t, we interpret $p_t(a, A)$ as the probability that an agent chooses alternative a from menu A if t is the maximum amount of time he is given to decide.[21]

An important alternative interpretation of t, especially when T is discrete and panel data are considered, is the number of times that the agent has been facing choice problem A, called experience level by McKelvey and Palfrey (1995).

We maintain the following assumptions:

Positivity *p_t satisfies Positivity for all $t \in T$.*

Choice Axiom *p_t satisfies the Choice Axiom for all $t \in T$.*

By Luce's Theorem, for each $t \in T$ there exists $u_t : X \to \mathbb{R}$ such that

$$p_t(a, A) = \frac{e^{u_t(a)}}{\sum_{b \in A} e^{u_t(b)}} \qquad \forall a \in A \in \mathcal{A}$$

Moreover, each of the u_t's is unique up to an additive constant, so that time-dependent utility differences are significant and captured by log-odds.[22]

[20] See Cerreia-Vioglio, Maccheroni, Marinacci and Rustichini (2017).

[21] E.g., by an experimenter, or by a script, or by a spouse. See Agranov, Caplin, and Tergiman (2015) for a simple protocol that allows to observe these probabilities for human agents.

[22] In fact, $\ell_t(a, b) = u_t(a) - u_t(b)$ for all $a, b \in X$ and all $t \in T$.

3.1 The Multinomial Logit Model

We start with the characterization of the special case of softmax corresponding to $T = (0, \infty)$ and $\lambda(t) = t$ for all $t \in T$. To this end, we add a few axioms.

Continuity $\lim_{t \to s} p_t(a, A)$ *exists for all* $A \in \mathcal{A}$*, all* $a \in A$*, all* $s \in [0, \infty]$*, and it coincides with* $p_s(a, A)$ *if* $s \in (0, \infty)$.

Continuity guarantees that, as t tends to either 0 or ∞, two limit random choice rules p_0 and p_∞ are defined that extend the domain of the random choice process $\{p_t\}$ to $[0, \infty]$. Notice that these rules satisfy the Choice Axiom if $\{p_t\}$ does, but they do not necessarily inherit the Positivity property.

Consistency *Given any* $a, b \in X$,

$$p_t(a, b) > p_t(b, a) \implies p_s(a, b) > p_s(b, a)$$

for all $s > t > 0$.

Consistency means that favorable odds for a against b remain favorable, formally

$$r_t(a, b) > 1 \implies r_s(a, b) > 1 \qquad 0 < t < s \leq \infty$$

This is coherent with the idea that correct, yet noisy, evidence is accumulating to inform the decision maker's choice between the two alternatives.

Asymptotic Uniformity *Given any* $a, b \in X$,

$$p_\infty(a, b) \neq 0, 1 \implies p_\infty(a, b) = 1/2$$

Asymptotic Uniformity postulates that, if the decision maker is unable to make up his mind between alternatives a and b irrespectively of the time available to do so, then he will choose by flipping a fair coin. This is a classical notion of indifference that can be traced back to the early days of experimental economics (see Davidson and Marschak, 1959).

Boundedness *Given any* $a, b \in X$,

$$\sup_{t, s \in (0, \infty)} |r_{t+s}(a, b) - r_t(a, b) r_s(a, b)| < \infty$$

Boundedness requires that the time variation of odds be not exponentially unbounded. It is a "grain of exponentiality" in odds' dynamics.[23]

Theorem 4 *A random choice process* $\{p_t\}_{t \in (0, \infty)}$ *satisfies Positivity, the Choice Axiom, Continuity, Consistency, Asymptotic Uniformity, and Boundedness if and only if there exists* $u : X \to \mathbb{R}$ *such that*

$$p_t(a, A) = \frac{e^{tu(a)}}{\sum_{b \in A} e^{tu(b)}} \tag{MNL}$$

for all $A \in \mathcal{A}$*, all* $a \in A$*, and all* $t \in (0, \infty)$.

In this case, u *is unique up to an additive constant.*

[23] See Frederic, Di Bacco, and Lad (2012) for a formal interpretation of the product of odds in terms of combination of evidence.

Although here t is interpreted as a time index, Theorem 4 yields a completely general characterization of the Multinomial Logit Model with scale parameter t and utility u.[24] For this reason, we call the process $\{p_t\}$ a *Multinomial Logit Model with utility* u. This model is the workhorse of Discrete Choice Analysis, where it is typically also assumed that

$$X \subseteq \mathbb{R}^n \text{ and } u(x) = \beta \cdot x \text{ for some } \beta \in \mathbb{R}^n$$

The following proposition characterizes this fundamental special case.[25]

Proposition 5 *Let X be a convex set and $\{p_t\}_{t\in(0,\infty)}$ be a Multinomial Logit Model with utility u. The following conditions are equivalent:*

1. *u is affine;*
2. *there exists $t \in (0,\infty)$ such that*
$$p_t(a,b) = p_{\frac{t}{\alpha}}(\alpha a + (1-\alpha)b, b)$$
for all $a, b \in X$ and all $\alpha \in (0,1)$;
3. *given any $t \in (0,\infty)$ and any $c \in X$,*
$$p_t(a,A) = p_{\frac{t}{\alpha}}(\alpha a + (1-\alpha)c, \alpha A + (1-\alpha)c)$$
for all $A \in \mathcal{A}$, all $a \in A$, and all $\alpha \in (0,1)$.

In Discrete Choice Analysis, the elements of X are often viewed as vectors of attributes.[26] The equivalence between points 1 and 3 above thus shows that the affinity of u corresponds to an inverse relation between the proximity of attribute levels and the degree of choice accuracy. To fix ideas, assume $A = \{a, b\}$ with $u(a) > u(b)$. Shrinkage by a factor $\alpha = 1/2$ of attribute levels in the direction of c then doubles the time to achieve the same probability of choosing the optimal alternative.

3.1.1 Stochastic dominance

Consider a Multinomial Logit Model with utility u and an arbitrarily fixed $A \in \mathcal{A}$. For each $t \in (0,\infty)$, by selecting alternatives according to $p_t(\cdot, A)$, the agent's payoffs (in utils) are described by the random variable

$$\begin{array}{rccc} U_t: & A & \to & \mathbb{R} \\ & a & \mapsto & u(a) \end{array}$$

The next proposition shows that U_s first order stochastically dominates U_t for all $s \geq t$, that is, the random process $\{U_t\}_{t\in(0,\infty)}$ of payoffs is increasing with respect to stochastic dominance.

[24] In the random utility interpretation of Multinomial Logit Models discussed in Section 1.1.2, u is called "systematic component of the utility".

[25] As well as the case in which X is a set of mixed actions and $u(x)$ is the expected payoff of x.

[26] For example, in a typical travel demand application the components x_0 and x_1 of x represent travel time and travel cost of alternative x, respectively (Ben-Akiva and Lerman, 1985, and Train, 2009).

Proposition 6 *Let* $\{p_t\}_{t\in(0,\infty)}$ *be a Multinomial Logit Model with utility* u. *Then*

$$p_s\left(\{a\in A: u(a)\geq h\}, A\right)\geq p_t\left(\{a\in A: u(a)\geq h\}, A\right) \qquad \forall h\in\mathbb{R}$$

for all $s>t$ *in* $(0,\infty)$ *and all* $A\in\mathcal{A}$.

3.1.2 *Curiosum:* Statistical Physics

The Boltzmann distribution law says that if the energy associated with some state or condition a of a system A is $\varepsilon(a)$ then the frequency with which that state or condition occurs, or the probability of its occurrence, in equilibrium, is

$$p_\tau(a, A)=\frac{e^{-\frac{1}{k\tau}\varepsilon(a)}}{\sum_{b\in A}e^{-\frac{1}{k\tau}\varepsilon(b)}}$$

where τ is the system's absolute temperature and k is the Boltzmann constant. Therefore, by identifying time with inverse temperature and utility with negative energy, Theorem 4 can be seen as a characterization of equilibrium distributions in large physical systems. The restatement of the axioms directly in terms of temperature – rather than time – is straightforward.

3.2 Continuous Multinomial Logit Processes

Every Multinomial Logit Process, with $T=(0,\infty)$ and increasing bijective λ, satisfies all the hypotheses of Theorem 4, except possibly Boundedness. A natural direction of investigation then consists in weakening this assumption. We start with an ordinal version of Boundedness.

Ordinal Boundedness *Given any* $a,b\in X$,

$$\sup_{t,s\in(0,\infty)}\left|r_{w(t+s)}(a,b)-r_{w(t)}(a,b)\,r_{w(s)}(a,b)\right|<\infty \tag{OB}$$

for some increasing bijection $w:(0,\infty)\to(0,\infty)$.

Ordinal Boundedness requires that time can be rescaled so that the corresponding variation of odds is not exponentially unbounded. It reduces to Boundedness when $w(t)=t$ for all t.[27] Our main representation results show that:

- replacing Boundedness with Ordinal Boundedness in Theorem 4 delivers representation MLP (Theorem 7);
- although an existential quantifier appears in its statement, Ordinal Boundedness is falsifiable using the primitive data $\{p_t\}$ (Proposition 9).

[27] Ordinal Boundedness stands to Boundedness as the Ordinal IIA assumption of Fudenberg, Iijima, and Strzalecki (2015) stands to IIA.

But, before formally stating the theorems, we introduce an alternative angle that delivers the separation of accuracy and utility by requiring the speed of preference discovery to be independent from the alternatives being compared.

Independence from Compared Alternatives *Given any $a, b, x, y \in X$,*

$$r_\tau(a,b) > r_t(a,b)\, r_s(a,b) \implies r_\tau(x,y) > r_t(x,y)\, r_s(x,y) \tag{ICA}$$

for all $s, t, \tau \in (0,\infty)$ such that $r_\tau(a,b) > 1$ and $r_\tau(x,y) > 1$.

A strengthening of Consistency is required for ICA to replace Ordinal Boundedness in our main result.

Discovered Preference Axiom *Given any $a, b \in X$,*

$$\begin{cases} r_0(a,b) = 1 \\ r_t(a,b) > 1 \implies r_s(a,b) > r_t(a,b) \\ r_t(a,b) \le 1 \implies r_s(a,b) \le r_t(a,b) \end{cases}$$

for all $s > t$ in $(0,\infty)$.

This axiom captures the spirit of the Discovered Preference Hypothesis of Plott (1996): discernment is born from reflection/practice, so it is impossible at $t = 0$ and increases over time. As evidence accumulates, favorable odds become more favorable, non-favorable ones become more so.

Theorem 7 *Let $\{p_t\}_{t\in(0,\infty)}$ be a random choice process. The following conditions are equivalent:*

1. *$\{p_t\}$ satisfies Positivity, the Choice Axiom, Continuity, Consistency, Asymptotic Uniformity, and Ordinal Boundedness;*

2. *$\{p_t\}$ satisfies Positivity, the Choice Axiom, Continuity, the Discovered Preference Axiom, Asymptotic Uniformity, and Independence from Compared Alternatives;*

3. *there exists $u : X \to \mathbb{R}$ and an increasing and bijective $\lambda : (0,\infty) \to (0,\infty)$ such that*

$$p_t(a,A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b\in A} e^{\lambda(t)u(b)}} \tag{MLP}$$

   for all $A \in \mathcal{A}$, all $a \in A$, and all $t \in (0,\infty)$.

In this case, u is cardinally unique, and λ is unique given u unless the latter is constant.

In this subsection, we call such a process a *Multinomial Logit Process with utility u and accuracy λ*. As anticipated in the introduction, these processes admit limit random choice rules

$$p_0(a,A) = \frac{1}{|A|} \quad \text{and} \quad p_\infty(a,A) = \frac{1}{|\arg\max_A u|}\delta_a(\arg\max_A u)$$

for all $A \in \mathcal{A}$ and all $a \in A$. So they yield pure randomness of choice as $t \to 0$ and hard maximization of utility as $t \to \infty$.

The next simple proposition characterizes the case of constant utility u.

Proposition 8 *Let $\{p_t\}_{t\in(0,\infty)}$ be a Multinomial Logit Process with utility u and accuracy λ. The following conditions are equivalent:*

1. *u is constant;*
2. *$p_t(a,A) = 1/|A|$ for all $A \in \mathcal{A}$, all $a \in A$, and all $t \in (0,\infty)$;*
3. *there are no $a, b \in X$ and $t \in (0,\infty)$ such that $r_t(a,b) > 1$.*

The importance of point 3 above is that, contrapositively, if there are $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0,\infty)$ such that $r_{\hat{t}}(\hat{a},\hat{b}) > 1$, then u is non-constant and λ identified (up to a strictly positive multiplicative constant). The next proposition shows that identification is straightforward,[28] and guarantees the direct falsifiability of Ordinal Boundedness.

Proposition 9 *Let $\{p_t\}_{t\in(0,\infty)}$ be a Multinomial Logit Process for which there exist $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0,\infty)$ such that $r_{\hat{t}}(\hat{a},\hat{b}) > 1$. Then setting*

$$\begin{cases} \hat{\lambda}(t) = \ell_t(\hat{a},\hat{b}) & \forall t \in (0,\infty) \\ \hat{u}(x) = \dfrac{\ell_{\hat{t}}(x,\hat{b})}{\ell_{\hat{t}}(\hat{a},\hat{b})} & \forall x \in X \end{cases}$$

the function $\hat{\lambda} : (0,\infty) \to (0,\infty)$ is increasing, bijective, and

$$p_t(a,A) = \frac{e^{\hat{\lambda}(t)\hat{u}(a)}}{\sum_{b\in A} e^{\hat{\lambda}(t)\hat{u}(b)}} \qquad \forall a \in A \in \mathcal{A} \quad \forall t \in (0,\infty) \tag{$\widehat{\text{MLP}}$}$$

Moreover, $\{p_t\}_{t\in(0,\infty)}$ satisfies Ordinal Boundedness with respect to the inverse $\hat{w}$ of $\hat{\lambda}$.

Operationally, one can choose $\hat{a}, \hat{b} \in X$ such that $p_{\hat{t}}(\hat{a},\hat{b}) > p_{\hat{t}}(\hat{b},\hat{a})$ for some $\hat{t} \in (0,\infty)$, and test Ordinal Boundedness against $\hat{w}$. Then, if all the other hypotheses of Theorem 7 are satisfied, it must be the case that $\widehat{\text{MLP}}$ holds.

Theoretically, Proposition 9 reveals the inverse relation between λ and w for Multinomial Logit Processes with non-constant utility. For example, logarithmic λ's, which appear both in Combinatorial Optimization and Psychophysics,[29] correspond to Exponential Boundedness,[30] and give softmax the special form

$$p_t(a,A) = \frac{(1+kt)^{u(a)}}{\sum_{b\in A}(1+kt)^{u(b)}} \qquad \forall a \in A \in \mathcal{A} \quad \forall t \in (0,\infty) \tag{LSP}$$

where k is a strictly positive constant.

Finally, it is easy to see that the stochastic dominance statement of Proposition 6 holds unchanged for Multinomial Logit Processes with utility u and accuracy λ, while Proposition 18 in the appendix shows how to modify Proposition 5 in order to obtain affinity of u even in the non-linear accuracy case.

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[28] Estimation of u and λ is standard, typically carried out by maximum likelihood. See, e.g., Ben-Akiva and Lerman (1985) on the econometric side and McKelvey and Palfrey (1995) on the game theoretic one.

[29] See, e.g., Romeo and Sangiovanni-Vincentelli (1991) and Ortega and Stocker (2016).

[30] That is, Ordinal Boundedness with $w(t) = (e^t - 1)/k$ for some strictly positive constant k yields $\lambda(t) = \ln(1+kt)$ in MLP.

3.3 General Multinomial Logit Processes

We close with the general case in which the index set T of the random choice process $\{p_t\}_{t\in T}$ is any non-singleton subset of $(0,\infty)$, for example a sequence $T=\{t_n\}_{n\in\mathbb{N}}$ or a time interval $(0,\tau]$.

Log-odds Ratio Invariance *Given any $t,s\in T$,*

$$\frac{\ell_t(a,c)}{\ell_t(b,c)}=\frac{\ell_s(a,c)}{\ell_s(b,c)}$$

for all $a,b,c\in X$ such that either ratio is well defined.

Since log-odds capture time-dependent utility differences, this axiom requires that relative utilities be time invariant. Recall that here T can also be a discrete set. For this reason, in our final theorem below, Continuity will be completely dispensed with. Accordingly, Consistency has to be weakened as follows:

Weak Consistency *Given any $a,b\in X$,*

$$p_t(a,b)>p_t(b,a)\implies p_s(a,b)>p_s(b,a)$$

for all $s>t$ in T.

Notice that if $T=(0,\infty)$ this requirement is equivalent to the one of Consistency except for the limit value $s=\infty$.

Theorem 10 *A random choice process $\{p_t\}_{t\in T}$ satisfies Positivity, the Choice Axiom, Weak Consistency, and Log-odds Ratio Invariance if and only if there exist $u:X\to\mathbb{R}$ and $\lambda:T\to(0,\infty)$ such that*

$$p_t(a,A)=\frac{e^{\lambda(t)u(a)}}{\sum_{b\in A}e^{\lambda(t)u(b)}} \tag{MLP}$$

for all $A\in\mathcal{A}$, all $a\in A$, and all $t\in T$.

In this case, u is cardinally unique, and λ is unique given u unless the latter is constant.

This result delivers in full generality the softmax representation that we discussed in the introduction. *Inter alia*, the possible non-monotonicity of λ allows us to capture fatigue in discrete choice experiments.[31] Finally, it is easy to see that the estimation technique $\widehat{\text{MLP}}$ continues to hold.

[31] At the same time, simple variations on the Discovered Preference Axiom permit to characterize increasing and decreasing λ's. Moreover, replacement of the clause "$s>t$" with the clause "s and t" in the Weak Consistency axiom permits to consider non-ordered index sets T (see Theorem 19 in the appendix).

4 Decision making procedures

So far, we regarded the components p_t of a random choice process $\{p_t\}$ as the output of a black box: our axioms characterize Multinomial Logit Processes, but remain silent about what decision procedure may generate the corresponding choice probabilities. In this final section, we address this issue by combining Markovian search of the menu and DDM pairwise comparison of its alternatives. As we anticipated in the introduction, the first assumption is in line with the experimental finding of Reutskaja, Nagel, Camerer, and Rangel (2011) that exploration is independent of alternatives' values, the second with the one of Russo and Rosen (1975) that the choice process is a sequence of pairwise comparisons.

Specifically, we consider a decision maker, with utility $u : X \to \mathbb{R}$, who aims to choose an optimal alternative from a menu $A \in \mathcal{A}$ within a stringent deadline t.

In what follows we first introduce the different parts of the decision procedure, we then assemble them. Notation is streamlined by assuming, without loss of generality, that $A = \{1, 2, ..., |A|\}$, with $|A| \geq 2$, and by identifying elements μ of $\Delta(A)$ with vectors in $\mathbb{R}^{|A|}$.

4.1 Exploration

Exploration of menu A has a classic Metropolis et al. (1953) format. The agent starts with a first (automatically accepted) candidate solution b drawn from an initial distribution $\mu \in \Delta(A)$. Then, given an incumbent solution b, the agent considers an alternative candidate solution $a \neq b$ with probability $Q(a \mid b)$. The only requirements we make on the probability transition matrix Q are irreducibility and symmetry (see, e.g., Madras, 2002). The independence of μ and Q from the utility function u are the formal counterparts of the aforementioned eye-tracking evidence.

A natural assumption that simultaneously guarantees both irreducibility and symmetry of Q is that the subjective distance which the decision maker perceives between alternatives be described by a *perceptual semimetric* d on A, and that $Q(a \mid b)$ be a strictly positive function of $d(a, b)$.[32] For example, d can be the discrete metric if A is a set of abstract alternatives, the Euclidean metric if A is a set of multiattribute alternatives (like transportation modes), the shortest-path distance if A is a connected graph (like a wine rack or a vending machine). For instance, Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2017) consider the latter case and the following parametric form

$$Q(a \mid b) = \frac{1}{|A| - 1} \frac{1}{d(a, b)^{\rho}} \qquad \forall a \neq b \text{ in } A$$

where $\rho \in (0, \infty)$ is an exploration aversion parameter: for very large ρ, the agent basically explores only the nearest neighbours of the incumbent solution; for very small ρ, $Q(\cdot \mid b)$ is essentially the uniform distribution on $A \setminus \{b\}$.[33]

[32] A semimetric has all the properties of a metric except the triangle inequality. It can be either a physical distance mechanically affecting attention or a psychological distance describing the mental landscape in which the decision maker organizes the elements of A (possibly incorporating similarity considerations), or a mix of the two. See Russo and Rosen (1975) and Roe, Busemeyer, and Townsend (2001).

[33] As usual, $Q(b \mid b)$ is residually defined by $1 - \sum_{c \in A \setminus \{b\}} Q(c \mid b)$.

4.2 Binary comparison

Once proposed, alternative a is compared with the incumbent b via the Drift Diffusion Model of Ratcliff (1978). According to this model, an alternative is selected as soon as the net evidence in its favor reaches a posited decision threshold $z \in (0, \infty)$, which in our case depends on the time constraint t via $z = \lambda(t)$.[34] Specifically, the comparison of a and b is believed to activate two neuronal populations whose activities (firing rates) provide evidence for the two alternatives. If their mean activities are $u(a)$ and $u(b)$, and each experiences instantaneous independent white noise fluctuations, then evidence accumulation in favor of a and b is represented by two uncorrelated Brownian motions with drift $V_a(\tau) = u(a)\tau + W_a(\tau)$ and $V_b(\tau) = u(b)\tau + W_b(\tau)$ defined on a sample space Ω. With this,

- the *net evidence* in favor of a against b is given by the difference

$$Z_{a,b}(\tau, \omega) = [u(a) - u(b)]\tau + \sqrt{2}W(\tau, \omega) \qquad \forall (\tau, \omega) \in (0, \infty) \times \Omega \tag{DDM}$$

  where W is the Wiener process $(W_a - W_b)/\sqrt{2}$;

- the comparison ends when $Z_{a,b}(\tau)$ reaches either the threshold z or $-z$; so the *response time* is the random variable

$$\mathrm{RT}^z_{a,b}(\omega) = \min\{\tau \in (0, \infty) : |Z_{a,b}(\tau, \omega)| = z\} \qquad \forall \omega \in \Omega$$

- at which time, if the upper bound z has been reached, the agent accepts proposal a; otherwise, the lower bound $-z$ has been reached, the proposal a is rejected and the agent maintains the incumbent b; so, the *comparison outcome* is the random variable

$$\mathrm{CO}^z_{a,b}(\omega) = \begin{cases} a & \text{if } Z_{a,b}\left(\mathrm{RT}^z_{a,b}(\omega), \omega\right) = z \\ b & \text{if } Z_{a,b}\left(\mathrm{RT}^z_{a,b}(\omega), \omega\right) = -z \end{cases}$$

The *acceptance probability* for a given b is then

$$\alpha^z(a \mid b) = \Pr\left\{\omega \in \Omega : \mathrm{CO}^z_{a,b}(\omega) = a\right\}$$

while the *rejection probability* is $1 - \alpha^z(a \mid b)$. As well known,[35] this quantity can be computed and has the logistic form

$$\alpha^z(a \mid b) = \frac{e^{zu(a)}}{e^{zu(a)} + e^{zu(b)}}$$

4.3 Decision

We now combine Metropolis exploration and DDM pairwise comparison. The resulting procedure describes an agent who – given time t to decide – automatically adjusts $z = \lambda(t)$ and begins exploring the menu according to Q and comparing alternatives according to CO^z. His search continues until the deadline is reached, at which point he chooses the incumbent solution.

[34] See Milosavljevic et al. (2010) and Karsilar, Simen, Papadakis, and Balci (2014, especially p. 14) for a discussion of the possible effects of deadlines on threshold reduction in the DDM.

[35] See, e.g., the original Ratcliff (1978), Bogacz et al. (2006), Smith and Ratcliff (2015), and Webb (2017).

Metropolis-DDM Algorithm

Input: *Given* $t \in (0, \infty)$ *and* $z = \lambda(t) \in (0, \infty)$.

Start: *Draw* a *from* A *according to* μ *and*

- *set* $\tau_0 = 0$,
- *set* $b_0 = a$;

Repeat: *Draw* a *from* A *according to* $Q(\cdot \mid b_n)$ *and compare it to* b_n *via the DDM, so:*

- *set* $\tau_{n+1} = \tau_n + \mathrm{RT}^z_{a,b_n}$,
- *set* $b_{n+1} = \mathrm{CO}^z_{a,b_n}$;

until $\tau_{n+1} > t$.

Stop: *Set* $b^* = b_n$.

Output: *Choose* b^* *from* A.

Since the evaluation of the signs of utility differences is performed according to the DDM, the algorithm we propose maintains intact the interpretation of preference discovery that we discussed for the DDM itself in the introduction.

At each iteration of the "repeat-until loop", proposal a is accepted as b_{n+1} with the softmax probability

$$\alpha^{\lambda(t)}(a \mid b_n) = \frac{e^{\lambda(t)u(a)}}{e^{\lambda(t)u(a)} + e^{\lambda(t)u(b_n)}} = p_t(a, b_n)$$

while a is rejected and b_n is maintained as b_{n+1} with the complementary probability $p_t(b_n, a)$.[36] Therefore, the choice behavior described by our procedure is the softmax counterpart of the one induced by random Brute-Force Optimization. The latter corresponds to the limit case of infinite accuracy $\lambda(t) = \infty$ in which no mistakes are allowed in comparing a and b_n at the acceptance stage.[37] In contrast, the adoption of the Drift Diffusion Model with a finite threshold $z = \lambda(t)$ entails an error probability $1/\left(1 + e^{\lambda(t)|u(a)-u(b_n)|}\right)$.[38]

By implementing the Metropolis-DDM algorithm, the resulting probability of selecting a given incumbent b is

$$P_t(a \mid b) = Q(a \mid b)\,\alpha^{\lambda(t)}(a \mid b) \qquad \forall a \neq b \text{ in } A$$

This transition probability combines the stochasticity of the proposal mechanism and that of the acceptance/rejection rule. Specifically, after n iterations of the "repeat-until loop", the probability that $b^* = a$ is the a-th component of the vector $P_t^n \mu$.

[36] The acceptance/rejection probabilities $\alpha^z(a \mid b)$ induced by the DDM coincide with the ones used by Barker (1965) to apply the Metropolis algorithm in the context of Plasma Physics, where the reciprocal $1/z$ of the evidence threshold z is called *temperature* (for the reasons we exposed in Section 3.1.2).

[37] The candidate a is accepted with probability 1 if $u(a) > u(b_n)$, $1/2$ if $u(a) = u(b_n)$, and 0 if $u(a) < u(b_n)$.

[38] A mistake consists in either accepting an inferior proposal or rejecting a superior one – events with probabilities $1/\left(1 + e^{\lambda(t)(u(b_n)-u(a))}\right)$ when $u(b_n) > u(a)$ and $1/\left(1 + e^{\lambda(t)(u(a)-u(b_n))}\right)$ when $u(a) > u(b_n)$, respectively.

Proposition 11 *Let $t \in (0, \infty)$ and $\lambda(t) \in (0, \infty)$. If Q is irreducible and symmetric, then P_t is aperiodic, irreducible, and its stationary distribution is the softmax*

$$p_t(a, A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b \in A} e^{\lambda(t)u(b)}} \qquad \forall a \in A \tag{MLP}$$

In particular, $\lim_{n \to \infty} P_t^n \mu = p_t(\cdot, A)$ *for all* $\mu \in \Delta(A)$.

This final result completes the interpretation of Multinomial Logit Processes as a theory of the Discovered Preference Hypothesis by presenting softmax as the ideal behavior of a decision maker that follows the Metropolis-DDM algorithm. Proposition 11 also suggests a neurobehavioral counterpart of Simulated Annealing: if accuracy $\lambda(t)$ diverges slowly enough as the time t available to decide increases, then the algorithm approximates the softmax distribution with increasing precision, asymptotically approaching the hardmax one

$$p_\infty(a, A) = \frac{1}{|\arg\max_A u|} \delta_a(\arg\max_A u) \qquad \forall a \in A$$

This is, indeed, the heuristics by which the celebrated algorithm of Kirkpatrick, Gelatt, and Vecchi (1983) searches for global optima within menu A, so our agents can be seen as *de facto* implementing it. We refer interested readers to Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2017) for an in-depth computational analysis of procedures of this kind.[39]

5 Concluding remarks

Summing up, in this paper we:

- Extend Luce's Model by maintaining the Independence from Irrelevant Alternatives assumption without requiring Full Support (Theorem 2).

- Provide an axiomatic foundation of the Multinomial Logit Model

$$\mathcal{M} = \left\{ \frac{e^{\lambda u(a)}}{\sum_{b \in A} e^{\lambda u(b)}} : a \in A \right\}_{\lambda \in (0,\infty), A \in \mathcal{A}} \tag{MNL}$$

  (Theorem 4) and characterize the linearity of u, which is the typical assumption of Discrete Choice Analysis when the set of possible alternatives consists of multiattribute vectors (Proposition 5).[40]

- Characterize Multinomial Logit Processes

$$p_t(a, A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b \in A} e^{\lambda(t)u(b)}} \qquad \forall a \in A \in \mathcal{A} \quad t \in T \tag{MLP}$$

  that is, processes $\{p_t\}_{t \in T}$ of random choice rules that have Multinomial Logit distributions with t-dependent scale parameter $\lambda(t)$ (Theorems 7, 10, and 19 in the appendix).

[39] *Inter alia*, they show that the possible asymmetry of thresholds does not affect stationarity of softmax and the same applies to the canonical discretization of the DDM (the Random Walk Model); they also formalize the sense in which $\lambda(t)$ must increase slowly with respect to t in order to guarantee convergence of the algorithm and provide a faster-than-the-clock simulation technique.

[40] Notice that, since u in MNL is unique up to an additive constant, affinity is equivalent to linearity.

We already observed that when t is the experience level of an agent and u is the expected payoff, MLP gives the logit specification of agents' mixed strategies in Quantal Response Equilibrium. Here we remark that, if instead u is the log-expected payoff, then MLP coincides with the Power Luce Model of Goeree, Holt and Palfrey (2016). Notice that in both cases u is independent of λ, thus permitting the separation of actions' payoffs and agents' degrees of rationality. In other words, Weak Consistency and Log-odds Ratio Invariance are as essential as Full Support and Independence from Irrelevant Alternatives for the basic laws of motions of selection probabilities in Quantal Response Equilibrium theory.

The same considerations apply to Discrete Choice Analysis whenever it is assumed that the systematic component u of utility is invariant across agents or dates while error variance is allowed to vary, that is, scale is heterogeneous (see Fiebig, Keane, Louviere, and Wasi, 2010).

Finally, in the paper we also:

- Show that softmax distributions emerge as stationary selection probabilities of a search algorithm *à la* Metropolis in which the acceptance/rejection rule is dictated by the Drift Diffusion Model, thus providing a neurophysiological foundation for the MNL.

The Metropolis-DDM algorithm that we present here relies on the simplest versions of Markovian exploration and of the Drift Diffusion Model. A natural extension of our neuropsychological analysis consists in considering more general two-alternatives forced-choice models, such as the ones discussed in Diederich and Busemeyer (2003), Ratcliff and Smith (2004), Bogacz et al. (2006), Krajbich, Armel, Rangel (2010), Rustichini and Padoa-Schioppa (2015).[41] This goes well beyond the scope of the present paper, whose objective is building a first bridge between the axiomatic and the neuro-computational approaches to random choice. But it is the object of current research (see also Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci, 2017).

An important extension of our axiomatic model consists in considering, rather than time, t as a *type*; for example, Cognitive Skill, or Intelligence, or Age, or Nationality. In its simplest mathematical form, this extension is obtained in Theorem 19 of the appendix where also the assumption that $T \subseteq (0, \infty)$ is dropped. A more challenging problem is the axiomatic modelling of the situation in which alternatives are multidimensional, for example, consumption streams or multiattribute vectors, and the speed of discovery is different across different dimensions. To illustrate, assume that X consists of pairs (x_0, x_1) indicating present and future consumption, and the speed of learning is different across the two, so that the inner product

$$\lambda_0 (t) \, u_0 (x_0) + \lambda_1 (t) \, u_1 (x_1)$$

replaces the scalar product $\lambda (t) \, [u_0 (x_0) + u_1 (x_1)]$. In an application of this richer choice setup to repeated games, actions affect current payoffs and continuation payoffs, and the type t of the decision maker (for example, Intelligence) affects discovery in these two dimensions in different ways (see Proto, Rustichini, and Sofianos, 2017).

[41] Also the modelling of menu exploration can be made more realistic, for example, by taking into account the visual saliency of alternatives (see Milosavljevic, Navalpakkam, Koch, and Rangel, 2012).

A Proofs

A.1 Proofs of the results of Section 2

Lemma 12 *Let $p : \mathcal{A} \to \Delta(X)$ be a random choice rule. The following conditions are equivalent:*

1. *p is such that, $p_A(C) = p_B(C)\, p_A(B)$ for all $C \subseteq B \subseteq A$ in $\mathcal{A}$;*
2. *p satisfies the Choice Axiom;*
3. *p is such that $p(b, B)\, p(a, A) = p(a, B)\, p(b, A)$ for all $B \subseteq A$ in $\mathcal{A}$ and all $a, b \in B$;*
4. *p satisfies Independence from Irrelevant Alternatives;*
5. *p is such that $p(Y \cap B, A) = p(Y, B)\, p(B, A)$ for all $B \subseteq A$ in $\mathcal{A}$ and all $Y \subseteq X$.*

Moreover, in this case, p satisfies Positivity if and only if it satisfies Full Support.

Proof *1 implies 2.* Choose as C the singleton a appearing in the statement of the axiom.

2 implies 3. Given any $B \subseteq A$ in $\mathcal{A}$ and any $a, b \in B$, by the Choice Axiom, $p(a, A) = p(a, B)\, p(B, A)$, but then $p(b, B)\, p(a, A) = p(a, B)\, p(b, B)\, p(B, A) = p(a, B)\, p(b, A)$ where the second equality follows from another application of the Choice Axiom.

3 implies 4. Let $A \in \mathcal{A}$ and arbitrarily choose $a, b \in A$ such that $p(a, A) / p(b, A) \neq 0/0$. By 3,

$$p(b, a)\, p(a, A) = p(b, \{a, b\})\, p(a, A) = p(a, \{a, b\})\, p(b, A) = p(a, b)\, p(b, A)$$

three cases have to be considered:

- $p(b, a) \neq 0$ and $p(b, A) \neq 0$, then $p(a, A) / p(b, A) = p(a, b) / p(b, a)$;
- $p(b, a) = 0$, then $p(a, b)\, p(b, A) = 0$, but $p(a, b) \neq 0$ (because $p(a, b) / p(b, a) \neq 0/0$), thus $p(b, A) = 0$ and $p(a, A) \neq 0$ (because $p(a, A) / p(b, A) \neq 0/0$); therefore

$$\frac{p(a, b)}{p(b, a)} = \infty = \frac{p(a, A)}{p(b, A)}$$

- $p(b, A) = 0$, then $p(b, a)\, p(a, A) = 0$, but $p(a, A) \neq 0$ (because $p(a, A) / p(b, A) \neq 0/0$), thus $p(b, a) = 0$ and $p(a, b) \neq 0$ (because $p(a, b) / p(b, a) \neq 0/0$); therefore

$$\frac{p(a, A)}{p(b, A)} = \infty = \frac{p(a, b)}{p(b, a)}$$

4 implies 3. Given any $B \subseteq A$ in $\mathcal{A}$ and any $a, b \in B$:

- If $p(a, A) / p(b, A) \neq 0/0$ and $p(a, B) / p(b, B) \neq 0/0$, then by IIA

$$\frac{p(a, A)}{p(b, A)} = \frac{p(a, b)}{p(b, a)} = \frac{p(a, B)}{p(b, B)}$$

- ◦ If $p(b,A) \neq 0$, then $p(b,B) \neq 0$ and $p(b,B)\,p(a,A) = p(a,B)\,p(b,A)$.
  - ◦ Else $p(b,A) = 0$, then $p(b,B) = 0$ and again $p(b,B)\,p(a,A) = p(a,B)\,p(b,A)$.

- Else, either $p(a,A)/p(b,A) = 0/0$ or $p(a,B)/p(b,B) = 0/0$ and in both cases

$$p(b,B)\,p(a,A) = p(a,B)\,p(b,A)$$

3 implies 5. Given any $B \subseteq A$ in $\mathcal{A}$ and any $Y \subseteq X$, since $p(B,B) = 1$, it follows $p(Y,B) = p(Y \cap B, B)$. Therefore

$$\begin{aligned}
p(Y \cap B, A) &= \sum_{y \in Y \cap B} p(y,A) = \sum_{y \in Y \cap B} \left(\sum_{x \in B} p(x,B) \right) p(y,A) = \sum_{y \in Y \cap B} \left(\sum_{x \in B} p(x,B)\,p(y,A) \right) \\
[\text{by } 3] &= \sum_{y \in Y \cap B} \left(\sum_{x \in B} p(y,B)\,p(x,A) \right) = \sum_{y \in Y \cap B} p(y,B) \left(\sum_{x \in B} p(x,A) \right) \\
&= \sum_{y \in Y \cap B} p(y,B)\,p(B,A) = p(Y \cap B, B)\,p(B,A) = p(Y,B)\,p(B,A)
\end{aligned}$$

5 implies 1. Take $Y = C$.

Finally, let p satisfy the Choice Axiom. Assume – *per contra* – Positivity holds and $p(a,A) = 0$ for some $A \in \mathcal{A}$ and some $a \in A$. Then $A \neq \{a\}$ and, for all $b \in A \setminus \{a\}$, the Choice Axiom implies $0 = p(a,A) = p(a,\{a,b\})\,p(\{a,b\},A) = p(a,b)\,(p(a,A) + p(b,A)) = p(a,b)\,p(b,A)$ whence $p(b,A) = 0$ (because $p(a,b) \neq 0$), contradicting $p(A,A) = 1$. Therefore Positivity implies Full Support. The converse is trivial. ■

If $p : \mathcal{A} \to \Delta(X)$ is a random choice rule, denote by $\sigma_p(A)$ the support of p_A, for all $A \in \mathcal{A}$.

Lemma 13 *If $p : \mathcal{A} \to \Delta(X)$ is a random choice rule that satisfies the Choice Axiom, then $\sigma_p : \mathcal{A} \to \mathcal{A}$ is a rational choice correspondence.*

Proof Clearly, $\varnothing \neq \sigma_p(A) \subseteq A$ for all $A \in \mathcal{A}$, then $\sigma_p : \mathcal{A} \to \mathcal{A}$ is a choice correspondence. Let $A, B \in \mathcal{A}$ be such that $B \subseteq A$ and assume that $\sigma_p(A) \cap B \neq \varnothing$.

We want to show that $\sigma_p(A) \cap B = \sigma_p(B)$. Since p satisfies the Choice Axiom, if $a \in \sigma_p(A) \cap B$, then $0 < p(a,A) = p(a,B)\,p(B,A)$. It follows that $p(a,B) > 0$, that is, $a \in \sigma_p(B)$. Thus, $\sigma_p(A) \cap B \subseteq \sigma_p(B)$. As to the converse inclusion, let $a \in \sigma_p(B)$, that is, $p(a,B) > 0$. By contradiction, assume that $a \notin \sigma_p(A) \cap B$. Since $a \in B$, it must be the case that $a \notin \sigma_p(A)$, that is, $p(a,A) = 0$. Since p satisfies the Choice Axiom, we then have $0 = p(a,A) = p(a,B)\,p(B,A)$. Since $p(a,B) > 0$, it must be the case that $p(B,A) = 0$, that is, $\sigma_p(A) \cap B = \varnothing$. This contradicts $\sigma_p(A) \cap B \neq \varnothing$; therefore, a belongs to $\sigma_p(A) \cap B$. Thus, $\sigma_p(B) \subseteq \sigma_p(A) \cap B$. ■

Lemma 14 *The function $p : \mathcal{A} \to \Delta(X)$ is a random choice rule that satisfies the Choice Axiom if and only if there exist a function $v : X \to (0,\infty)$ and a rational choice correspondence $\Gamma : \mathcal{A} \to \mathcal{A}$ such that, for all $x \in X$ and $A \in \mathcal{A}$*

$$p(x,A) = \begin{cases} \dfrac{v(x)}{\sum_{b \in \Gamma(A)} v(b)} & \textit{if } x \in \Gamma(A) \\ 0 & \textit{else} \end{cases} \tag{1}$$

and $p(Y,A)=\sum_{y\in Y} p(y,A)$ *for all* $Y\subseteq X$.

In this case, Γ *is unique and coincides with* σ_p.

Proof *If.* Let p be given by (1) with Γ a rational choice correspondence and $v: X\to(0,\infty)$. It is easy to check that p is a well defined random choice rule and $p(Y,A)=\sum_{y\in Y\cap\Gamma(A)}\dfrac{v(y)}{\sum_{d\in\Gamma(A)} v(d)}$ for all $Y\subseteq X$. Let $A,B\in\mathcal{A}$ be such that $B\subseteq A$ and $a\in B$. We have two cases:

(i) If $\Gamma(A)\cap B\neq\varnothing$, since Γ satisfies WARP, $\Gamma(A)\cap B=\Gamma(B)$.

- If $a\in\Gamma(B)$, then $a\in\Gamma(A)$ and $p(a,B)=v(a)/\sum_{b\in\Gamma(B)} v(b)$, it follows that

$$p(a,A)=\frac{v(a)}{\sum_{d\in\Gamma(A)} v(d)}=\frac{v(a)}{\sum_{b\in\Gamma(B)} v(b)}\frac{\sum_{b\in\Gamma(A)\cap B} v(b)}{\sum_{d\in\Gamma(A)} v(d)}=p(a,B)\,p(B,A)$$

- Else $a\notin\Gamma(B)$, and since $a\in B$, it must be the case that $a\notin\Gamma(A)$, so $p(a,A)=0=p(a,B)=p(a,B)\,p(B,A)$.

(ii) Else $\Gamma(A)\cap B=\varnothing$. It follows that $a\notin\Gamma(A)$ and $p(B,A)=0=p(a,A)$; again, we have $p(a,A)=p(a,B)\,p(B,A)$.

Cases (i) and (ii) prove that p satisfies the Choice Axiom.

Only if. Let $p:\mathcal{A}\to\Delta(X)$ be a random choice rule that satisfies the Choice Axiom and, given any $a,b\in X$, set $a\succsim b \iff a\in\sigma_p(\{a,b\})$. By Lemma 13, σ_p is a rational choice correspondence. Therefore, $\succsim$ is a weak order on X and its symmetric part $\sim$ is an equivalence relation such that

$$a\sim b \iff p(a,b)>0 \text{ and } p(b,a)>0 \iff r(a,b)\in(0,\infty)$$

Moreover, by Theorem 3 of Arrow (1959), it follows that

$$\sigma_p(A)=\{a\in A: a\succsim b \quad \forall b\in A\} \qquad \forall A\in\mathcal{A} \tag{2}$$

in particular, all elements of $\sigma_p(A)$ are equivalent with respect to $\sim$, and

$$\sigma_p(S)=S \tag{3}$$

for all $S\in\mathcal{A}$ consisting of equivalent elements.

Let $\{X_i: i\in I\}$ be the family of all equivalence classes of $\sim$ in X. Choose $a_i\in X_i$ for all $i\in I$. For each $x\in X$, there exists one and only one $i=i_x$ such that $x\in X_i$, set

$$v(x)=r(x,a_i) \tag{4}$$

Since $x\sim a_i$, then $r(x,a_i)\in(0,\infty)$; and so $v: X\to(0,\infty)$ is well defined. Given any $x\sim y$ in X and any $S\in\mathcal{A}$ consisting of equivalent elements and containing x and y, by (3) and the Choice Axiom,

$$0<p(x,S)=p(x,y)\,p(\{x,y\},S)$$
$$0<p(y,S)=p(y,x)\,p(\{x,y\},S)$$

yielding that

$$p(x,y)\,p(y,x)\,p(x,S)\,p(y,S)>0 \text{ and } \frac{p(x,S)}{p(y,S)}=\frac{p(x,y)}{p(y,x)}=r(x,y) \tag{5}$$

We are ready to conclude our proof, that is, to show that (1) holds with $\Gamma=\sigma_p$. Let $a\in X$ and $A\in\mathcal{A}$. If $a\notin\sigma_p(A)$, then $p(a,A)=0$ because $\sigma_p(A)$ is the support of p_A. Else, $a\in\sigma_p(A)$, and, by (2), all the elements in $\sigma_p(A)$ are equivalent with respect to $\sim$ and therefore they are equivalent to some a_i with $i\in I$. It follows that $\sigma_p(A)\cup\{a_i\}\in\mathcal{A}$ and it is such that $\sigma_p(A)\cup\{a_i\}\subseteq X_i$. By (3), we have that $\sigma_p(\sigma_p(A)\cup\{a_i\})=\sigma_p(A)\cup\{a_i\}$, that is, $p(x,\sigma_p(A)\cup\{a_i\})>0$ for all $x\in\sigma_p(A)\cup\{a_i\}$ and $p(\sigma_p(A),\sigma_p(A)\cup\{a_i\})>0$. By the Choice Axiom and, since $p(\sigma_p(A),A)=1$, it follows that

$$p(a,A)=p(a,\sigma_p(A))\,p(\sigma_p(A),A)=p(a,\sigma_p(A))=\frac{p(a,\sigma_p(A)\cup\{a_i\})}{p(\sigma_p(A),\sigma_p(A)\cup\{a_i\})}=\frac{\frac{p(a,\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}{\frac{p(\sigma_p(A),\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}$$

applying (5) to the pairs $(x,y)=(a,a_i)$ and $(x,y)=(b,a_i)$, with $b\in\sigma_p(A)$, in $S=\sigma_p(A)\cup\{a_i\}\subseteq X_i$, we can conclude that

$$\frac{\frac{p(a,\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}{\frac{p(\sigma_p(A),\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}=\frac{\frac{p(a,\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}{\sum_{b\in\sigma_p(A)}\frac{p(b,\sigma_p(A)\cup\{a_i\})}{p(a_i,\sigma_p(A)\cup\{a_i\})}}=\frac{r(a,a_i)}{\sum_{b\in\sigma_p(A)}r(b,a_i)}=\frac{v(a)}{\sum_{b\in\sigma_p(A)}v(b)}$$

as wanted.

As for the uniqueness part, it is clear that, by (1), $\Gamma=\sigma_p$. ■

Theorem 2 and **Corollary 3** follow immediately.

A.2 Proofs of the results of Section 3

Lemma 15 *Let $\{p_t\}_{t\in(0,\infty)}$ be a random choice process that satisfies Positivity, the Choice Axiom, Continuity, Consistency, and Asymptotic Uniformity. Then:*

(i) the relation defined on X by $a\succsim b$ if and only if $p_\infty(a,b)>0$ is a weak order, and

$$a\succsim b \iff p_\infty(a,b)\geq p_\infty(b,a)$$

(ii) given any $a,b\in X$, the function $\varphi_{a,b}:(0,\infty)\to(0,\infty)$ defined by

$$\varphi_{a,b}(t)=r_t(a,b)\qquad\forall t\in(0,\infty)$$

is continuous and either constantly equal to 1 (if $a\sim b$), or divergent at ∞ as $t\to\infty$ (if $a\succ b$), or vanishing as $t\to\infty$ (if $b\succ a$);

(iii) if $\{p_t\}_{t\in(0,\infty)}$ further satisfies the Discovered Preference Axiom, then $\varphi_{a,b}(t)\to 1$ as $t\to 0$ and it is strictly monotonic when not constant (increasing if $a\succ b$ and decreasing if $b\succ a$).

Proof (i) By Theorem 1, for each $t \in (0, \infty)$, there exists $v_t : X \to (0, \infty)$ such that

$$p_t(a, A) = \frac{v_t(a)}{\sum_{b \in A} v_t(b)} \qquad \forall a \in A \in \mathcal{A} \tag{6}$$

While p_∞ satisfies the Choice Axiom because it is defined by Continuity and p_t satisfies the Choice Axiom for all $t \in (0, \infty)$. By Theorem 2, there exist $v : X \to (0, \infty)$ and a rational choice correspondence $\Gamma : \mathcal{A} \to \mathcal{A}$ such that

$$p_\infty(a, A) = \begin{cases} \dfrac{v(a)}{\sum_{b \in \Gamma(A)} v(b)} & \text{if } a \in \Gamma(A) \\ 0 & \text{else} \end{cases}$$

for all $A \in \mathcal{A}$ and all $a \in A$; moreover, $\Gamma(A) = \operatorname{supp}(p_\infty)_A$ for all $A \in \mathcal{A}$. Given any $A \in \mathcal{A}$ and any $a, b \in \operatorname{supp}(p_\infty)_A = \Gamma(A)$, we have that $p_\infty(a, A), p_\infty(b, A) \neq 0$ and so $p_\infty(a, A) / p_\infty(b, A)$ is well defined. By Lemma 12 (the equivalence of the Choice Axiom and Independence from Irrelevant Alternatives), we have that

$$\frac{v(a)}{v(b)} = \frac{p_\infty(a, A)}{p_\infty(b, A)} = \frac{p_\infty(a, b)}{p_\infty(b, a)}$$

If $a \neq b$, since $v(a), v(b) \in (0, \infty)$, it must be the case that $p_\infty(a, b) \neq 0, 1$, then, by Asymptotic Uniformity, $p_\infty(a, b) = 1/2 = p_\infty(b, a)$ and so $v(a) = v(b)$. Since the choice of $A \in \mathcal{A}$ and $a, b \in \operatorname{supp}(p_\infty)_A$ was arbitrary, it follows that v is constant on $\Gamma(A)$ for all $A \in \mathcal{A}$, and so p_∞ is a uniform random choice rule. But then Corollary 3 guarantees that the relation defined on X by

$$a \succsim b \iff p_\infty(a, b) > 0 \iff p_\infty(a, b) \geq p_\infty(b, a) \tag{7}$$

is a weak order.[42]

(ii) Given any $t \in (0, \infty)$,

$$\varphi_{a,b}(t) = r_t(a, b) = \frac{p_t(a, b)}{p_t(b, a)} = \frac{v_t(a)}{v_t(b)} \in (0, \infty)$$

for all $a, b \in X$, thus $\varphi_{a,b} : (0, \infty) \to (0, \infty)$ is well defined. Moreover, by Continuity, $\varphi_{a,b}$ is continuous on $(0, \infty)$ too.

- If $a \sim b$, and *per contra* $\varphi_{a,b}(t) \neq 1$ for some $t \in (0, \infty)$, then
  - either $\varphi_{a,b}(t) > 1$, thus $p_t(a, b) > p_t(b, a)$ and, by Consistency with $s = \infty$, $p_\infty(a, b) > p_\infty(b, a)$, contradicting $a \sim b$,
  - or $\varphi_{a,b}(t) < 1$, thus $p_t(a, b) < p_t(b, a)$ and, by Consistency with $s = \infty$, $p_\infty(a, b) < p_\infty(b, a)$, contradicting $a \sim b$,

  so we can conclude $\varphi_{a,b}(t) = 1$ for all $t \in (0, \infty)$.

[42] And $a \sim b$ if and only if $p_\infty(a, b) = p_\infty(b, a)$, while $a \succ b$ if and only if $p_\infty(a, b) > p_\infty(b, a)$.

- If $a \succ b$, by (7) it follows $p_\infty(b,a) = 0$ and $p_\infty(a,b) = 1$, then

$$\lim_{t\to\infty} \varphi_{a,b}(t) = \lim_{t\to\infty} \frac{p_t(a,b)}{p_t(b,a)} = \frac{p_\infty(a,b)}{p_\infty(b,a)} = \infty$$

thus $\varphi_{a,b}$ diverges at ∞ as $t \to \infty$.

- If $b \succ a$, then (since $\varphi_{a,b} = 1/\varphi_{b,a}$ for all $a, b \in X$)

$$\lim_{t\to\infty} \varphi_{b,a}(t) = \infty \implies \lim_{t\to\infty} \varphi_{a,b}(t) = \lim_{t\to\infty} \frac{1}{\varphi_{b,a}(t)} = 0$$

thus $\varphi_{a,b}$ is vanishing as $t \to \infty$.

(iii) By Continuity, and the Discovered Preference Axiom,

$$\lim_{t\to 0} \varphi_{a,b}(t) = \lim_{t\to 0} \frac{p_t(a,b)}{p_t(b,a)} = \frac{p_0(a,b)}{p_0(b,a)} = r_0(a,b) = 1$$

Moreover, if $\varphi_{a,b}$ is not constant, then either $a \succ b$ or $b \succ a$. In the first case ($a \succ b$), $\varphi_{a,b}$ diverges at ∞ as $t \to \infty$, if – *per contra* – there are $s > t$ in $(0,\infty)$ such that $\varphi_{a,b}(s) \le \varphi_{a,b}(t)$, then by the Discovered Preference Axiom $\varphi_{a,b}(t) \le 1$, by the Discovered Preference Axiom again $\varphi_{a,b}(\tau) \le \varphi_{a,b}(t)$ for all $\tau \in (t,\infty)$, contradicting divergence. While, in the second case ($b \succ a$), $\varphi_{b,a}$ is strictly increasing by the previous argument, thus $\varphi_{a,b}$ is strictly decreasing. ∎

Lemma 16 *A random choice process $\{p_t\}_{t\in(0,\infty)}$ satisfies Positivity, the Choice Axiom, Continuity, Consistency, Asymptotic Uniformity, and Ordinal Boundedness if and only if there exist $u : X \to \mathbb{R}$ and an increasing bijective $\lambda : (0,\infty) \to (0,\infty)$ such that*

$$p_t(a,A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b\in A} e^{\lambda(t)u(b)}} \tag{8}$$

for all $A \in \mathcal{A}$, all $a \in A$, and all $t \in (0,\infty)$.

In this case,

(i) $p_0(a,A) = \dfrac{1}{|A|}$ *for all $A \in \mathcal{A}$ and all $a \in A$;*

(ii) $p_\infty(a,A) = \dfrac{1}{|\arg\max_A u|}\delta_a(\arg\max_A u)$ *for all $A \in \mathcal{A}$ and all $a \in A$;*

(iii) u is cardinally unique;

(iv) if u is non-constant, then λ is unique given u;

(v) $w = \lambda^{-1}$ is increasing, bijective, and $r_{\lambda^{-1}(t+s)}(a,b) = r_{\lambda^{-1}(t)}(a,b)\, r_{\lambda^{-1}(s)}(a,b)$ for all $a, b \in X$ and all $s, t \in (0,\infty)$, thus w realizes OB.

Proof Let $\{p_t\}$ be a random choice process that satisfies Positivity, the Choice Axiom, Continuity, Consistency, Asymptotic Uniformity, and Ordinal Boundedness. As in Lemma 15, we define, for all $a, b \in X$,

$$\varphi_{a,b}(t) = r_t(a,b) \qquad \forall t \in (0,\infty)$$

and we show that, thanks to the additional assumption of Ordinal Boundedness,

$$\varphi_{a,b}(w(t+s)) = \varphi_{a,b}(w(t))\,\varphi_{a,b}(w(s)) \qquad \forall t, s \in (0,\infty) \tag{9}$$

Three cases have to be considered, depending on whether $a \sim b$, $a \succ b$, or $b \succ a$ according to the weak order $\succsim$ defined in Lemma 15.

- If $a \sim b$, then $\varphi_{a,b}(t) = 1$ for all $t \in (0,\infty)$ and (9) holds.
- If $a \succ b$, then $\varphi_{a,b}$ is unbounded above and so is $\varphi_{a,b} \circ w : (0,\infty) \to (0,\infty)$. Moreover, by Ordinal Boundedness, there exists $M > 0$ such that

  $$|\varphi_{a,b}(w(t+s)) - \varphi_{a,b}(w(t))\,\varphi_{a,b}(w(s))| < M \qquad \forall t, s \in (0,\infty) \tag{10}$$

  But $(0,\infty)$ is a semigroup with respect to addition and $\varphi_{a,b} \circ w$ is unbounded above. Therefore, Baker (1980, Theorem 1) implies that (9) holds.
- Else, $b \succ a$, then the previous point shows

  $$\varphi_{b,a}(w(t+s)) = \varphi_{b,a}(w(t))\,\varphi_{b,a}(w(s))$$

  for all $t, s \in (0,\infty)$, but then

  $$\varphi_{a,b}(w(t+s)) = \frac{1}{\varphi_{b,a}(w(t+s))} = \frac{1}{\varphi_{b,a}(w(t))\,\varphi_{b,a}(w(s))} = \varphi_{a,b}(w(t))\,\varphi_{a,b}(w(s))$$

  for all $t, s \in (0,\infty)$, and (9) holds also in this case.

We conclude that the functional equation (9) holds for all $a, b \in X$. Continuity of $\varphi_{a,b} \circ w$, its strict positivity, and (9), imply that

$$\varphi_{a,b}(w(t)) = e^{h(a,b)t} \qquad \forall t \in (0,\infty)$$

for a unique $h(a,b) \in \mathbb{R}$ (see, e.g., Aczel, 1966, Theorem 2.1.2.1, p. 38). Setting $\lambda = w^{-1}$ it follows that $\varphi_{a,b}(s) = e^{h(a,b)\lambda(s)}$ for all $s \in (0,\infty)$, and λ is an increasing bijection like w.

Now fix some $a^* \in X$ and define $u : X \to \mathbb{R}$ by $u(x) = h(x, a^*)$ for all $x \in X$. Given any $t \in (0,\infty)$ and any $x, y \in X$, by (6),

$$\varphi_{x,y}(t) = \frac{v_t(x)}{v_t(y)} = \frac{v_t(x)}{v_t(a^*)}\frac{v_t(a^*)}{v_t(y)} = \frac{\varphi_{x,a^*}(t)}{\varphi_{y,a^*}(t)} = \frac{e^{h(x,a^*)\lambda(t)}}{e^{h(y,a^*)\lambda(t)}} = \frac{e^{u(x)\lambda(t)}}{e^{u(y)\lambda(t)}}$$

Therefore, for every $t \in (0,\infty)$, $A \in \mathcal{A}$, and $a \in A$, arbitrarily choosing $y \in A$,

$$p_t(a,A) = \frac{v_t(a)}{\sum_{b\in A} v_t(b)} = \frac{\frac{v_t(a)}{v_t(y)}}{\sum_{b\in A}\frac{v_t(b)}{v_t(y)}} = \frac{\frac{e^{u(a)\lambda(t)}}{e^{u(y)\lambda(t)}}}{\sum_{b\in A}\frac{e^{u(b)\lambda(t)}}{e^{u(y)\lambda(t)}}} = \frac{e^{\lambda(t)u(a)}}{\sum_{b\in A} e^{\lambda(t)u(b)}}$$

and (8) holds.

Annotation 1 *Notice that from the axioms we derived (8) with* $\lambda = w^{-1}$.

Since p_0 and p_∞ are defined by Continuity and λ vanishes as $t \to 0$ while it diverges at ∞ as $t \to \infty$, then (8) implies (i) and (ii).

As to uniqueness of u and λ, notice that, if also $\bar{u}$ and $\bar{\lambda}$ represent $\{p_t\}$ in the sense of (8), then

$$e^{\lambda(t)(u(a)-u(b))} = \frac{e^{\lambda(t)u(a)}}{e^{\lambda(t)u(b)}} = r_t(a,b) = \frac{p_t(a,b)}{p_t(b,a)} = e^{\bar{\lambda}(t)(\bar{u}(a)-\bar{u}(b))}$$

for all $t \in (0,\infty)$ and all $a, b \in X$. Therefore $\lambda(t)(u(a) - u(b)) = \bar{\lambda}(t)(\bar{u}(a) - \bar{u}(b))$ for all $t \in (0,\infty)$ and all $a, b \in X$. Choose $t = 1$ and arbitrarily fix $b^* \in X$ to conclude that

$$\bar{u}(a) = \frac{\lambda(1)}{\bar{\lambda}(1)}(u(a) - u(b^*)) + \bar{u}(b^*) = ku(a) + h \qquad \forall a \in A$$

with $k > 0$ and $h \in \mathbb{R}$. Point (iii) follows. If u is not constant, by choosing $a, b \in X$ with $u(a) \neq u(b)$, the previous argument, yields $\lambda(t)(u(a) - u(b)) = \bar{\lambda}(t)(ku(a) - ku(b))$ for all $t \in (0,\infty)$, so that $\bar{\lambda} = k^{-1}\lambda$. If $\bar{u} = u$ is given, then $k = 1$ and $\bar{\lambda} = \lambda$. This proves point (iv).

Clearly, λ^{-1} is increasing and bijective, and by (8)

$$r_{\lambda^{-1}(t+s)}(a,b) = r_{\lambda^{-1}(t)}(a,b)\, r_{\lambda^{-1}(s)}(a,b) \qquad \forall a, b \in X \quad \forall t, s \in (0,\infty)$$

which implies point (v).

The rest is trivial. ■

Lemma 17 *Let* $\{p_t\}_{t\in(0,\infty)}$ *be a random choice process that satisfies Positivity, the Choice Axiom, Continuity, the Discovered Preference Axiom, Asymptotic Uniformity, and Independence from Compared Alternatives. Then:*

(i) $\{p_t\}_{t\in(0,\infty)}$ *satisfies Consistency;*

(ii) if $\hat{a}, \hat{b} \in X$ *and* $\hat{t} \in (0,\infty)$ *are such that* $r_{\hat{t}}(\hat{a},\hat{b}) > 1$, *the function*

$$\lambda(t) = \ell_t(\hat{a},\hat{b}) \qquad \forall t \in (0,\infty)$$

is an increasing bijection from $(0,\infty)$ *to* $(0,\infty)$ *such that*

$$r_{\lambda^{-1}(t+s)}(a,b) = r_{\lambda^{-1}(t)}(a,b)\, r_{\lambda^{-1}(s)}(a,b) \qquad \forall a, b \in X \quad \forall t, s \in (0,\infty)$$

(iii) $\{p_t\}_{t\in(0,\infty)}$ *satisfies Ordinal Boundedness.*

Proof (i) Let $a, b \in X$ and $t \in (0,\infty)$ be such that $r_t(a,b) > 1$. Then the Discovered Preference Axiom implies $r_s(a,b) > r_t(a,b)$, whence $r_s(a,b) > 1$, for all $s \in (t,\infty)$. By the same argument, $t < \tau < s < \infty$ implies $r_\tau(a,b) > 1$ and the Discovered Preference Axiom again implies $r_s(a,b) > r_\tau(a,b)$. Therefore $r_t(a,b) > 1$ implies $\tau \mapsto r_\tau(a,b)$ is strictly increasing on $[t,\infty)$; by Continuity $r_\infty(a,b) = \lim_{\tau\to\infty} r_\tau(a,b) \geq r_t(a,b) > 1$. Thus, we have that $r_t(a,b) > 1$ implies $r_s(a,b) > 1$ for all $s \in (t,\infty]$ and Consistency holds.[43]

(ii) Lemma 15 guarantees that, given any $a, b \in X$, the function $\varphi_{a,b} : (0,\infty) \to (0,\infty)$ satisfies one and only one of the following conditions:[44]

[43] Notice that this argument does not use ICA.
[44] Trichotomy follows from the fact that $\succsim$ is a weak order.

$a \succ b)$ $\varphi_{a,b}$ is continuous and strictly increasing with $\lim_{t\to 0} \varphi_{a,b}(t) = 1$ and $\lim_{t\to\infty} \varphi_{a,b}(t) = \infty$. In particular, $\varphi_{a,b}(0,\infty) = (1,\infty)$.

$a \sim b)$ $\varphi_{a,b}$ is constantly equal to 1. In particular, $\varphi_{a,b}(0,\infty) = \{1\}$.

$b \succ a)$ $\varphi_{a,b}$ is continuous and strictly decreasing with $\lim_{t\to 0} \varphi_{a,b}(t) = 1$ and $\lim_{t\to\infty} \varphi_{a,b}(t) = 0$. In particular, $\varphi_{a,b}(0,\infty) = (0,1)$.

If $r_{\hat{t}}(a,b) > 1$ (resp., $= 1$) for some $\hat{t} \in (0,\infty)$, then we are in the first (resp., second) case.

Therefore, if $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0,\infty)$ are such that $r_{\hat{t}}(\hat{a},\hat{b}) > 1$, then $\varphi_{\hat{a},\hat{b}} : (0,\infty) \to (1,\infty)$ is an increasing bijection, and

$$\lambda(t) = \ell_t(\hat{a},\hat{b}) = \ln r_t(\hat{a},\hat{b}) = \ln \varphi_{\hat{a},\hat{b}}(t) \qquad \forall t \in (0,\infty)$$

defines an increasing bijection from $(0,\infty)$ to $(0,\infty)$.

Moreover, given any $t, s \in (0,\infty)$, we have

$$\ell_{\lambda^{-1}(t+s)}(\hat{a},\hat{b}) = \lambda\left(\lambda^{-1}(t+s)\right) = t+s = \lambda\left(\lambda^{-1}(t)\right) + \lambda\left(\lambda^{-1}(s)\right) = \ell_{\lambda^{-1}(t)}(\hat{a},\hat{b}) + \ell_{\lambda^{-1}(s)}(\hat{a},\hat{b})$$

thus

$$\ell_{\lambda^{-1}(t+s)}(\hat{a},\hat{b}) = \ell_{\lambda^{-1}(t)}(\hat{a},\hat{b}) + \ell_{\lambda^{-1}(s)}(\hat{a},\hat{b}) \qquad \forall t, s \in (0,\infty) \tag{11}$$

Next we show that (11) and Independence from Compared Alternatives imply

$$\ell_{\lambda^{-1}(t+s)}(a,b) = \ell_{\lambda^{-1}(t)}(a,b) + \ell_{\lambda^{-1}(s)}(a,b) \qquad \forall a, b \in X \quad \forall t, s \in (0,\infty) \tag{12}$$

exponentiation can then be used to conclude the proof of this point.

By ICA, given any $c, d, x, y \in X$ and any $s, t, \tau \in (0,\infty)$ such that $r_\tau(c,d) > 1$ and $r_\tau(x,y) > 1$, it follows

$$r_\tau(c,d) > r_t(c,d)\, r_s(c,d) \iff r_\tau(x,y) > r_t(x,y)\, r_s(x,y)$$

(the roles of (c,d) and (x,y) are symmetric). As observed above, if $r_{\hat{t}}(c,d) > 1$ and $r_{\hat{s}}(x,y) > 1$ for some $\hat{t}, \hat{s} \in (0,\infty)$, then $r_\tau(c,d) > 1$ and $r_\tau(x,y) > 1$ for all $\tau \in (0,\infty)$. Therefore, under the other assumptions of this lemma, ICA implies that: given any $c, d, x, y \in X$,

- if $r_{\hat{t}}(c,d) > 1$ and $r_{\hat{s}}(x,y) > 1$ for some $\hat{t}, \hat{s} \in (0,\infty)$, then, given any $s, t, \tau \in (0,\infty)$, it follows
$$r_\tau(c,d) \le r_t(c,d)\, r_s(c,d) \iff r_\tau(x,y) \le r_t(x,y)\, r_s(x,y)$$
and, by passing to the logarithms, we have
$$\ell_\tau(c,d) \le \ell_t(c,d) + \ell_s(c,d) \iff \ell_\tau(x,y) \le \ell_t(x,y) + \ell_s(x,y) \tag{13}$$

Moreover, as we have shown above, the functions $f(t) = \ell_t(c,d)$ and $g(t) = \ell_t(x,y)$ are increasing bijections from $(0,\infty)$ to $(0,\infty)$ and (13) implies

$$\tau \le f^{-1}(f(t) + f(s)) \iff \tau \le g^{-1}(g(t) + g(s))$$

for all $s, t, \tau \in (0, \infty)$. But then $f^{-1}(f(t) + f(s)) = g^{-1}(g(t) + g(s))$ for all $s, t \in (0, \infty)$. Hence, for all $s, t, \tau \in (0, \infty)$,

$$\tau = f^{-1}(f(t) + f(s)) \iff \tau = g^{-1}(g(t) + g(s))$$

Therefore:

- if $r_{\hat{t}}(c, d) > 1$ and $r_{\hat{s}}(x, y) > 1$ for some $\hat{t}, \hat{s} \in (0, \infty)$, then, given any $s, t, \tau \in (0, \infty)$, it holds
$$\ell_\tau(c, d) = \ell_t(c, d) + \ell_s(c, d) \iff \ell_\tau(x, y) = \ell_t(x, y) + \ell_s(x, y)$$
- if $r_{\hat{t}}(c, d) > 1$ and $r_{\hat{s}}(x, y) < 1$ for some $\hat{t}, \hat{s} \in (0, \infty)$, then, $r_{\hat{s}}(y, x) > 1$ and, given any $s, t, \tau \in (0, \infty)$, it holds
$$\begin{aligned} \ell_\tau(c, d) = \ell_t(c, d) + \ell_s(c, d) &\iff \ell_\tau(y, x) = \ell_t(y, x) + \ell_s(y, x) \\ &\iff -\ell_\tau(y, x) = -\ell_t(y, x) - \ell_s(y, x) \\ &\iff \ell_\tau(x, y) = \ell_t(x, y) + \ell_s(x, y) \end{aligned}$$
- if $r_{\hat{t}}(c, d) > 1$ and $r_{\hat{s}}(x, y) = 1$ for some $\hat{t}, \hat{s} \in (0, \infty)$, then, as we observed above, $r_\tau(x, y) = r_t(x, y) = r_s(x, y) = 1$, for all $s, t, \tau \in (0, \infty)$, thus , given any $s, t, \tau \in (0, \infty)$, it holds
$$\ell_\tau(c, d) = \ell_t(c, d) + \ell_s(c, d) \implies \ell_\tau(x, y) = \ell_t(x, y) + \ell_s(x, y)$$

Summing up, if $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0, \infty)$ are such that $r_{\hat{t}}(\hat{a}, \hat{b}) > 1$, then, given any $s, t, \tau \in (0, \infty)$

$$\ell_\tau(\hat{a}, \hat{b}) = \ell_t(\hat{a}, \hat{b}) + \ell_s(\hat{a}, \hat{b}) \implies \ell_\tau(x, y) = \ell_t(x, y) + \ell_s(x, y) \qquad \forall x, y \in X \tag{14}$$

which yields the desired relation between (11) and (12).

(iii) If there exist $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0, \infty)$ such that $r_{\hat{t}}(\hat{a}, \hat{b}) > 1$, and the increasing bijection $\lambda : (0, \infty) \to (0, \infty)$ defined in the previous point is considered, then $w = \lambda^{-1}$ realizes OB. Otherwise, $r_t(a, b) = 1$ for all $a, b \in X$ and all $t \in (0, \infty)$,[45] and Boundedness is satisfied (*a fortiori* Ordinal Boundedness is satisfied too). ■

Theorem 7 follows immediately, **Theorem 4** is obtained as the special case in which $w(t) = t$ for all $t \in (0, \infty)$,[46] **Propositions 8** and **9** are simple corollaries of Theorem 7 (we leave the routine proofs to the reader).

The next proposition provides the form **Proposition 5** takes in the more general case of Multinomial Logit Processes. In reading it notice that points 2 and 3 can be equivalently stated in terms of λ or $w = \lambda^{-1}$.

Proposition 18 *Let X be a convex set and $\{p_t\}_{t \in (0,\infty)}$ be a Multinomial Logit Process with utility u and accuracy λ. The following conditions are equivalent:*

[45] If $r_{\hat{t}}(\hat{a}, \hat{b}) < 1$ for some $\hat{a}, \hat{b} \in X$ and $\hat{t} \in (0, \infty)$, then $r_{\hat{t}}(\hat{b}, \hat{a}) > 1$.

[46] See Annotation 1.

1. *u is affine;*

2. *there exists $t \in (0,\infty)$ such that*

$$p_t(a,b) = p_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(\alpha a + (1-\alpha)b, b) \tag{15}$$

for all $a, b \in X$ and all $\alpha \in (0,1)$;

3. *given any $t \in (0,\infty)$ and any $c \in X$,*

$$p_t(a,A) = p_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(\alpha a + (1-\alpha)c, \alpha A + (1-\alpha)c)$$

for all $A \in \mathcal{A}$, all $a \in A$, and all $\alpha \in (0,1)$.

Proof *2 implies 1.* Let $t \in (0,\infty)$ be such that (15) holds, then for all $a \neq b$ in X, and all $\alpha \in (0,1)$,

$$\begin{aligned} p_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(b, \alpha a + (1-\alpha)b) &= 1 - p_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(\alpha a + (1-\alpha)b, b) \\ &= 1 - p_t(a,b) \\ &= p_t(b,a) \end{aligned}$$

whence

$$\begin{aligned} e^{\frac{\lambda(t)}{\alpha}[u(\alpha a+(1-\alpha)b)-u(b)]} &= r_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(\alpha a + (1-\alpha)b, b) \\ &= r_t(a,b) = e^{\lambda(t)[u(a)-u(b)]} \end{aligned}$$

so

$$\begin{aligned} \frac{1}{\alpha}[u(\alpha a + (1-\alpha)b) - u(b)] &= u(a) - u(b) \\ u(\alpha a + (1-\alpha)b) &= \alpha u(a) + (1-\alpha)u(b) \end{aligned}$$

which delivers affinity of u.

1 implies 3. Given any $t \in (0,\infty)$ and any $c \in X$, we have that, for all $A \in \mathcal{A}$, all $a \in A$, and all $\alpha \in (0,1)$,

$$\begin{aligned} p_{\lambda^{-1}\left(\frac{\lambda(t)}{\alpha}\right)}(\alpha a + (1-\alpha)c, \alpha A + (1-\alpha)c) &= \frac{e^{\frac{\lambda(t)}{\alpha}u(\alpha a+(1-\alpha)c)}}{\sum_{b\in A} e^{\frac{\lambda(t)}{\alpha}u(\alpha b+(1-\alpha)c)}} = \frac{e^{\frac{\lambda(t)}{\alpha}\alpha u(a)+\frac{\lambda(t)}{\alpha}(1-\alpha)u(c)}}{\sum_{b\in A} e^{\frac{\lambda(t)}{\alpha}\alpha u(b)+\frac{\lambda(t)}{\alpha}(1-\alpha)u(c)}} \\ &= \frac{e^{\lambda(t)u(a)}}{\sum_{b\in A} e^{\lambda(t)u(b)}} = p_t(a,A) \end{aligned}$$

3 implies 2 is trivial. ■

Proof of Proposition 6 Arbitrarily choose $A \in \mathcal{A}$. If we prove that, given any $h \in \mathbb{R}$ such that $\varnothing \subsetneq \{c \in A : u(c) \geq h\} \subsetneq A$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\{c \in A : u(c) \geq h\}, A) > 0 \qquad \forall t \in (0,\infty) \tag{16}$$

then the statement follows. Indeed, in this case, the function $t \mapsto p_t(\{c \in A : u(c) \geq h\}, A)$ is strictly increasing on $(0, \infty)$ for all $h \in \mathbb{R}$ such that $\varnothing \subsetneq \{c \in A : u(c) \geq h\} \subsetneq A$, while it is constantly equal to 0 or 1 if h is such that $\{c \in A : u(c) \geq h\} = \varnothing$ or $\{c \in A : u(c) \geq h\} = A$.

Next we show that (16) holds. Set $[u \geq h] = \{c \in A : u(c) \geq h\}$, assume $h \in \mathbb{R}$ is such that $\varnothing \subsetneq [u \geq h] \subsetneq A$, and notice that in this case $[u < h]$ must be non-empty. Given any $t \in (0, \infty)$, with the abbreviation $\sum\limits_{u(c) \geq h} = \sum\limits_{c \in A : u(c) \geq h}$, we have

$$
\begin{aligned}
& 0 < \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sum\limits_{u(c)\geq h} e^{tu(c)}}{\sum\limits_{b\in A} e^{tu(b)}}\right) = \frac{\left(\sum\limits_{b\in A} e^{tu(b)}\right)\sum\limits_{u(c)\geq h} u(c)\, e^{tu(c)} - \sum\limits_{u(c)\geq h} e^{tu(c)}\left(\sum\limits_{b\in A} u(b)\, e^{tu(b)}\right)}{\left(\sum\limits_{b\in A} e^{tu(b)}\right)^2} \\
\iff\ & 0 < \sum_{u(c)\geq h} u(c)\, e^{tu(c)} \sum_{u(b)<h} e^{tu(b)} - \sum_{u(c)<h} u(c)\, e^{tu(c)} \sum_{u(b)\geq h} e^{tu(b)} \\
\iff\ & 0 < \frac{\sum\limits_{u(c)\geq h} u(c)\, e^{tu(c)}}{\sum\limits_{u(b)\geq h} e^{tu(b)}} - \frac{\sum\limits_{u(c)<h} u(c)\, e^{tu(c)}}{\sum\limits_{u(b)<h} e^{tu(b)}} \\
\iff\ & 0 < \sum_{u(c)\geq h} u(c)\left(\frac{e^{tu(c)}}{\sum\limits_{u(b)\geq h} e^{tu(b)}}\right) - \sum_{u(c)<h} u(c)\left(\frac{e^{tu(c)}}{\sum\limits_{u(b)<h} e^{tu(b)}}\right) \\
\iff\ & 0 < \sum_{c\in[u\geq h]} u(c)\, p_t(c, [u\geq h]) - \sum_{c\in[u<h]} u(c)\, p_t(c, [u<h])
\end{aligned}
$$

and this concludes the proof, because – in the last step above – the minuend is an average (i.e., a convex combination) of values $u(c) \geq h$, so it cannot be smaller than h itself, the subtrahend is an average of values $u(c) < h$, so it is strictly smaller than h itself, hence the difference on the right hand side is strictly positive, irrespectively of the value of t. ■

Proof of Theorem 10 *Only if.* Since $\{p_t\}_{t\in T}$ satisfies Positivity and the Choice Axiom, by Theorem 1, for each $t \in T$, there exists $u_t : X \to \mathbb{R}$ such that

$$
p_t(a, A) = \frac{e^{u_t(a)}}{\sum_{b\in A} e^{u_t(b)}} \qquad \forall a \in A \in \mathcal{A} \tag{17}
$$

Arbitrarily choose $\bar{c} \in X$ and replace each u_t with $u_t - u_t(\bar{c})$. With this, $u_t(\bar{c}) = 0$ for all $t \in T$ and (17) still holds.

If, for all $t \in T$, u_t is constant, then MLP holds (e.g., with $u(x) = 0$ for all $x \in X$). Otherwise, there exists $\bar{t} \in T$ such that $u_{\bar{t}}$ is not constant, so that $u_{\bar{t}}(\bar{b}) \neq 0 = u_{\bar{t}}(\bar{c})$ for some $\bar{b} \in X$. This implies that

$$
\frac{\ell_{\bar{t}}(a, \bar{c})}{\ell_{\bar{t}}(\bar{b}, \bar{c})} = \frac{u_{\bar{t}}(a) - u_{\bar{t}}(\bar{c})}{u_{\bar{t}}(\bar{b}) - u_{\bar{t}}(\bar{c})} = \frac{u_{\bar{t}}(a)}{u_{\bar{t}}(\bar{b})}
$$

is a well defined real number for all $a \in X$. By Log-odds Ratio Invariance, $\ell_t(a, \bar{c}) / \ell_t(\bar{b}, \bar{c})$ is well defined too, and

$$
\frac{u_t(a)}{u_t(\bar{b})} = \frac{\ell_t(a, \bar{c})}{\ell_t(\bar{b}, \bar{c})} = \frac{\ell_{\bar{t}}(a, \bar{c})}{\ell_{\bar{t}}(\bar{b}, \bar{c})} = \frac{u_{\bar{t}}(a)}{u_{\bar{t}}(\bar{b})} \in \mathbb{R} \qquad \forall (a, t) \in X \times T
$$

Therefore, $u_t(\bar{b}) \neq 0 = u_t(\bar{c})$ for all $t \in T$, and

$$u_t(a) = \frac{u_t(\bar{b})}{u_{\bar{t}}(\bar{b})} u_{\bar{t}}(a) \qquad \forall (a,t) \in X \times T. \tag{18}$$

Consider the case in which $u_{\bar{t}}(\bar{b}) > 0 = u_{\bar{t}}(\bar{c})$. If $t > \bar{t}$, then (17) and Weak Consistency imply

$$p_{\bar{t}}(\bar{b},\bar{c}) > p_{\bar{t}}(\bar{c},\bar{b}) \implies p_t(\bar{b},\bar{c}) > p_t(\bar{c},\bar{b}) \implies u_t(\bar{b}) > u_t(\bar{c}) = 0$$

thus $u_t(\bar{b})/u_{\bar{t}}(\bar{b}) > 0$. This is clearly true also if $t = \bar{t}$. Else $t < \bar{t}$, assume *per contra* $u_t(\bar{b}) < 0 = u_t(\bar{c})$, then (17) and Weak Consistency imply

$$p_t(\bar{c},\bar{b}) > p_t(\bar{b},\bar{c}) \implies p_{\bar{t}}(\bar{c},\bar{b}) > p_{\bar{t}}(\bar{b},\bar{c}) \implies u_{\bar{t}}(\bar{b}) < u_{\bar{t}}(\bar{c}) = 0$$

a contradiction. Thus $u_t(\bar{b})/u_{\bar{t}}(\bar{b}) > 0$ holds for all $t \in T$ provided $u_{\bar{t}}(\bar{b}) > 0$. It is easy to show that the same is true if $u_{\bar{t}}(\bar{b}) < 0$.[47] This shows that

$$\begin{array}{rccc} \lambda: & T & \to & (0,\infty) \\ & t & \mapsto & \frac{u_t(\bar{b})}{u_{\bar{t}}(\bar{b})} \end{array}$$

is well defined. Moreover, the function $u = u_{\bar{t}} : X \to \mathbb{R}$ is non-constant and relation (18) implies

$$u_t(a) = \lambda(t)\, u(a) \qquad \forall (a,t) \in X \times T$$

which together with (17) shows that the axioms imply representation MLP.

If. It is easy to verify that the converse implication holds too. For the sake of completeness, we check that representation MLP implies Log-odds Ratio Invariance. Let $t, s \in T$ and $a, b, c, x, y \in X$. Notice that

$$\ell_t(x,y) = \lambda(t)\,[u(x) - u(y)]$$

so that $\ell_t(x,y) = 0$ if and only if $u(x) = u(y)$, and the same considerations hold with s in place of t. Assume $\ell_s(a,c)/\ell_s(b,c)$ is well defined:

- If $\ell_s(b,c) = 0$, then $u(b) = u(c)$ and $\ell_s(a,c) \neq 0$, so $u(a) \neq u(c)$, then
  - $\ell_s(a,c) = \lambda(s)\,[u(a) - u(c)] \neq 0$ and since $\lambda(s) > 0$, then

$$\frac{\ell_s(a,c)}{\ell_s(b,c)} = \frac{\lambda(s)\,[u(a) - u(c)]}{0} = \frac{u(a) - u(c)}{0}$$

[47] If $t > \bar{t}$, then, by (17) and Weak Consistency, we have

$$u_{\bar{t}}(\bar{b}) < 0 = u_{\bar{t}}(\bar{c}) \implies p_{\bar{t}}(\bar{c},\bar{b}) > p_{\bar{t}}(\bar{b},\bar{c}) \implies p_t(\bar{c},\bar{b}) > p_t(\bar{b},\bar{c}) \implies u_t(\bar{b}) < u_t(\bar{c}) = 0$$

thus $u_t(\bar{b})/u_{\bar{t}}(\bar{b}) > 0$. This is clearly true also if $t = \bar{t}$. Else $t < \bar{t}$, assume *per contra* $u_t(\bar{b}) > 0 = u_t(\bar{c})$, then (17) and Weak Consistency imply

$$p_t(\bar{b},\bar{c}) > p_t(\bar{c},\bar{b}) \implies p_{\bar{t}}(\bar{b},\bar{c}) > p_{\bar{t}}(\bar{c},\bar{b}) \implies u_{\bar{t}}(\bar{b}) > u_{\bar{t}}(\bar{c}) = 0$$

a contradiction. Thus $u_t(\bar{b})/u_{\bar{t}}(\bar{b}) > 0$ holds for all $t \in T$.

- $\ell_t(b,c) = \lambda(t)[u(b) - u(c)] = 0$, because $u(b) = u(c)$,
- $\ell_t(a,c) = \lambda(t)[u(a) - u(c)] \neq 0$, because $u(a) \neq u(c)$, and since $\lambda(t) > 0$, then

$$\frac{\ell_t(a,c)}{\ell_t(b,c)} = \frac{\lambda(t)[u(a) - u(c)]}{0} = \frac{u(a) - u(c)}{0} = \frac{\ell_s(a,c)}{\ell_s(b,c)}$$

- Else $\ell_s(b,c) \neq 0$, then $u(b) \neq u(c)$ and $\ell_t(b,c) = \lambda(t)[u(b) - u(c)] \neq 0$, so that

$$\frac{\ell_s(a,c)}{\ell_s(b,c)} = \frac{\lambda(s)[u(a) - u(c)]}{\lambda(s)[u(b) - u(c)]} = \frac{u(a) - u(c)}{u(b) - u(c)} = \frac{\lambda(t)[u(a) - u(c)]}{\lambda(t)[u(b) - u(c)]} = \frac{\ell_t(a,c)}{\ell_t(b,c)}$$

The case in which $\ell_t(a,c)/\ell_t(b,c)$ is well defined is analogous.

As to uniqueness of u and λ, notice that, if also $\bar{u}$ and $\bar{\lambda}$ represent $\{p_t\}_{t\in T}$ in the sense of MLP, then

$$e^{\lambda(t)(u(a)-u(b))} = r_t(a,b) = e^{\bar{\lambda}(t)(\bar{u}(a)-\bar{u}(b))}$$

for all $t \in T$ and all $a,b \in X$. Therefore $\lambda(t)(u(a) - u(b)) = \bar{\lambda}(t)(\bar{u}(a) - \bar{u}(b))$ for all $t \in T$ and all $a,b \in X$. Arbitrarily choose $t^* \in T$ and $b^* \in X$ to conclude that

$$\bar{u}(a) = \frac{\lambda(t^*)}{\bar{\lambda}(t^*)}(u(a) - u(b^*)) + \bar{u}(b^*) = ku(a) + h \qquad \forall a \in X$$

with $k > 0$ and $h \in \mathbb{R}$. Since the converse is also true,[48] cardinal uniqueness of u follows. Moreover, if u is not constant, choosing $a,b \in X$ with $u(a) \neq u(b)$, the previous argument yields $\lambda(t)(u(a) - u(b)) = \bar{\lambda}(t)(ku(a) - ku(b))$ for all $t \in T$, so that $\bar{\lambda} = k^{-1}\lambda$ if $\bar{u} = ku + h$. Finally, if $\bar{u} = u$ is given, so that $k = 1$, it follows $\bar{\lambda} = \lambda$. ■

Inspection of the proof shows that it is possible to obtain the softmax representation for any non-singleton index set T by replacing Weak Consistency with:

Strong Consistency *Given any $a,b \in X$,*

$$p_t(a,b) > p_t(b,a) \implies p_s(a,b) > p_s(b,a)$$

for all s and t in T.

As discussed in Section 5, in the theorem below we read t as type.

Theorem 19 *A collection $\{p_t\}_{t\in T}$ of random choice rules satisfies Positivity, the Choice Axiom, Strong Consistency, and Log-odds Ratio Invariance if and only if there exist $u : X \to \mathbb{R}$ and $\lambda : T \to (0,\infty)$ such that*

$$p_t(a,A) = \frac{e^{\lambda(t)u(a)}}{\sum_{b\in A} e^{\lambda(t)u(b)}} \qquad \text{(softmax)}$$

for all $A \in \mathcal{A}$, all $a \in A$, and all $t \in T$.

In this case, u is cardinally unique, and λ is unique given u unless the latter is constant.

[48] That is, if $\bar{u} = ku + h$ with $k > 0$ and $h \in \mathbb{R}$, then $\bar{u}$ and $\bar{\lambda} = k^{-1}\lambda$ represent $\{p_t\}_{t\in T}$ in the sense of MLP.

A.3 Proofs of the results of Section 4

Proof of Proposition 11 This result is essentially due to Barker (1965), here we report a simple proof for completeness. Given any $t, \lambda(t) \in (0,\infty)$, the explicit form of P_t is

$$P_t(a \mid b) = \begin{cases} Q(a \mid b)\dfrac{e^{\lambda(t)u(a)}}{e^{\lambda(t)u(a)} + e^{\lambda(t)u(b)}} & \text{if } a \neq b \\ 1 - \sum_{c\in A\setminus\{b\}} Q(c \mid b)\dfrac{e^{\lambda(t)u(c)}}{e^{\lambda(t)u(c)} + e^{\lambda(t)u(b)}} & \text{if } a = b \end{cases}$$

and so P_t is irreducible because Q is. Moreover, again by irreducibility of Q,

$$\sum_{c\in A\setminus\{b\}} Q(c \mid b) > 0 \qquad \forall b \in A$$

(otherwise it would follow $Q(b \mid b) = 1$ for some $b \in A$, violating irreducibility). But then $P_t(b \mid b) > 0$ for all $b \in A$, which implies aperiodicity of P_t.

Next we show stationarity of $p_t(\cdot, A)$. Notice that, for all $a \neq b$ in A,

$$\begin{aligned} P_t(a \mid b)\, p_t(b, A) &= Q(a \mid b)\frac{e^{\lambda(t)u(a)}}{e^{\lambda(t)u(a)} + e^{\lambda(t)u(b)}}\frac{e^{\lambda(t)u(b)}}{\sum_{x\in A} e^{\lambda(t)u(x)}} \\ &= \frac{Q(a \mid b)}{\sum_{x\in A} e^{\lambda(t)u(x)}}\frac{e^{\lambda(t)(u(a)+u(b))}}{e^{\lambda(t)u(a)} + e^{\lambda(t)u(b)}} = P_t(b \mid a)\, p_t(a, A) \end{aligned}$$

while if $a = b$, then $P_t(a \mid b)\, p_t(b, A) = P_t(b \mid a)\, p_t(a, A)$ is obvious, thus

$$P_t(a \mid b)\, p_t(b, A) = P_t(b \mid a)\, p_t(a, A) \qquad \forall a, b \in A$$

Therefore, P_t is reversible with respect to $p_t(\cdot, A)$ and *a fortiori* $P_t p_t(\cdot, A) = p_t(\cdot, A)$ (see, e.g., Madras, 2002, Proposition 4.4).

Finally, since P_t is aperiodic and irreducible, then stationarity of $p_t(\cdot, A)$ implies $P_t^n \mu \to p_t(\cdot, A)$ as $n \to \infty$ for all $\mu \in \Delta(A)$ (see, e.g., Madras, 2002, Theorem 4.2). ∎

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