# Ambiguity Aversion and Wealth Effects*

Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci
Università Bocconi and IGIER

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**Abstract**

We study how changes in wealth affect ambiguity attitudes. We define a decision maker decreasing (resp., increasing) absolute ambiguity averse if he becomes less (resp., more) ambiguity averse as he becomes richer. Our definition is behavioral. We provide different characterizations of these attitudes for a large class of preferences: monotone and continuous preferences which satisfy risk independence. We then specialize our results for different subclasses of preferences. Inter alia, our characterizations provide alternative ways to test experimentally the validity of some of the models of choice under uncertainty.

## 1 Introduction

Beginning with the seminal work of David Schmeidler, several choice models have been proposed in the past thirty years in the large literature on choice under uncertainty that deals with ambiguity, that is, with Ellsberg-type phenomena.[1] At the same time, many papers have investigated the economic consequences of ambiguity. Our purpose in this paper is to study a basic economic problem: How the ambiguity attitudes of a decision maker change as his wealth changes. In other words, our purpose is to study absolute and relative ambiguity attitudes in terms of changes in *wealth*. This is an alternative and complementary approach to some of the existing literature, discussed at the end of this Introduction, which in contrast studies how ambiguity attitudes change in terms of utility shifts rather than wealth

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[1] See Gilboa and Marinacci [19] for a survey.

shifts. Put differently, this literature studies absolute and relative ambiguity attitudes in terms of changes in *utility*.

To fix ideas and understand our main motivation, one should think of how central is in many fields of Economics the relationship between wealth and agents' attitudes toward risk (e.g., portfolio allocation problems and insurance demand). For example, in his seminal work [3, p. 96], Arrow, in discussing measures of absolute and relative risk attitudes, mentions that "The behaviour of these measures as wealth changes is of the greatest importance for prediction of economic reactions in the presence of uncertainty".

To the best of our knowledge, no systematic study has been done in exploring a similar relation between *wealth* and *ambiguity* attitudes, despite the large and growing use in applications of models that are nonneutral toward ambiguity. In this context, Arrow's comment would seem to apply all the more. The challenge of our work, compared to the analysis done under risk by Arrow and Pratt, is that their study has been restricted to the expected utility model. Under ambiguity, instead, there are by now several alternative models, thus moving the analysis well beyond expected utility. In characterizing how ambiguity attitudes change with wealth, our results might provide some guidance in choosing between these models, as the standard theory of absolute risk aversion of Arrow and Pratt provides guidance in the choice of the von Neumann-Morgenstern utility function. For example, our results will show that a researcher who believes that agents are not constant absolute ambiguity averse – be that due to experimental evidence and/or personal introspection as for Arrow's assumption of decreasing absolute risk aversion – can rule out the use of some models: for example, $\alpha$-maxmin, Choquet expected utility, and variational preferences under risk neutrality. Similarly, for a researcher relying on the smooth ambiguity model, behavioral assumptions on absolute and relative ambiguity attitudes translate into corresponding choices of the model's parameters. For instance, if risk attitudes are assumed to be CRRA and risk averse, as common in Macroeconomics,[2] and relative ambiguity attitudes are assumed to be constant as well (irrespective of the prior $\mu$), then our results yield that $\phi$ must be either CARA or CRRA, depending on the von Neumann-Morgenstern function being either the logarithm or the power function.

Finally, our work provides alternative and useful methods to falsify models of choice under ambiguity as well as testable implications. For example, on the one hand, under the assumption that agents are CARA,[3] falsifying our notion of constant absolute ambiguity attitudes yields that preferences cannot be invariant biseparable preferences (e.g., $\alpha$-maxmin and Choquet expected utility). On the other hand, under the assumption that agents are

[2] More formally, consequences are elements of $(0,\infty)$ and the von Neumann-Morgenstern utility function over consumption/money is often set to be $\bar{v}_\gamma(c) = c^\gamma$ if $\gamma \in (0,1)$ and $\bar{v}_\gamma(c) = \log c$ if $\gamma = 0$.

[3] For a portfolio-choice experiment estimating ambiguity aversion in a CARA setup, see e.g., Ahn et al. [1].

CRRA,[4] if we observe that the share invested in the uncertain asset is not constant with wealth, then we can conclude that preferences cannot be invariant biseparable.

**Our methodological approach** We consider a standard Anscombe and Aumann setup.[5] This choice is motivated by our aim to study how wealth effects change *ambiguity* attitudes, thus we want to control for the effects due to risk attitudes. We denote by $\mathcal{F}$ the set of all Anscombe and Aumann acts $f : S \to \Delta_0(\mathbb{R})$, where $S$ is a state space and $\Delta_0(\mathbb{R})$ is the set of simple monetary lotteries. Methodologically, we think that the idea of defining decreasing absolute ambiguity aversion rests on answering the following question: How do we formalize the idea that the poorer the decision maker is, the more ambiguity averse he is? In order to do that, we need two ingredients:

1. A notion of preference at different wealth levels $w$, in symbols, $\succsim^w$.

2. A notion of comparative ambiguity aversion.

Given any formalization of points 1 and 2, then one can say that $\succsim$ is decreasing absolute ambiguity averse if ambiguity aversion is higher at lower wealth levels, that is, $w' > w$ yields that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$. Clearly, when preferences are defined over lotteries, this approach exactly mirrors how decreasing absolute risk attitudes are defined. Conversely, under ambiguity the above approach might lead to different definitions depending on which formal notion one translates points 1 and 2 into. In our case, preferences over final wealth levels are modelled by a binary relation $\succsim$ on $\mathcal{F}$. Given a wealth level $w$ and an act $f$, we define by $f^w$ the act whose final monetary outcomes are the outcomes of $f$ shifted by $w$ (see Section 2.1, for a formal definition). Thus, as for 1, we define preferences at wealth level $w$ by

$$f \succsim^w g \overset{def}{\iff} f^w \succsim g^w.$$

As for 2, we rely on the comparative notion of Ghirardato and Marinacci [17].[6] Finally, in a similar fashion, we also define the notions of increasing and constant absolute ambiguity aversion (see Definition 3).

Before proceeding, we reiterate that there are other rather different approaches that formalize absolute ambiguity attitudes in terms of *utility shifts* rather than *wealth shifts* (see the related literature below). Since our goal here is to talk about wealth effects, we will always talk about these attitudes as wealth decreasing (resp., increasing, constant) absolute ambiguity averse attitudes and we will refer to them as $w$-DAAA (resp., $w$-IAAA, $w$-CAAA), thus adding the qualifier of *w*ealth to our notion.

[4] For an empirical study of constant relative risk attitudes using portfolio composition data, see Chiappori and Paiella [13].

[5] The relevant decision theoretic and mathematical notions are introduced in Section 2 and Appendix A.

[6] See Epstein [14] for a different comparative notion of ambiguity attitudes. See also Section 3.6.

**The class of preferences studied** In the paper, we characterize wealth absolute ambiguity attitudes for the class of rational preferences. This class of preferences is large and contains several models of choice which are common in the literature (e.g., maxmin, $\alpha$-maxmin, smooth ambiguity, and variational preferences). Rational preferences are known to admit a representation of the form $V : \mathcal{F} \to \mathbb{R}$ such that

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F}, \tag{1}$$

where $u$ is a von Neumann-Morgenstern expected utility functional over $\Delta_0(\mathbb{R})$ and $I$ is a normalized and monotone functional that maps utility profiles $s \mapsto u(f(s))$ into the real line. This decomposition of the utility function $V$ dates back to Schmeidler [27].[7] From a behavioral point of view, this decomposition is particularly useful since the pair $(u, I)$, other than representing $\succsim$ as in (1), characterizes the attitudes of the decision maker toward risk and ambiguity: Namely, $u$ characterizes the risk attitudes of the decision maker, while $I$ describes the ambiguity attitudes. This specific feature of this decomposition has been emphasized by Ghirardato and Marinacci [16] and exploited several times in the literature.[8] Also in this work the two functions $u$ and $I$ will play a key role.

**Wealth classifiable preferences** As in the risk case, it is not hard to show that wealth absolute attitudes do not provide an exhaustive class of categories with which we can classify rational preferences. In other words, there exist rational preferences that are neither wealth decreasing, nor increasing, nor constant absolute ambiguity averse. When a rational preference relation $\succsim$ exhibits one of these three absolute ambiguity attitudes, we will say that $\succsim$ is *wealth classifiable*. Of course, being classifiable in terms of absolute attitudes toward ambiguity has neither a positive nor a negative connotation. Non-classifiable preferences simply fall outside our analysis.

Our first result (Proposition 4) states that if $\succsim$ is a wealth classifiable rational preference, then it must be constant absolute risk averse (henceforth, CARA). Conceptually, this is important because, in this way, absolute risk attitudes do not intrude in wealth effects and all the differences in terms of attitudes toward uncertainty can be then rightfully attributed to attitudes toward ambiguity. At the same time, this can be a serious limitation. The aforementioned utility approach allows for more general risk attitudes. Nevertheless, this comes at the potential cost of studying utility shifts rather than wealth effects.

With this in mind, we proceed by characterizing wealth absolute ambiguity attitudes using the decomposition $(u, I)$ (Theorem 2 and Corollary 1). The following table provides an informal summary of our characterization for a wealth classifiable $\succsim$:

[7] In [27] it plays a key role in characterizing Choquet expected utility preferences (the functional $I$ is indeed a Choquet integral).

[8] For example, it has been useful in characterizing comparative ambiguity attitudes, as in Ghirardato and Marinacci [17].

| | Risk averse | Risk loving | Risk neutral |
|---|---|---|---|
| $w$-DAAA | $I$ superhomogeneous | $I$ subhomogeneous | $I$ constant superadditive |
| $w$-IAAA | $I$ subhomogeneous | $I$ superhomogeneous | $I$ constant subadditive |
| $w$-CAAA | $I$ homogeneous | $I$ homogeneous | $I$ constant additive |

The table should be read as follows: Under the assumption of wealth classifiability, the rows specify the absolute ambiguity attitudes while the columns specify the risk attitudes, be they averse, loving, or neutral.[9] Each cell then provides a full characterization in terms of the functional $I$. For example, consider a preference relation which is wealth decreasing absolute ambiguity averse and risk averse. By Theorem 2, $I$ is superhomogeneous. On the other hand, if $I$ is assumed to be superhomogeneous, the table shows that there are only two possibilities for a wealth classifiable preference: either $\succsim$ is risk averse and $w$-DAAA or $\succsim$ is risk loving and $w$-IAAA.

The table also shows that (Corollary 3) invariant biseparable preferences – so in particular $\alpha$-maxmin and Choquet expected utility preferences – are wealth classifiable if and only if they are wealth constant absolute ambiguity averse. The reason is simple: For this class of preferences, the functional $I$ is both positively homogeneous and constant additive.

These are two dichotomic properties of the functional $I$. They characterize absolute attitudes toward ambiguity in the risk neutral and nonneutral cases and are most evident for $w$-CAAA preferences. We argue that they are the by-product of a unit of account problem. In fact, though wealth effects are in monetary units (as traditional in Economics), for each act $f$ the number $I(u(f))$ is in von Neumann-Morgenstern utils.[10] In contrast, if $v$ denotes the von Neumann-Morgenstern utility function on monetary outcomes of $u$, then the map $c : \mathcal{F} \to \mathbb{R}$ defined by

$$c(f) = v^{-1}(I(u(f))) \qquad \forall f \in \mathcal{F}$$

is a *monetary* certainty equivalent. Clearly, $c$ is expressed in the same unit of account of wealth $w$. We show that monetary certainty equivalents emerge as the proper representation for absolute attitudes (Proposition 5); for example, $\succsim$ is $w$-DAAA if and only if $\succsim$ is CARA and $c$ is wealth superadditive, that is,

$$c(f^w) \geq c(f) + w \qquad \forall w \geq 0$$

for every act $f$. To sum up, a consistent use of the unit of account allows for a clear-cut characterization of wealth absolute ambiguity attitudes.

[9] Being wealth classifiable, $\succsim$ must be CARA (Proposition 4). Thus, the von Neumann-Morgenstern utility function over monetary outcomes can be normalized to be either $v(c) = -\frac{1}{\alpha}e^{-\alpha c}$ with $\alpha \neq 0$ or $v(c) = c$.

[10] Since $I$ is normalized, if an act $f$ is such that, for some scalar $k$, $u(f(s)) = k$ for all $s \in S$, then $I(u(f)) = k$.

We then proceed to characterize wealth absolute attitudes toward ambiguity by focusing on the subclass of uncertainty averse preferences. For this class, we provide a characterization of these attitudes in terms of their dual representation, that is, in terms of properties of their ambiguity aversion index (Theorem 3). For this particular class, we are able to show how wealth constant absolute ambiguity attitudes are characterized by two radically different models: variational preferences, under risk neutrality, and homothetic preferences under risk nonneutrality (Corollaries 4 and 7).

In Section 3.5, we also study some portfolio implications of absolute attitudes toward ambiguity. Our portfolio application adapts Arrow's portfolio exercise to our setting.

**Relative attitudes** Finally, in Section 4 we conduct a similar analysis for relative ambiguity aversion. Though our analysis rests on the same arguments and intuitions used for the absolute case, we report the main definitions and characterizations because of the relevance of relative attitudes in applied work. For example, a preference is wealth decreasing relative ambiguity averse if, at a higher *proportional* wealth level, it becomes comparatively less averse to ambiguity. Similarly to the absolute case, we obtain that a proper analysis of (wealth) *relative* attitudes toward ambiguity requires that the underlying risk preference on lotteries be constant relative risk averse (CRRA, a popular assumption in Macroeconomics and Finance), so that *relative* risk attitudes do not intrude in proportional wealth effects. Our analysis of wealth relative attitudes reinforces our main message: It is fundamental to keep track of risk attitudes (i.e., risk aversion/love) in studying ambiguity attitudes, be they absolute or relative. Also for relative attitudes, we perform a portfolio exercise. In a two asset allocation problem, we obtain that wealth constant relative ambiguity attitudes yield that the share of wealth invested in the non risk free asset does not vary with wealth. Thus, the empirical evidence on individuals' portfolio allocations in favor of CRRA preferences might be consistent with both CRRA and wealth constant relative ambiguity attitudes (see Section 4.3).

**Related literature** Absolute attitudes toward uncertainty have been previously studied in a few insightful papers. On the one hand, Cherbonnier and Gollier [12] propose and characterize a preferential definition of wealth absolute attitudes toward uncertainty (being the sum of risk and ambiguity) within the $\alpha$-maxmin and the smooth ambiguity models while Wakker and Tversky [30, Propositions 9.5 and 9.6] characterize wealth constant attitudes over gains within the prospect theory model. The latter paper shows that wealth constant attitudes, be those either absolute or relative, within the prospect theory model, translate into the same properties (i.e., either CARA or CRRA) of the corresponding von Neumann-Morgenstern utility $v$. This is perfectly in line with our Corollary 3, despite having been derived in a different setting and for a specific model. Instead, for the former paper, the key differences with our work are that Cherbonnier and Gollier focus on the portfolio implications of their characterizations and, since they do not operate in an Anscombe and Aumann setup, they are not able to disentangle risk and ambiguity attitudes, which is essential to our analysis. Moreover, their analysis is limited to two particular classes of preferences. On the other hand, Grant and Polak [21] start from the following observation: "Constant absolute risk aversion says that if we add or subtract the same constant both to a random variable and to a sure outcome to which it is preferred, then the preference is maintained". They consider an Anscombe and Aumann setting where lotteries are not necessarily monetary. They identify random variables with acts and constants with constant acts (i.e., lotteries). Then, they observe that in such a setting formal standard additions are not allowed,[11] but convex combinations are. Hence, they replace the former with the latter. In this way, constant absolute ambiguity aversion becomes the following property: for any act $f$ in $\mathcal{F}$ and any three lotteries $x$, $y$, and $z$, and any $\alpha$ in $(0,1)$,

$$\alpha f + (1-\alpha)\, x \succsim \alpha z + (1-\alpha)\, x \implies \alpha f + (1-\alpha)\, y \succsim \alpha z + (1-\alpha)\, y. \tag{2}$$

For rational preferences with $\operatorname{Im} u = \mathbb{R}$, (2) turns out to be equivalent to the Weak C-Independence Axiom (e.g., variational preferences as in [24] and vector expected utility preferences as in [28]), which in turn is equivalent to the constant additivity of $I$, irrespective of any other property of $u$ and its risk attitudes. From a comparative point of view, their analysis would be equivalent to the following approach. Consider a rational preference with representation as in (1). As in [21], assume that $\operatorname{Im} u = \mathbb{R}$. Define a preference relation $\succcurlyeq$ over utility profiles by

$$u(f) \succcurlyeq u(g) \overset{def}{\iff} f \succsim g.$$

It turns out that the binary relation $\succcurlyeq$ is a well defined monotone preference over simple real-valued random variables. For, since $\operatorname{Im} u = \mathbb{R}$, for each simple real-valued random variable $\varphi$ there exists an act $f \in \mathcal{F}$ such that $u(f) = \varphi$. This fact and the definition of $\succcurlyeq$ allow for defining a derived preference $\succcurlyeq^k$ over utility profiles by imposing that

$$u(f) \succcurlyeq^k u(g) \overset{def}{\iff} u(f) + k \succcurlyeq u(g) + k.$$

The binary relation $\succcurlyeq^k$ is interpreted as the preference of the decision maker at a utility level $k \in \mathbb{R}$. In other words, in this analysis, adding or subtracting the same constant is done at a utility level. With this in mind, constant absolute ambiguity aversion of Grant and Polak [21] would be equivalent to say that $\succcurlyeq^k$ is as ambiguity averse as $\succcurlyeq^{k'}$ for any two utility levels $k$ and $k'$. In other words, their notion can be constructed in a similar fashion to ours, but rather than defining preferences at different wealth levels, one has to define preferences at

[11] In other words, in order to define the sum of an act $f$ and a lottery $x$, we would need to define the sum, state by state, of two lotteries, namely $f(s)$ and $x$, which is clearly something nonstandard.

different *utility* levels (i.e., $k$ and $k'$). This way of reasoning would lead to notions of *utility* absolute ambiguity attitudes, where their case is the constant one.

In the same spirit, Xue [31] and [32] considers more general attitudes, namely utility decreasing and increasing absolute attitudes, by suitably weakening (2) and by axiomatizing a constant superadditive version of variational preferences as well as two equivalent representations of uncertainty averse preferences (see also Chambers et al. [10] for a similar definition). Independently of Xue, Ghirardato and Siniscalchi [18] studied a similar notion of utility absolute ambiguity attitudes for the class of rational preferences. Relative to these papers, the key difference with our work is that we directly address the effect of baseline monetary shifts. As mentioned, in the latter four papers instead, absolute ambiguity attitudes are defined in terms of utility shifts rather than wealth shifts.

A similar approach was already present in Klibanoff, Marinacci, and Mukerji [23]. As a consequence, our analysis is consistent with their results under risk neutrality: this is the only case when additive wealth shifts coincide with additive utility shifts. In general, standard shifts in wealth considered in Economics do not generate well behaved shifts in utility and, apart from the risk neutral case, our analysis leads to different results. For example, homothetic preferences in this literature would be classified as (utility) constant *relative* ambiguity averse while in our case they turn out to be (wealth) constant *absolute* ambiguity averse under risk nonneutrality and CARA.

One major benefit of the utility approach is, of course, that if a researcher believes that agents exhibit absolute ambiguity attitudes in terms of utility shifts, then the same researcher has no limitation on how to model the risk attitudes of the decision maker. If instead the same is assumed about absolute ambiguity attitudes in terms of wealth shifts, then the decision maker over risk, necessarily must be CARA.

Finally, to the best of our knowledge, the only experimental paper testing absolute/relative ambiguity attitudes is Baillon and Placido [4]. They discuss their findings using both definitions: the one based on utility shifts as well as the one based on wealth shifts. They observe that roughly 60% of their subjects are CARA. Within this group, the majority of risk neutral subjects was wealth constant absolute ambiguity averse, followed by wealth decreasing absolute ambiguity averse. Risk averse agents were, instead, mostly wealth increasing absolute ambiguity averse.[12]

[12] Given our results, for a risk neutral agent, for example, this would be consistent with either having invariant biseparable or variational preferences. For a risk averse agent, instead, this would rule out risk averse variational preferences.

## 2 Preliminaries

### 2.1 Setup

We begin by considering a generalized version of the Anscombe and Aumann [2] setup with a nonempty set $S$ of *states of the world*, an algebra $\Sigma$ of subsets of $S$ called *events*, and a nonempty convex set $X$ of *consequences*. We will confine our attention to sets $X$ of monetary simple lotteries: for wealth absolute ambiguity attitudes $X = \Delta_0(\mathbb{R})$, while for wealth relative ambiguity attitudes $X = \Delta_0(\mathbb{R}_{++})$.[13] Since part of the analysis of these two cases is in common, some of the results use the abstract notion of consequence set $X$ (e.g., Theorem 1). We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f : S \to X$ that are $\Sigma$-measurable and take on finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act that takes value $x$. Thus, with the usual slight abuse of notation, we identify $X$ with the subset of constant acts in $\mathcal{F}$. Using the linear structure of $X$, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$. Given a binary relation $\succsim$ on $\mathcal{F}$ (a *preference*), for each $f \in \mathcal{F}$ we denote by $x_f \in X$ a certainty equivalent of $f$, that is, $x_f \sim f$.[14] Given a function $u : X \to \mathbb{R}$, we denote by $\operatorname{Im} u$ the set $u(X)$; in particular, observe that $u \circ f \in B_0(\Sigma)$ when $f \in \mathcal{F}$. The mathematical notions used in the main text, but not defined there, are collected in Appendix A.

The paper relies on the following comparative notion of Ghirardato and Marinacci [17].

**Definition 1** *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$, we say that $\succsim_1$ is more ambiguity averse than $\succsim_2$ if, for each $f \in \mathcal{F}$ and $x \in X$, $f \succsim_1 x$ implies $f \succsim_2 x$.*

An important example of a convex consequence set $X$ is that of all *simple monetary lotteries*:

$$\Delta_0(\mathbb{R}) = \left\{ x \in [0,1]^{\mathbb{R}} : x(c) \neq 0 \text{ for finitely many } c \in \mathbb{R} \text{ and } \sum_{c \in \mathbb{R}} x(c) = 1 \right\}.$$

Any wealth level $w \in \mathbb{R}$ defines a map ${}^w : \Delta_0(\mathbb{R}) \to \Delta_0(\mathbb{R})$ which is bijective and affine: for each $x$ in $\Delta_0(\mathbb{R})$, $x^w$ is the lottery such that $x^w(c) = x(c-w)$ for all $c \in \mathbb{R}$. We thus interpret the outcome of a lottery, $c \in \mathbb{R}$, as a final wealth level. Thus, given $x$ in $\Delta_0(\mathbb{R})$, if the decision maker has wealth $w$, we interpret $x^w$ as being the distribution on final wealth levels. In fact, lottery $x$ yields a consequence $d \in \mathbb{R}$ (on top of $w$) with probability $x(d)$ and

[13] We define $\Delta_0(\mathbb{R})$ right below and $\Delta_0(\mathbb{R}_{++})$ in Section 4.

[14] In a monetary framework when $X$ is either $\Delta_0(\mathbb{R})$ or $\Delta_0(\mathbb{R}_{++})$, note that given $f$, $x_f$ is a lottery that, received with certainty in each state $s$, is indifferent to $f$. Thus, $x_f$ is a *risky* prospect which is independent of the realization on $S$.

the probability of having as final wealth $w + d$, that is $x^w(w + d)$, is equal to $x(d)$. This implies that $x^w(w + d) = x(d)$ for all $d \in \mathbb{R}$, which is equivalent to our definition of $x^w$. Note that the map $^w$ admits a natural extension to $\mathcal{F}$ when $X = \Delta_0(\mathbb{R})$, where $f \mapsto f^w$ is defined by $f^w(s) = f(s)^w$ for all $s \in S$.

### 2.2 Axioms and representations

We will consider the following classes of preferences $\succsim$ on $\mathcal{F}$: rational preferences (Cerreia-Vioglio et al. [6]), uncertainty averse preferences (Cerreia-Vioglio et al. [7]), invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci [15]), variational preferences (Maccheroni, Marinacci, and Rustichini [24]), and maxmin preferences (Gilboa and Schmeidler [20]). They rely on the following axioms, discussed in the original papers as well as in Gilboa and Marinacci [19].

**Axiom A. 1 (Weak Order)** *$\succsim$ is nontrivial, complete, and transitive.*

**Axiom A. 2 (Monotonicity)** *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

**Axiom A. 3 (Continuity)** *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed.*

**Axiom A. 4 (Risk Independence)** *If $x, y, z \in X$ and $\alpha \in (0,1)$,*

$$x \sim y \Longrightarrow \alpha x + (1-\alpha) z \sim \alpha y + (1-\alpha) z.$$

**Axiom A. 5 (Convexity)** *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succsim f.$$

**Axiom A. 6 (Weak C-Independence)** *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Longrightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

**Axiom A. 7 (C-Independence)** *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$,*

$$f \succsim g \iff \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.$$

**Axiom A. 8 (Unboundedness)** *There exist $x$ and $y$ in $X$ such that $x \succ y$ and for each $\alpha \in (0,1)$ there exists $z \in X$ that satisfies*

$$\text{either } y \succ \alpha z + (1-\alpha)x \text{ or } \alpha z + (1-\alpha)y \succ x.$$

The following omnibus result collects some of the results that the above papers proved for the classes of preferences that they studied.

**Theorem 1 (Omnibus)** *A preference $\succsim$ on $\mathcal{F}$ satisfies Weak Order, Monotonicity, Continuity, and Risk Independence if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ such that the criterion $V : \mathcal{F} \to \mathbb{R}$, given by*

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F} \tag{3}$$

*represents $\succsim$. The function $u$ is cardinally unique and, given $u$, $I$ is the unique normalized, monotone, and continuous functional satisfying (3). In this case, we say that $\succsim$ is a* rational preference. *A rational preference satisfies:*

*(i) C-Independence if and only if $I$ is constant linear; in this case, we say that $\succsim$ is an* invariant biseparable preference.[15]

*(ii) Convexity if and only if $I$ is quasiconcave; in this case, we say that $\succsim$ is an* uncertainty averse preference.[16]

*(iii) Convexity and Weak C-Independence if and only if $I$ is quasiconcave and constant additive; in this case, we say that $\succsim$ is a* variational preference.

*(iv) Convexity and C-Independence if and only if $I$ is quasiconcave and constant linear; in this case, we say that $\succsim$ is a* maxmin preference.

*(v) Unboundedness if and only if* $\operatorname{Im} u$ *is unbounded.*

Given $u$ and $I$ as in Theorem 1, we call $(u, I)$ a (*canonical*) *representation* of the rational preference $\succsim$.[17]

We say that $\succsim$ on $\mathcal{F}$ is a *homothetic* (*uncertainty averse*) *preference* if there exists a canonical representation $(u, I)$, with $\operatorname{Im} u$ equal to either $(-\infty, 0)$ or $(0, \infty)$, such that

$$I(\varphi) = \min_{p \in \Delta} \int \varphi c(p)^{-\operatorname{sgn} \varphi} dp = \begin{cases} \min_{p \in \Delta} \frac{\int \varphi dp}{c(p)} & \text{if } \operatorname{Im} u = (0, \infty) \\ \min_{p \in \Delta} c(p) \int \varphi dp & \text{if } \operatorname{Im} u = (-\infty, 0) \end{cases}$$

where $c : \Delta \to [0, 1]$ is normalized, upper semicontinuous, and quasiconcave.[18] Note that $I$ is positively homogeneous. These preferences, proposed by Chateauneuf and Faro [11],

[15] Invariant biseparable preferences correspond to the general class of $\alpha(f)$-maxmin preferences of Ghirardato, Maccheroni, and Marinacci [15], which, inter alia, includes the Choquet expected utility preferences of Schmeidler [27].

[16] Uncertainty averse preferences, within the rational preferences class, are distinguished by further satisfying the Convexity axiom. A slightly stronger version of this axiom was termed Uncertainty Aversion by Schmeidler [27, p. 582], since it captures a preference for diversification/hedging. According to our terminology, these preferences should be called ambiguity averse preferences, yet we opted to use their original name as in [7].

[17] In Appendix B, we discuss more in detail the uniqueness features of canonical representations.

[18] The function $c$ is normalized if and only if $\max_{p \in \Delta} c(p) = 1$. Observe also that since $\operatorname{Im} u$ is equal to either $(-\infty, 0)$ or $(0, \infty)$, then $-\operatorname{sgn} \varphi = 1$ or $-\operatorname{sgn} \varphi = -1$, yielding that $c(p)$ can be factored out the integral.

are a natural counterpart to variational preferences with positive homogeneity in place of constant additivity. As [11] showed, positive homogeneity is implied by a form of homotheticity/independence with respect to a worst consequence, when such a consequence exists (something that in this paper we do not allow for; this is why these preferences are not included in the omnibus theorem).

### 2.3 A twist: Anscombe and Aumann setup and wealth

One possible way of justifying the use of Anscombe and Aumann acts to study wealth effects is to relate this framework to model uncertainty. To this end, assume there exists an underlying measurable space $(\Omega, \mathcal{A})$ of payoff relevant states, a collection $M = \{m_s\}_{s \in S}$ of "models" (probability measures on the $\sigma$-algebra $\mathcal{A}$).[19] The "primitive" alternatives of the decision maker are (simple and measurable) state-contingent payoffs $h : \Omega \to \mathbb{R}$; in symbols $h \in B_0(\mathcal{A})$. If $S$ is finite, then for each simple $\mathcal{A}$-measurable and real valued $h$, there corresponds an Anscombe and Aumann act

$$\begin{array}{rccc} \hat{h}: & S & \to & \Delta_0(\mathbb{R}) \\ & s & \mapsto & m_s \circ h^{-1} \end{array}$$

that maps $s$ to the distribution of $h$ under $m_s$. The agent is assumed to have a preference over state-contingent payoffs, that is, over $B_0(\mathcal{A})$. If he is indifferent between state-contingent payoffs that are identically distributed with respect to each model (i.e., the agent is *consequentialist*), then his "primitive" preference over $B_0(\mathcal{A})$ can be directly expressed by a preference on Anscombe and Aumann acts. More formally:[20]

**Proposition 1** *If $M = \{m_s\}_{s \in S}$ is a finite set of probability measures on $(\Omega, \mathcal{A})$, then*

$$\left\{\hat{h} : h \in B_0(\mathcal{A})\right\} \subseteq \mathcal{F}. \tag{4}$$

*with equality if and only if the elements of $M$ are orthogonal and nonatomic.*

For our exercise, it is interesting to observe that, given any $h \in B_0(\mathcal{A})$ for each $w \in \mathbb{R}$ and each $c \in \mathbb{R}$

$$\left(\widehat{h+w}\right)(s)(c) = m_s(\omega \in \Omega : (h+w)(\omega) = c) = m_s(\omega \in \Omega : h(\omega) = c - w) = (\hat{h})^w(s)(c)$$

for all $s \in S$. That is, $\hat{h}^w = \widehat{h+w}$ is the Anscombe and Aumann act that corresponds to a wealth shift $w$ of the state-contingent payoff $h$. This permits to interpret our Anscombe and Aumann analysis in terms of model uncertainty.

[19] Only for this section, the word *state* is reserved to the elements $\omega$ of $\Omega$ while the elements $s$ of $S$ are best seen as *parameters*.

[20] For the sake of brevity, we omit the proof of Proposition 1. It is available upon request.

# 3 Results

## 3.1 Induced preferences

In the rest of the paper (with the exception of Section 4) we specialize the set of consequences $X$ to be made of monetary lotteries, that is $X = \Delta_0(\mathbb{R})$. A preference $\succsim$ on $\mathcal{F}$ induces, through a wealth level $w \in \mathbb{R}$, a preference $\succsim^w$ on $\mathcal{F}$ given by

$$f \succsim^w g \Longleftrightarrow f^w \succsim g^w.$$

The induced preference inherits some of the properties of the original preference.

**Proposition 2** *Let $\succsim$ be a preference on $\mathcal{F}$ and $w \in \mathbb{R}$. Then:*

*(i) If $\succsim$ is a rational preference, so is $\succsim^w$.*

*(ii) If $\succsim$ is an uncertainty averse preference, so is $\succsim^w$.*

Next, we compare the ambiguity aversion of different induced preferences.

**Proposition 3** *Let $\succsim$ be a rational preference on $\mathcal{F}$ and $w$ and $w'$ two wealth levels. If $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$, then $u_w$ is a positive affine transformation of $u_{w'}$.*[21]

Before proceeding, note that an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$ takes the form $u(x) = \sum_{c \in \mathbb{R}} v(c)\, x(c)$, where $v : \mathbb{R} \to \mathbb{R}$. Throughout the paper we make the following assumption.

**Assumption** The function $v$ is strictly increasing and continuous.

In this monetary setup, we have the following classic notion.

**Definition 2** *A preference $\succsim$ on $\mathcal{F}$ is* constant absolute risk averse *(CARA) if, for any two levels $w$ and $w'$ of wealth, the induced preferences $\succsim^w$ and $\succsim^{w'}$ agree on $\Delta_0(\mathbb{R})$.*

This behavioral definition amounts to say that preferences over lotteries are unaffected by the level of wealth $w$. A routine argument shows that, if $\succsim$ (on lotteries) is represented by an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$, then $\succsim$ is CARA if and only if there exist $\alpha \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v(c) = v_\alpha(c) = \begin{cases} -a\frac{1}{\alpha}e^{-\alpha c} + b & \text{if } \alpha \neq 0 \\ ac + b & \text{if } \alpha = 0 \end{cases}, \tag{5}$$

[21] Here, $u_w$ and $u_{w'}$ are part of a canonical representation for, respectively, $\succsim^w$ and $\succsim^{w'}$.

that is, $v_\alpha$ is either exponential or affine. In the former case, $\succsim$ is a CARA preference which is not risk neutral; in particular, it is (strictly) risk averse if $\alpha > 0$ and (strictly) risk loving if $\alpha < 0$.[22] Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \alpha > 0 \\ (b, +\infty) & \text{if } \alpha < 0 \\ (-\infty, +\infty) & \text{if } \alpha = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\succsim$ is risk averse and $b = \inf \operatorname{Im} u$ when $\succsim$ is risk loving. Momentarily, this extremum role of $b$ will play a key role in Theorem 2.

### 3.2 Rational preferences

Wealth absolute ambiguity attitudes describe how the decision maker's preferences over uncertain monetary alternatives vary as his wealth changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational and for uncertainty averse preferences.

**Definition 3** *A preference $\succsim$ on $\mathcal{F}$ is wealth decreasing (increasing, constant) absolute ambiguity averse if, for any two levels $w$ and $w'$ of wealth, $w' > w$ implies that $\succsim^w$ is more (less, equally) ambiguity averse than $\succsim^{w'}$.*[23]

As this classification is not exhaustive, we say that a preference is (absolutely) *wealth classifiable* in terms of absolute ambiguity aversion if it can be classified according to this definition, that is, if it is either wealth decreasing or increasing or constant absolute ambiguity averse. The next result shows that being CARA is a necessary condition for a preference in order to be wealth classifiable: in fact, in this way absolute risk attitudes do not intrude in wealth effects.

**Proposition 4** *A rational preference $\succsim$ is wealth classifiable only if it is CARA.*

We first characterize wealth absolute ambiguity attitudes for rational preferences.

**Theorem 2** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. The following statements are equivalent:*

*(i) $\succsim$ is wealth decreasing absolute ambiguity averse;*

[22] In what follows, we omit "strictly" since a CARA preference is either risk neutral ($\alpha = 0$) or strictly risk averse ($\alpha > 0$) or strictly risk loving ($\alpha < 0$).

[23] Clearly, $\succsim^w$ is less ambiguity averse than $\succsim^{w'}$ if and only if $\succsim^{w'}$ is more ambiguity averse than $\succsim^w$. Similarly, equally ambiguity averse means that $\succsim^w$ is, at the same time, more and less ambiguity averse than $\succsim^{w'}$.

*(ii) $\succsim$ is CARA and $I$ is:*

*(a) concave (convex) at $b$ provided $\succsim$ is risk averse (loving);*

*(b) constant superadditive provided $\succsim$ is risk neutral.*

*(iii) $\succsim$ is wealth classifiable and $I$ satisfies (a) or (b).*

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (a) concavity (convexity) at $b$ reduces to positive superhomogeneity (subhomogeneity).[24]

Dual versions of this theorem are easily seen to hold for wealth increasing and constant absolute ambiguity aversion (for this latter case see Corollary 1). In particular, by keeping the same premises, Theorem 2 takes a similar form with (i), (ii), and (iii) replaced by:

(i)' $\succsim$ is wealth increasing absolute ambiguity averse;

(ii)' $\succsim$ is CARA and $I$ is:

(a) convex (concave) at $b$ provided $\succsim$ is risk averse (loving);

(b) constant subadditive provided $\succsim$ is risk neutral.

(iii)' $\succsim$ is wealth classifiable and $I$ satisfies (a) or (b).

The next result characterizes wealth constant absolute ambiguity aversion for wealth classifiable rational preferences. At the same time, the result still holds if instead of requiring $\succsim$ being wealth classifiable we only require $\succsim$ to be CARA.[25]

**Corollary 1** *Let $\succsim$ be a wealth classifiable rational preference on $\mathcal{F}$ with representation $(u, I)$. Then:*

*(i) If $\succsim$ is risk neutral, it is wealth constant absolute ambiguity averse if and only if $I$ is constant additive.*[26]

*(ii) If $\succsim$ is not risk neutral, it is wealth constant absolute ambiguity averse if and only if $I$ is affine at $b$.*[27]

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (ii) the affinity at $b$ reduces to positive homogeneity, that is, $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\lambda > 0$. Risk neutrality and risk aversion of $\succsim$ may thus translate wealth constant absolute ambiguity aversion in, respectively,

[24] See also Appendix A for the notions of concavity/convexity at $b$.

[25] Recall that, by Proposition 4, wealth classifiable and rational preferences are CARA.

[26] Recall that $\operatorname{Im} u = \mathbb{R}$ in the risk neutral case.

[27] Recall that $b = \sup \operatorname{Im} u$ when $\succsim$ is risk averse and $b = \inf \operatorname{Im} u$ when $\succsim$ is risk loving.

constant additivity and positive homogeneity of $I$ which are two mathematically and decision theoretically distinct properties.

Indeed, constant additivity and positive homogeneity can be obtained jointly by assuming C-Independence. The assumption of C-Independence could be equivalently rewritten as: for each $f, g \in \mathcal{F}$, $x, y \in \Delta_0(\mathbb{R})$, and $\alpha, \beta \in (0,1]$

$$\alpha f + (1-\alpha)\, x \succsim \alpha g + (1-\alpha)\, x \implies \beta f + (1-\beta)\, y \succsim \beta g + (1-\beta)\, y.$$

Thus, as argued in [24, p. 1454], C-Independence actually involves two types of independence: independence relative to mixing with constants and independence relative to the weights used in such mixing. The first type (Weak C-Independence) corresponds to $I$ being constant additive while the second type, in the presence of a worst consequence, corresponds to $I$ being positively homogeneous (see Chateauneuf and Faro [11]). Another class of preferences that satisfy Weak C-Independence, but do not necessarily satisfy Convexity, is the class of vector expected utility preferences (see Siniscalchi [28]).[28]

**Corollary 2** *A risk neutral rational preference is wealth constant absolute ambiguity averse if and only if it satisfies Weak C-Independence.*

Along with Corollary 1, the next result shows that invariant biseparable preferences are a class of rational preferences that, when wealth classifiable, are wealth constant absolute ambiguity averse regardless of their risk attitudes.

**Corollary 3** *Let $\succsim$ be an invariant biseparable preference $\succsim$ on $\mathcal{F}$. The following conditions are equivalent:*

*(i)* $\succsim$ *is wealth classifiable;*

*(ii)* $\succsim$ *is wealth constant absolute ambiguity averse;*

*(iii)* $\succsim$ *is CARA.*

As mentioned in the Introduction, Corollary 2 (and Corollary 4 below) show that our analysis is consistent, under risk neutrality, with the approach of Grant and Polak [21].

Since we are dealing with acts yielding monetary lotteries, it is also possible to discuss *monetary* certainty equivalents. Given a canonical representation $(u, I)$, we can define the functional $c : \mathcal{F} \to \mathbb{R}$ by the rule $c(f) = v^{-1}(I(u(f)))$. Note that, given $f \in \mathcal{F}$, the scalar $c(f)$ is the monetary amount that, received with certainty in each state of the world, makes

[28] Vector expected utility preferences, on top of being rational and satisfying Weak C-Independence, satisfy two other axioms of independence/invariance and an extra continuity axiom. In terms of framework, $\Sigma$ is required to be countably generated.

the decision maker indifferent between $f$ and the constant (risk free) act paying $c(f)$. We will say that $c$ is wealth superadditive (resp., subadditive, additive) if and only if for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) \geq c(f) + w \qquad (\text{resp., } \leq, = ).$$

**Proposition 5** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. Then:*

*(i) $\succsim$ is wealth decreasing absolute ambiguity averse if and only if $c$ is wealth superadditive and $\succsim$ is CARA.*

*(ii) $\succsim$ is wealth increasing absolute ambiguity averse if and only if $c$ is wealth subadditive and $\succsim$ is CARA.*

*(iii) $\succsim$ is wealth constant absolute ambiguity averse if and only if $c$ is wealth additive and $\succsim$ is CARA.*

### 3.3 Uncertainty averse preferences

Assume that $\succsim$ is an uncertainty averse preference. By definition, $\succsim$ is also rational. If $(u, I)$ is a (rational) representation of $\succsim$, then there exists a unique minimal linearly continuous $G \in \mathcal{G}(\operatorname{Im} u \times \Delta)$ such that $I(\psi) = \inf_{p \in \Delta} G\left(\int \psi dp, p\right)$ for all $\psi \in B_0(\Sigma, \operatorname{Im} u)$. Uncertainty averse preferences are thus characterized by the pair $(u, G)$. In particular, the function $G$ is an index of ambiguity aversion.[29]

Now we characterize wealth absolute ambiguity attitudes for uncertainty averse preferences in terms of the pair $(u, G)$.

**Theorem 3** *Let $\succsim$ be an uncertainty averse preference on $\mathcal{F}$ with representation $(u, G)$. The following statements are equivalent:*

*(i) $\succsim$ is wealth decreasing absolute ambiguity averse;*

*(ii) $\succsim$ is CARA and $G$ is such that:*

*(a) $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t, p) + (1-\lambda) b$ ($\leq$) for all $(t, p) \in \operatorname{Im} u \times \Delta$ and for all $\lambda \in (0,1)$ provided $\succsim$ is risk averse (loving);*

*(b) $G(t + k, p) \geq G(t, p) + k$ for all $(t, p) \in \operatorname{Im} u \times \Delta$ and for all $k \geq 0$ provided $\succsim$ is risk neutral.*

*(iii) $\succsim$ is wealth classifiable and $G$ satisfies (a) or (b).*

[29] These facts can be found in [7] (see also Appendix A). Because of the minimality of $G$, we have $G(t, p) = \sup_{f \in \mathcal{F}} \left\{ u(x_f) : \int u(f)\, dp \leq t \right\}$ for all $(t, p) \in \operatorname{Im} u \times \Delta$. The function $G$ is unique given $u$.

As mentioned in the Introduction and above, our analysis is consistent, under risk neutrality, with the approach of Grant and Polak [21]. Indeed, the role of constant superadditivity in Theorem 3 shows that a similar consistency holds with the results of Xue [31] and [32] where decreasing absolute ambiguity aversion is modelled in terms of utility shifts.

In Theorem 3 as well, dual versions of this result hold in the wealth increasing and constant absolute ambiguity averse case (with, respectively, opposite inequalities and equalities).

The next corollary shows that the behavioral characterization established in Corollary 2 leads to variational preferences when preferences are uncertainty averse.

**Corollary 4** *A risk neutral uncertainty averse preference is wealth constant absolute ambiguity averse if and only if it is a variational preference.*

The next result reports a noteworthy consequence of the previous theorem for uncertainty averse preferences which feature a concave $G$ (or, equivalently, a concave $I$).

**Corollary 5** *Let $\succsim$ be an uncertainty averse preference which is CARA and risk averse. If $G$ is concave, then $\succsim$ is wealth decreasing absolute ambiguity averse.*

This corollary can be sharpened for the class of variational preferences that are not maxmin, and so in particular are not invariant biseparable. This class features a concave $G$.

**Corollary 6** *A variational preference, which is not maxmin and not risk neutral, satisfies:*

*(i) wealth decreasing absolute ambiguity aversion if and only if it is CARA and risk averse;*

*(ii) wealth increasing absolute ambiguity aversion if and only if it is CARA and risk loving.*

In order to characterize wealth constant absolute ambiguity attitudes when the preference is not risk neutral, we need to consider homothetic preferences.

**Corollary 7** *A risk nonneutral uncertainty averse preference is wealth constant absolute ambiguity averse if and only if it is CARA and homothetic.*

To sum up, depending on risk attitudes, either homothetic or variational preferences characterize wealth constant absolute ambiguity attitudes for uncertainty averse preferences.

### 3.4 Smooth ambiguity preferences

Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be a strictly increasing and continuous function, and $\mu$ a Borel probability measure over $\Delta$. The preferences represented by a pair $(u, I)$, where

$$I(\varphi) = \phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right) \tag{6}$$

are called *smooth ambiguity preferences* (Klibanoff, Marinacci and, Mukerji [23]). They are uncertainty averse when $\phi$ is concave.

**Proposition 6** *Let $\succsim$ be a CARA smooth ambiguity preference and $\phi(t) = -e^{-\gamma t}$ with $\gamma > 0$. Then,*

*(i) If $\succsim$ is risk neutral, then it is wealth constant absolute ambiguity averse.*

*(ii) If $\succsim$ is risk averse, then it is wealth decreasing absolute ambiguity averse.*

In our setup an exponential $\phi$ thus yields wealth constant absolute ambiguity aversion, as argued in [23], as long as $\succsim$ is risk neutral. In the next result, using Theorem 2, we provide a full characterization of wealth decreasing absolute ambiguity aversion within the smooth ambiguity model. Before doing so, we need to introduce some additional notions and terminology.

Given $\phi : \mathbb{R} \to \mathbb{R}$ and $w \in \mathbb{R}$, we define $\phi_w : \mathbb{R} \to \mathbb{R}$ to be such that $\phi_w(t) = \phi(t + w)$ for all $t \in \mathbb{R}$. Similarly, given $\phi : (-\infty, 0) \to \mathbb{R}$ (resp., $\phi : (0, \infty) \to \mathbb{R}$) and $\nu > 0$, we define $\phi_\nu(t) = \phi(\nu t)$ for all $t < 0$ (resp., $t > 0$).

**Definition 4** *Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be strictly increasing and continuous.*

*(i) If $\operatorname{Im} u = \mathbb{R}$, we say that $\phi$ is DARA if for each $w', w \in \mathbb{R}$, with $w' > w$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_w = f \circ \phi_{w'}$.*

*(ii) If $\operatorname{Im} u = (-\infty, 0)$, we say that $\phi$ is IRRA if for each $\nu, \eta > 0$, with $\nu > \eta$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_\nu = f \circ \phi_\eta$.*

*(iii) If $\operatorname{Im} u = (0, \infty)$, we say that $\phi$ is DRRA if for each $\nu, \eta > 0$, with $\nu > \eta$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_\eta = f \circ \phi_\nu$.*

Consider a function $\phi : \operatorname{Im} u \to \mathbb{R}$ which is twice continuously differentiable and such that $\phi' > 0$. Clearly, $\phi$ is DARA if and only if $-\phi''(t)/\phi'(t)$ is decreasing and similarly, $\phi$ is DRRA (resp., IRRA) if and only if $-t\phi''(t)/\phi'(t)$ is decreasing (resp., increasing).

**Proposition 7** *Let $\succsim$ be a CARA smooth ambiguity preference with $b = 0$ in (5) and assume that $\Sigma$ is nontrivial. Then,*

*(i) If $\succsim$ is risk neutral, $\succsim$ is wealth decreasing absolute ambiguity averse for all $\mu$ if and only if $\phi$ is DARA.*

*(ii) If $\succsim$ is risk averse, $\succsim$ is wealth decreasing absolute ambiguity averse for all $\mu$ if and only if $\phi$ is IRRA.*

*(iii) If $\succsim$ is risk loving, $\succsim$ is wealth decreasing absolute ambiguity averse for all $\mu$ if and only if $\phi$ is DRRA.*

This result provides some behavioral guidance in the specification of the function $\phi$, as the standard theory of absolute risk aversion of Arrow and Pratt provides guidance in the choice of the von Neumann-Morgenstern utility function.

**Remark 1** Cherbonnier and Gollier [12, Proposition 2 and Corollary 1], in a different framework, characterize wealth decreasing absolute *uncertainty* aversion (being the sum of risk and ambiguity) for the smooth ambiguity model. Under the assumption that $\phi$ is concave, they show that a smooth ambiguity preference is wealth decreasing absolute *uncertainty* averse if and only if $v$ and $\phi \circ v$ are both DARA. The characterization in Proposition 7, where $v$ is CARA, is consistent with their findings. At the same time, in our case, $\phi$ does not have to be concave.

Let $c_f(p) \in \mathbb{R}$ be the monetary certainty equivalent of act $f$ under $p$, that is, $c_f(p) = v^{-1}\left(\int u(f)\, dp\right)$. By setting $w = \phi \circ v : \mathbb{R} \to \mathbb{R}$, the smooth ambiguity representation can be written as

$$\begin{aligned} V(f) &= \left(v \circ w^{-1}\right)\left(\int w\left(c_f(p)\right) d\mu\right) \\ &= \left(v \circ w^{-1}\right)\left(\int \left(w \circ v^{-1}\right)\left(\int u(f)\, dp\right) d\mu\right). \end{aligned}$$

The function $w$ can be interpreted as aversion to epistemic uncertainty.[30] When $v$ is the identity, we have $\phi = w$ and so point (i) of the Proposition 6 can be interpreted in terms of constant attitudes toward such uncertainty. When both $v(c) = -e^{-\alpha c}$ and $w(c) = -e^{-\beta c}$ are risk averse exponentials, with $\beta > \alpha > 0$, then $\phi(t) = -(-t)^{\frac{\beta}{\alpha}}$. The condition $\beta > \alpha$ can be interpreted as higher aversion to epistemic uncertainty than to risk (both being constant absolute averse). The next result shows that in this double exponential case the resulting wealth absolute ambiguity aversion is decreasing.

**Proposition 8** *Let $\succsim$ be a CARA smooth ambiguity preference, with $b \leq 0$ in (5), and suppose $\phi(t) = -(-t)^{\gamma}$ for all $t < 0$ with $\gamma > 1$. If $\succsim$ is risk averse, then it is wealth decreasing absolute ambiguity averse.*

### 3.5 Portfolio problem and absolute attitudes

In this section we study how wealth absolute ambiguity attitudes affect portfolio choices. To do so, we adapt to our setting the standard portfolio exercise of Arrow which originally was carried out in a risk domain, as an illustration of the implications of absolute and relative risk attitudes (see Section 4.3 for the study of relative attitudes). Assume that $f : S \to \Delta_0(\mathbb{R})$ is

[30] See Marinacci [25] for a discussion of this version of the smooth ambiguity model. The context should clarify that here $w$ is a function and not a wealth level.

a purely ambiguous asset, that is, for each state of the world the act $f$ yields a deterministic consequence, interpreted as a return. Formally, as an Anscombe and Aumann act, we have that $f(s) = \delta_{r_s}$ for all $s \in S$ where $r_s > 0$ is the return in state $s$.[31] The risk free asset is instead modelled by the act $g$ such that $g(s) = \delta_{r_f}$ for all $s \in S$ where $r_f > 0$ is the return on the risk free asset. The agent faces the following portfolio problem: he has wealth $w > 0$ which he has to allocate between the ambiguous asset and the risk free asset. We denote by $\beta$ the amount of wealth invested in the ambiguous asset and by $w - \beta$ the amount invested in the risk free one. We assume that the agent cannot short any of the two securities and therefore $\beta \in [0, w]$. Note that the allocation $(\beta, w - \beta)$ generates an Anscombe and Aumann act that in each state of the world yields $\delta_{\beta r_s + (w-\beta) r_f}$ where $\beta r_s + (w - \beta) r_f = w r_f + \beta (r_s - r_f)$ is the final wealth level in state $s$. We denote the real-valued measurable random variable $s \mapsto r_s$ by $r$. Similarly, with a small abuse of notation, we denote by $r_f$ the constant random variable that in each state $s$ assumes value $r_f$.

In terms of preferences, we assume that the agent has rational preferences $\succsim$ on $\mathcal{F}$ with canonical representation $(u, I)$ and von Neumann-Morgenstern function $v : \mathbb{R} \to \mathbb{R}$. The portfolio problem amounts to

$$\max I \left( v \left( \beta r + (w - \beta) r_f \right) \right) \text{ subject to } \beta \in [0, w]. \tag{7}$$

In what follows, we assume that this problem admits a unique solution for all $w > 0$, denoted by $\beta^* (w)$.

**Proposition 9** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. If $\succsim$ is wealth constant absolute ambiguity averse, then*

$$w' > w > 0 \implies \beta^* \left( w' \right) \geq \beta^* (w).$$

*If, in addition, $\beta^* (w) \in (0, w)$ with $w > 0$ and $\succsim$ is risk averse and uncertainty averse, then*

$$w' > w \implies \beta^* \left( w' \right) = \beta^* (w).$$

Before discussing the result, we comment on its generality. From a theoretical point of view, note that, differently from what happens under risk, the subclass of preferences which exhibit wealth constant absolute attitudes is quite large. Under risk and the expected utility model, constant absolute attitudes coincide to a very specific form of $v$. In contrast, under ambiguity wealth constant absolute attitudes encompass a family of preferences: $\alpha$-maxmin, Choquet expected utility, variational under risk neutrality, vector expected utility under risk neutrality, homothetic under risk nonneutrality as well as the risk averse CARA smooth ambiguity preferences $\succsim$ of Proposition 8 when $b = 0$.[32]

[31] As usual, $x = \delta_c$ is the degenerate lottery at $c$, that is, $x(d) = 1$ if $d = c$, and $x(d) = 0$ otherwise.

[32] That said, the relevance of this family to describe the behavior of decision makers is, in a final analysis, an empirical question.

We next discuss the second part of the statement. The result is indeed in line with intuition. If the decision maker is risk and uncertainty averse, then his preferences are convex in $\beta$, so the agent values diversification. It follows that if $\beta^*(w)$ is an interior solution, then an intermediate subjective optimal balance has been found between the certainty provided by the risk free asset and the potentially higher, yet uncertain, returns of the ambiguous asset. At the same time, if $w' > w$ and $\succsim$ is wealth constant absolute ambiguity averse, then wealth does not impact the ambiguity attitudes of the decision maker. In other words, the increment in wealth $(w' - w) r_f$ is factored out and, as a consequence, the previously optimal balance between the risk free asset and the ambiguous one is unaffected, that is, $\beta^*(w') = \beta^*(w)$. In the first part of the statement, we only obtain a weak inequality since we impose no restriction on $\beta^*(w)$. This is easy to understand if, for example, we think of the case where $r_s > r_f$ for all $s \in S$. In such a case, the decision maker would always choose $\beta^*(w) = w$, no matter what, and the inequality would trivially follow.

Note that the second part of the statement provides a testable implication for wealth constant absolute ambiguity aversion, which could be brought to portfolio composition data. Indeed, under the assumption the agent is risk averse and uncertainty averse as well as $\beta^*(w) > 0$, wealth constant absolute ambiguity aversion yields that the share of wealth invested in the non risk free asset decreases with the agent's wealth, that is,

$$w' > w > 0 \implies \frac{\beta^*(w)}{w} \geq \frac{\beta^*(w')}{w'}.$$

It eluded us to which extent a general portfolio result holds for wealth decreasing absolute ambiguity aversion. We were able to prove such a result for two important classes of preferences: 1) risk neutral smooth ambiguity preferences and 2) CARA multiplier preferences. This is still somehow surprising in light of the negative result of Yaari [33, p. 322 and Figure 2].[33]

In the first case, by Proposition 7 and since $\succsim$ is risk neutral, by choosing $v$ to be the identity, we know that wealth decreasing absolute ambiguity aversion amounts to impose $\phi$ being DARA, provided $\Sigma$ is nontrivial.

**Proposition 10** *Let $\succsim$ be a CARA smooth ambiguity preference with $\phi$ twice continuously differentiable and such that $\phi' > 0$. If $\succsim$ is risk neutral, $\phi$ is concave and DARA, and $\beta^*(w) \in (0, w)$ with $w > 0$, then*

$$w' > w \implies \beta^*\left(w'\right) \geq \beta^*(w). \tag{8}$$

[33] In a nutshell, Yaari, in a mildly different framework, provides an example of convex preferences which are wealth decreasing absolute *uncertainty* averse for which the investment in the uncertain security, at least locally, decreases as wealth increases.

The second result instead deals with Hansen and Sargent [22] multiplier preferences. Recall that $\succsim$ is a multiplier preference if it admits a rational representation $(u, I)$ where

$$I(\varphi) = -\frac{1}{\theta}\log\left(\int e^{-\theta\varphi}dq\right) = \min_{p\in\Delta}\left\{\int \varphi dp + \frac{1}{\theta}R(p||q)\right\}$$

where $\theta > 0$, $q$ is a countably additive element of $\Delta$, and $R(p||q)$ is the relative entropy of $p$ with respect to $q$.[34] Multiplier preferences are variational. By Corollary 6, if $\succsim$ is risk averse then $\succsim$ is wealth decreasing absolute ambiguity averse.

**Proposition 11** *Let $\succsim$ be a CARA multiplier preference. If $\succsim$ is risk averse and $\beta^*(w) \in (0, w)$ with $w > 0$, then (8) holds.*

**Remark 2** Cherbonnier and Gollier [12] carried out a portfolio analysis for wealth decreasing absolute *uncertainty* averse smooth and $\alpha$-maxmin preferences. It is, however, a different exercise than ours, based also on assumptions on returns. Combined with the differences in the frameworks, this makes their results not directly comparable with ours. In particular, in our setting also for the smooth ambiguity model we have a monotonicity result in wealth (cf. [12, Proposition 5]).

### 3.6 Two alternative approaches

In the Introduction,[35] we mentioned that, in defining wealth decreasing absolute ambiguity aversion ($w$-DAAA), the way we formalize the idea that the poorer the decision maker is, the more ambiguity averse he is, rests on two key ingredients:

1. A notion of preference at different wealth levels $w$, in symbols, $\succsim^w$.

2. A notion of comparative ambiguity aversion.

We feel that the notion we use for point 1 merely formalizes the idea of wealth shift. Thus, any alternative notion of $w$-DAAA should rest on a different definition of comparative ambiguity aversion. Next, we mention two possible alternatives one could follow and their potential drawbacks. Before doing so, we introduce a piece of terminology. We say that an act $f \in \mathcal{F}$ is purely ambiguous if and only if $f(s)$ is a degenerate lottery for all $s \in S$. We denote the subset of purely ambiguous acts $\mathcal{F}_{\text{pa}}$. With this in mind, two possible alternative definitions of "more ambiguity averse" are:

a. decision maker 1 is more ambiguity averse than decision maker 2 if for each $f \in \mathcal{F}_{\text{pa}}$ and each $x \in \Delta_0(\mathbb{R})$

$$f \succsim_1 x \implies f \succsim_2 x.$$

[34] See also Maccheroni, Marinacci, and Rustichini [24, Section 4.2.1] as well as Strzalecki [29].

[35] This section has been written in collaboration with Giacomo Cattelan.

b. decision maker 1 is more ambiguity averse than decision maker 2 if for each $A \in \Sigma$, each $\alpha \in [0,1]$, and each $c, d \in \mathbb{R}$

$$\delta_c A \delta_d \succsim_1 \alpha\delta_c + (1-\alpha)\,\delta_d \implies \delta_c A \delta_d \succsim_2 \alpha\delta_c + (1-\alpha)\,\delta_d.$$ [36]

The notion in *a* is rather simple: indeed it amounts to say that if decision maker 1 is bold enough to choose (pure) ambiguity over risk, so does decision maker 2. Compared to Ghirardato and Marinacci [17], by focusing on purely ambiguous acts, this notion does not immediately force the risk attitudes of the decision makers to be the same. The notion in *b*, instead, follows the same structure of *a* but focuses the comparisons on purely ambiguous binary acts vs binary lotteries. This latter definition does also not force risk attitudes to coincide, nevertheless it is rather weak. For, ambiguity preferences are not fully determined by willingness to bet (i.e., by preferences over binary acts), at least in general.[37] It is so for very specific models like Choquet expected utility.

To sum up, using Ghirardato and Marinacci notion of comparative ambiguity aversion allows for arbitrary ambiguity preferences, but forces risk preferences to be CARA. Using the notion in *b*, one restricts ambiguity preferences, but allows for arbitrary risk preferences. On the other hand, using the notion in *a* seems a good compromise. Nevertheless, it would surprisingly lead to the same results for a large class of preferences. Namely, preferences will still turn out to be CARA, provided we restrict attention to invariant biseparable preferences which admit an essential event.[38]

# 4 Relative ambiguity aversion

## 4.1 Relative analysis

In this section we briefly explore relative ambiguity aversion. Due to the relevance of relative attitudes in applied work, we report the main definitions and characterizations. For this reason, we focus on lotteries which yield only strictly positive numbers, interpreted as returns: $X = \Delta_0(\mathbb{R}_{++})$.[39] As before, we consider a group of transformations on $X$, but this time, it

[36] Here, $\delta_c A \delta_d$ is the Anscombe and Aumann act that yields the degenerate lottery $\delta_c$ if $s \in A$ and $\delta_d$ otherwise.

[37] To wit, one could construct two maxmin preferences which are equivalently ambiguity averse according to definition *b*, but where the set of probabilities of the first preference is strictly contained in the second one: a fact which seems counterintuitive. Namely, one would like to say that the larger is the set over which the min is taken, the more ambiguity averse is the decision maker.

[38] An event $A \in \Sigma$ is essential if and only if there exist two consequences $x$ and $y$ such that $x \succ xAy \succ y$ where $xAy$ is the act that yields $x$ if $s \in A$ and $y$ otherwise.

[39] Proofs follow closely the ones carried out for the absolute case and are therefore omitted for brevity. The formal definition of $\Delta_0(\mathbb{R}_{++})$ is easily obtained by replacing $\mathbb{R}$ in the definition of $\Delta_0(\mathbb{R})$ with $\mathbb{R}_{++}$ (see Section 2.1).

is indexed by $\mathbb{R}_{++}$. In particular, given $\nu > 0$, we denote by $^{\nu} : \Delta_0(\mathbb{R}_{++}) \to \Delta_0(\mathbb{R}_{++})$ the affine and onto map such that $x^{\nu}(\nu c) = x(c)$ for all $c \in \mathbb{R}_{++}$ and for all $x \in \Delta_0(\mathbb{R}_{++})$. Given wealth $\nu > 0$, the lottery $x^{\nu}$ is interpreted as the distribution of final wealth if $\nu$ is invested in $x$. A preference $\succsim$ on $\mathcal{F}$ thus induces, through a wealth level $\nu \in \mathbb{R}_{++}$, a preference $\succsim^{\nu}$ on $\mathcal{F}$ given by

$$f \succsim^{\nu} g \Longleftrightarrow f^{\nu} \succsim g^{\nu}$$

where $f^{\nu}$ is the act $s \mapsto f(s)^{\nu}$ for all $s \in S$. In this monetary setup, we have the following classic notion.

**Definition 5** *A preference $\succsim$ on $\mathcal{F}$ is* constant relative risk averse *(CRRA) if, for any two strictly positive levels $\nu$ and $\eta$ of wealth, the induced preferences $\succsim^{\nu}$ and $\succsim^{\eta}$ agree on $\Delta_0(\mathbb{R}_{++})$.*

This behavioral definition amounts to say that preferences over lotteries yielding returns are unaffected by changes in invested wealth. A routine argument shows that, if $\succsim$ is represented by an affine utility function $u : \Delta_0(\mathbb{R}_{++}) \to \mathbb{R}$,[40] then $\succsim$ is CRRA if and only if there exist $\gamma \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v_{\gamma}(c) = \begin{cases} a\gamma c^{\gamma} + b & \text{if } \gamma \neq 0 \\ a \log c + b & \text{if } \gamma = 0 \end{cases}, \tag{9}$$

that is, $v_{\gamma}$ is either a power or the logarithm. Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \gamma < 0 \\ (b, +\infty) & \text{if } \gamma > 0 \\ (-\infty, +\infty) & \text{if } \gamma = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\gamma < 0$ and $b = \inf \operatorname{Im} u$ when $\gamma > 0$. Again, this extremum role of $b$ will play a key role momentarily (Theorem 4).

## 4.2 Relative ambiguity attitudes

Relative ambiguity attitudes describe how the decision maker's preferences over uncertain monetary *returns* vary as the wealth invested changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational preferences.

**Definition 6** *A preference $\succsim$ on $\mathcal{F}$ is wealth decreasing (increasing, constant) relative ambiguity averse if, for any two strictly positive levels $\nu$ and $\eta$ of wealth, $\nu > \eta$ implies that $\succsim^{\eta}$ is more (less, equally) ambiguity averse than $\succsim^{\nu}$.*

[40] Even in this section, we maintain the assumption that if $\succsim$ on $\Delta_0(\mathbb{R}_{++})$ is represented by an affine utility function, then its von Neumann-Morgenstern utility function is strictly increasing and continuous.

Since also this classification of preferences is not exhaustive, we say that a preference is *relatively wealth classifiable* (in terms of relative ambiguity aversion) if it can be classified according to this definition, that is, if it is either wealth decreasing or increasing or constant relative ambiguity averse. The next result shows that being CRRA is a necessary condition for a preference to be relatively wealth classifiable: indeed, in this way *relative risk attitudes* do not intrude in wealth's proportionality effects.

**Proposition 12** *A rational preference $\succsim$ is relatively wealth classifiable only if it is CRRA.*

We next characterize wealth decreasing relative ambiguity aversion for rational preferences.

**Theorem 4** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. The following statements are equivalent:*

*(i) $\succsim$ is wealth decreasing relative ambiguity averse;*

*(ii) $\succsim$ is CRRA and $I$ is:*

*(a) concave (convex) at $b$ provided $\gamma < 0$ ($\gamma > 0$);*

*(b) constant superadditive provided $\gamma = 0$.*

*(iii) $\succsim$ is relatively wealth classifiable and $I$ satisfies (a) or (b).*

Similar characterizations hold for wealth increasing and constant relative ambiguity aversion.[41] We next provide a formal statement of a result mentioned in the Introduction which shows that our results provide some behavioral guidance in the choice of the parameters of specific functional representations.

**Proposition 13** *Let $\succsim$ be a CRRA smooth ambiguity preference with $b = 0$ in (9), $\gamma \in [0, 1)$, and assume that $\Sigma$ is nontrivial. Then,*

*(i) If $\gamma = 0$, $\succsim$ is wealth constant relative ambiguity averse for all $\mu$ if and only if $\phi$ is CARA.*[42]

[41] If we replace wealth decreasing relative ambiguity aversion with wealth increasing relative ambiguity aversion, then we must invert the role of concavity and convexity at $b$ as well as change constant superadditivity in constant subadditivity. Similarly, if we replace wealth decreasing relative ambiguity aversion with wealth constant relative ambiguity aversion, then concavity and convexity at $b$ (resp., constant superadditivity) will become affinity at $b$ (resp., constant additivity).

[42] That is, $\phi : \mathbb{R} \to \mathbb{R}$ is a positive affine transformation of either $-\frac{1}{\beta}e^{-\beta t}$ where $\beta \neq 0$ or the identity.

*(ii) If $\gamma \in (0,1)$, $\succsim$ is wealth constant relative ambiguity averse for all $\mu$ if and only if $\phi$ is CRRA.*[43]

Also in this case, it is possible to introduce *monetary* certainty equivalents. Given a canonical representation $(u, I)$, we can again define the functional $c : \mathcal{F} \to \mathbb{R}_{++}$ by the rule $c(f) = v^{-1}(I(u(f)))$. We will say that $c$ is wealth superproportional (resp., subproportional, proportional) if and only if for each $f \in \mathcal{F}$ and for each $\nu \geq 1$

$$c(f^{\nu}) \geq \nu c(f) \qquad \text{(resp., } \leq, = \text{)}.$$

**Proposition 14** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. Then:*

*(i) $\succsim$ is wealth decreasing relative ambiguity averse if and only if $c$ is wealth superproportional and $\succsim$ is CRRA.*

*(ii) $\succsim$ is wealth increasing relative ambiguity averse if and only if $c$ is wealth subproportional and $\succsim$ is CRRA.*

*(iii) $\succsim$ is wealth constant relative ambiguity averse if and only if $c$ is wealth proportional and $\succsim$ is CRRA.*

### 4.3 Portfolio problem and relative attitudes

We again consider the portfolio problem of Section 3.5. In a nutshell, we consider an agent with rational preferences $(u, I)$ and von Neumann-Morgenstern function $v : \mathbb{R}_{++} \to \mathbb{R}$.[44] The decision maker is choosing an optimal portfolio which can consist of a mixture between a purely ambiguous asset, yielding returns $r_s > 0$ for all $s \in S$, and a risk free asset, yielding a constant return $r_f > 0$. The agent has wealth $w > 0$ which has to be allocated between these two assets. The number $\beta$ denotes the absolute amount of wealth invested in the ambiguous asset. The agent cannot short any of the two securities and therefore $\beta \in [0, w]$. Formally, the portfolio problem takes the form:

$$\max I\left(v\left(\beta r + (w - \beta) r_f\right)\right) \text{ subject to } \beta \in [0, w]. \tag{10}$$

Also here, we assume that this problem always admits a unique solution for all $w > 0$, denoted by $\beta^*(w)$.

**Proposition 15** *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation $(u, I)$. If $\succsim$ is wealth constant relative ambiguity averse, then*

$$w' > w > 0 \implies \frac{\beta^*(w')}{w'} = \frac{\beta^*(w)}{w}.$$

[43] That is, $\phi : (0, \infty) \to \mathbb{R}$ is a positive affine transformation of either $\rho t^{\rho}$ where $\rho \neq 0$ or $\log t$.

[44] In Section 3.5, since we were studying absolute attitudes, $v$ was defined over the entire real line. Nevertheless, given our assumptions on returns the set $(0, \infty)$ suffices.

In order to understand the previous proposition, we recall the standard result under risk for constant relative risk attitudes. In that case, if the decision maker is CRRA and expected utility, then the share of his wealth invested in the risky asset does not depend on the wealth level $w$. Our result is saying that if the risky asset is indeed perceived as ambiguous by the agent, then wealth constant relative ambiguity attitudes would yield the same prediction. In other words, the share of wealth invested in the non risk free asset does not vary with the agent's wealth. It is interesting to note that some papers in the literature exactly look at the share invested in the risky asset to test if CRRA preferences are consistent with the empirical evidence (see, e.g., Brunnermeier and Nagel [5] as well as Chiappori and Paiella [13]). Thus, the empirical evidence in favor of CRRA preferences might be indeed consistent with both CRRA and wealth constant relative ambiguity attitudes. Recall that rational preferences, which are also wealth constant relative ambiguity averse, are necessarily CRRA. So, examples of rational preferences that are wealth constant relative ambiguity averse are: $\alpha$-maxmin, Choquet expected utility, variational if $v$ is the logarithm, and vector expected utility if $v$ is the logarithm.

# A Appendix: Mathematics

We denote by $B_0(\Sigma)$ the set of all real-valued $\Sigma$-measurable simple functions. If $T$ is an interval of the real line, set $B_0(\Sigma, T) = \{\psi \in B_0(\Sigma) : \psi(s) \in T \text{ for all } s \in S\}$. We endow both $B_0(\Sigma)$ and $B_0(\Sigma, T)$ with the topology induced by the supnorm.

With a small abuse of notation, we denote by $k$ both the real number and the constant function on $S$ that takes value $k$. Let $\varphi, \psi \in B_0(\Sigma, T)$. A functional $I : B_0(\Sigma, T) \to \mathbb{R}$ is:

(i) *normalized* if $I(k) = k$ for all $k \in T$;

(ii) *monotone* if $\varphi \geq \psi$ implies $I(\varphi) \geq I(\psi)$;

(iii) *quasiconcave* if $I(\lambda\varphi + (1-\lambda)\psi) \geq \min\{I(\varphi), I(\psi)\}$ for all $\lambda \in (0,1)$;

(iv) *positively superhomogeneous* (*subhomogeneous*) if $I(\lambda\varphi) \geq (\leq) \lambda I(\varphi)$ for all $\lambda \in (0,1)$ such that $\lambda\varphi \in B_0(\Sigma, T)$;

(v) *positively homogeneous* if it is both: positively superhomogeneous and subhomogeneous;[45]

(vi) *concave* (*convex*) at $k \in \mathrm{cl}(T)$ if $I(\lambda\varphi + (1-\lambda)k) \geq (\leq) \lambda I(\varphi) + (1-\lambda)k$ for all $\lambda \in (0,1)$;

[45] When either $T = (-\infty, 0)$ or $T = (0, \infty)$ or $T = \mathbb{R}$, then $I$ is positively homogeneous if and only if $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\varphi \in B_0(\Sigma, T)$ and for all $\lambda > 0$. Often, in this paper, in talking about positive homogeneity properties of $I$, we will either say $I$ is (sup/sub)homogeneous, dropping the qualifier positive, or equivalently say it is positive (sup/sub)homogeneous as well as positively (sup/sub)homogeneous.

(vii) *affine* at $k \in \mathrm{cl}(T)$ if it is both concave and convex at $k$;

(viii) *constant superadditive* (*subadditive*) if $I(\varphi + k) \geq (\leq) I(\varphi) + k$ for all $k \geq 0$ such that $\varphi + k \in B_0(\Sigma, T)$.

(ix) *constant additive* if $I$ is both constant superadditive and subadditive;[46]

(x) *constant linear* if $I(\lambda\varphi + k) = \lambda I(\varphi) + k$ for all $\lambda \in (0, 1]$ and $k \in \mathbb{R}$ such that $\lambda\varphi + k \in B_0(\Sigma, T)$. If $T$ is either $(-\infty, 0)$ or $(0, \infty)$ or $\mathbb{R}$, this amounts to impose that $I$ is constant additive and positively homogeneous.

When $k = 0$, concavity (convexity) at $k$ reduces to positive superhomogeneity (subhomogeneity).

As well known, the norm dual space of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on $(S, \Sigma)$. The set of probabilities in $ba(\Sigma)$ is denoted by $\Delta$ and is a (weak*) compact and convex subset of $ba(\Sigma)$. Elements of $\Delta$ are denoted by $p$ or $q$. We endow $\Delta$ and any of its subsets with the weak* topology.

Functions of the form $G : T \times \Delta \to (-\infty, \infty]$, where $T$ is an interval of the real line, play an important role in Section 3.3. We denote by $\mathcal{G}(T \times \Delta)$ the class of these functions such that:

(i) $G$ is quasiconvex on $T \times \Delta$,

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$,

(iii) $\inf_{p \in \Delta} G(t, p) = t$ for all $t \in T$.

A function $G : T \times \Delta \to (-\infty, \infty]$ is *linearly continuous* if the map

$$\psi \mapsto \inf_{p \in \Delta} G\left(\int \psi dp, p\right)$$

from $B_0(\Sigma, T)$ to $[-\infty, \infty]$ is extended-valued continuous. Finally, given a function, say $u : X \to \mathbb{R}$, we will denote its image, that is $u(X)$, by $\mathrm{Im}\, u$.

# B Appendix: Proofs and related material

We begin with a preliminary result that will be used in the appendix.

**Lemma 1** *Let $\succsim_1$ and $\succsim_2$ be two rational preferences on $\mathcal{F}$ with representations $(u_1, I_1)$ and $(u_2, I_2)$. The following statements are equivalent:*

[46] Note that $I$ is constant additive if and only if $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma, T)$ and for all $k \in \mathbb{R}$ such that $\varphi + k \in B_0(\Sigma, T)$. In other words, if $I(\varphi + k) = I(\varphi) + k$ holds for positive constants, then it also holds for $k < 0$, provided $\varphi, \varphi + k \in B_0(\Sigma, T)$.

*(i)* $\succsim_1$ *is more ambiguity averse than* $\succsim_2$*;*

*(ii) There exist* $a > 0$ *and* $b \in \mathbb{R}$ *such that* $u_1 = au_2 + b$ *and* $I_1 \le I_2$ *(provided* $u_1 = u_2$*).*

## B.1 Generic set of consequences and shifted preferences

**Proof of Proposition 2.** Clearly, $\succsim^w$ is well defined. Moreover, we have

$$f \succ^w g \iff f \succsim^w g \text{ and } g \not\succsim^w f \iff f^w \succsim g^w \text{ and } g^w \not\succsim f^w \iff f^w \succ g^w.$$

(i). *Weak Order.* Since $\succsim$ satisfies Weak Order and Monotonicity, it follows that there exist $\bar{x}$ and $\bar{y}$ in $X$ such that $\bar{x} \succ \bar{y}$. Since $^w$ is bijective, it follows that there exist $x, y \in X$ such that $\bar{x} = x^w$ and $\bar{y} = y^w$. By definition of $\succsim^w$, we have that

$$\bar{x} \succ \bar{y} \implies x^w \succ y^w \implies x \succ^w y,$$

proving that $\succsim^w$ is nontrivial. Consider $f, g \in \mathcal{F}$. Since $f^w, g^w \in \mathcal{F}$ and $\succsim$ satisfies Weak Order, we have that either $f^w \succsim g^w$ or $g^w \succsim f^w$. By definition of $\succsim^w$, this implies that either $f \succsim^w g$ or $g \succsim^w f$ or both, thus proving that $\succsim^w$ is complete. Next, consider $f, g, h \in \mathcal{F}$ and assume that $f \succsim^w g$ and $g \succsim^w h$. By definition of $\succsim^w$, we have that $f^w \succsim g^w$ and $g^w \succsim h^w$. Since $\succsim$ satisfies Weak Order, we can conclude that $f^w \succsim h^w$, that is, $f \succsim^w h$, proving that $\succsim^w$ is transitive. We can conclude that $\succsim^w$ satisfies Weak Order.

*Monotonicity.* Consider $f, g \in \mathcal{F}$ and assume that $f(s) \succsim^w g(s)$ for all $s \in S$. By definition of $\succsim^w$ and $^w$, it follows that $f^w(s) = f(s)^w \succsim g(s)^w = g^w(s)$ for all $s \in S$. Since $\succsim$ satisfies Monotonicity, we have that $f^w \succsim g^w$, that is, $f \succsim^w g$.

*Continuity.* Consider $f, g, h \in \mathcal{F}$ and a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [0,1]$ such that $\alpha_n \to \alpha$ and $\alpha_n f + (1-\alpha_n) g \succsim^w h$ for all $n \in \mathbb{N}$. By definition of $\succsim^w$ and since $^w$ is affine, we have $\alpha_n f^w + (1-\alpha_n) g^w = (\alpha_n f + (1-\alpha_n) g)^w \succsim h^w$ for all $n \in \mathbb{N}$. Since $\succsim$ satisfies Mixture Continuity, we have that $(\alpha f + (1-\alpha) g)^w = \alpha f^w + (1-\alpha) g^w \succsim h^w$. We can conclude that $\alpha f + (1-\alpha) g \succsim^w h$. Thus, the set $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim^w h\}$ is closed. A symmetric argument yields the closure of $\{\alpha \in [0,1] : h \succsim^w \alpha f + (1-\alpha) g\}$.

*Risk Independence.* Consider $x, y, z \in X$, $\alpha \in (0,1)$, and assume that $x \sim^w y$. It follows that $x^w \sim y^w$. Since $\succsim$ satisfies Risk Independence and $^w$ is affine, we have that

$$(\alpha x + (1-\alpha) z)^w = \alpha x^w + (1-\alpha) z^w \sim \alpha y^w + (1-\alpha) z^w = (\alpha y + (1-\alpha) z)^w,$$

proving that $\alpha x + (1-\alpha) z \sim^w \alpha y + (1-\alpha) z$.

(ii). We only need to show that $\succsim^w$ also satisfies Convexity.

*Convexity.* Consider $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$ and assume that $f \sim^w g$. It follows that $f^w \sim g^w$. Since $\succsim$ satisfies Convexity and $^w$ is affine, we have that $(\alpha f + (1-\alpha) g)^w = \alpha f^w + (1-\alpha) g^w \succsim f^w$, that is, $\alpha f + (1-\alpha) g \succsim^w f$. ∎

**Proof of Proposition 3.** Let $w, w' \in \mathbb{R}$. By Proposition 2, both preferences $\succsim^w$ and $\succsim^{w'}$ are rational preferences. By Theorem 1, both preferences have a canonical representation: $(u_w, I_w)$ and $(u_{w'}, I_{w'})$. In particular, $u_w$ and $u_{w'}$ are nonconstant and affine. Since $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$, we have that $y \succsim^w x$ implies $y \succsim^{w'} x$. Thus, we conclude that $u_w(y) \geq u_w(x)$ implies $u_{w'}(y) \geq u_{w'}(x)$. By [15, Corollary B.3], the statement follows. ■

The next result will be instrumental in proving Theorem 2.

**Proposition 16** *Let $(u, I)$ and $(\bar{u}, \bar{I})$ be two canonical rational representations. The two representations $(u, I)$ and $(\bar{u}, \bar{I})$ represent the same rational preference $\succsim$ if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$\bar{u} = au + b \text{ and } \bar{I}(\cdot) = aI\left(\frac{\cdot - b}{a}\right) + b.$$

*Moreover,*

*(i) $I$ is concave if and only if $\bar{I}$ is concave.*

*(ii) $I$ is concave (convex, affine) at $c$ if and only if $\bar{I}$ is concave (convex, affine) at $ac + b$.*

*(iii) $I$ is constant superadditive (subadditive, additive) if and only if $\bar{I}$ is constant superadditive (subadditive, additive), provided* $\operatorname{Im} u$ *is unbounded from above.*

**Proof.** The first part of the statement follows from [6, Proposition 1]. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = at + b$ for all $t \in \mathbb{R}$. Define $T : B_0(\Sigma, \operatorname{Im}\bar{u}) \to B_0(\Sigma, \operatorname{Im} u)$ as $T(\varphi) = \frac{\varphi - b}{a}$ for all $\varphi \in B_0(\Sigma, \operatorname{Im}\bar{u})$. Note that both functions are bijective and $\bar{I} = f \circ I \circ T$ as well as $I = f^{-1} \circ \bar{I} \circ T^{-1}$.

(i). "Only if". Assume that $I$ is concave. Since $f$ and $T$ are monotone and affine and $\bar{I} = f \circ I \circ T$, it follows that $\bar{I}$ is concave. "If". Note that $I = f^{-1} \circ \bar{I} \circ T^{-1}$. Assume that $\bar{I}$ is concave. Since $f^{-1}$ and $T^{-1}$ are monotone and affine, it follows that $I$ is concave.

(ii). "Only if". Assume that $I$ is concave (convex, affine) at $c \in \operatorname{cl}(\operatorname{Im} u)$. Note that $\bar{c} = ac + b \in \operatorname{cl}(\operatorname{Im}\bar{u})$. It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im}\bar{u})$ and for each $\lambda \in (0, 1)$

$$\begin{aligned}
\bar{I}(\lambda\varphi + (1-\lambda)\bar{c}) &= aI\left(\frac{\lambda\varphi + (1-\lambda)\bar{c} - b}{a}\right) + b = aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)\frac{\bar{c} - b}{a}\right) + b \\
&= aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)\frac{ac + b - b}{a}\right) + b \\
&= aI\left(\lambda\frac{\varphi - b}{a} + (1-\lambda)c\right) + b \\
&\geq (\leq, =)\, a\left(\lambda I\left(\frac{\varphi - b}{a}\right) + (1-\lambda)c\right) + b \\
&= \lambda\left(aI\left(\frac{\varphi - b}{a}\right) + b\right) + (1-\lambda)(ac + b) = \lambda\bar{I}(\varphi) + (1-\lambda)\bar{c},
\end{aligned}$$

proving that $\bar{I}$ is concave (convex, affine) at $\bar{c}$. "If". Assume that $\bar{I}$ is concave (convex, affine) at $\bar{c} = ac + b$. It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $\lambda \in (0,1)$

$$
\begin{aligned}
I(\lambda\varphi + (1-\lambda)c) &= \frac{1}{a}\bar{I}(a(\lambda\varphi + (1-\lambda)c) + b) - \frac{b}{a} \\
&= \frac{1}{a}\bar{I}(\lambda(a\varphi + b) + (1-\lambda)(ac + b)) - \frac{b}{a} \\
&= \frac{1}{a}\bar{I}(\lambda(a\varphi + b) + (1-\lambda)\bar{c}) - \frac{b}{a} \\
&\geq (\leq, =) \frac{1}{a}\left(\lambda\bar{I}(a\varphi + b) + (1-\lambda)\bar{c}\right) - \frac{b}{a} \\
&= \lambda\left(\frac{1}{a}\bar{I}(a\varphi + b) - \frac{b}{a}\right) + (1-\lambda)\left(\frac{\bar{c}}{a} - \frac{b}{a}\right) = \lambda I(\varphi) + (1-\lambda)c,
\end{aligned}
$$

proving that $I$ is concave (convex, affine) at $c$.

(iii). "Only if". Assume that $I$ is constant superadditive (subadditive, additive). It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} \bar{u})$ and for each $k \geq 0$

$$
\begin{aligned}
\bar{I}(\varphi + k) &= aI\left(\frac{\varphi + k - b}{a}\right) + b = aI\left(\frac{\varphi - b}{a} + \frac{k}{a}\right) + b \\
&\geq (\leq, =)\, a\left(I\left(\frac{\varphi - b}{a}\right) + \frac{k}{a}\right) + b = aI\left(\frac{\varphi - b}{a}\right) + b + k = \bar{I}(\varphi) + k,
\end{aligned}
$$

proving that $\bar{I}$ is constant superadditive (subadditive, additive). "If". Assume that $\bar{I}$ is constant superadditive (subadditive, additive). It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $k \geq 0$

$$
\begin{aligned}
I(\varphi + k) &= \frac{1}{a}\bar{I}(a(\varphi + k) + b) - \frac{b}{a} = \frac{1}{a}\bar{I}((a\varphi + b) + ak) - \frac{b}{a} \\
&\geq (\leq, =) \frac{1}{a}\left(\bar{I}(a\varphi + b) + ak\right) - \frac{b}{a} = \left(\frac{1}{a}\bar{I}(a\varphi + b) - \frac{b}{a}\right) + k = I(\varphi) + k,
\end{aligned}
$$

proving that $I$ is constant superadditive (subadditive, additive). ■

## B.2 Monetary consequences

We next prove a couple of ancillary facts. Moreover, when $\succsim$ (on $\Delta_0(\mathbb{R})$) is represented by an affine $u$ and is CARA, we first assume that $v$ of $u$ corresponds to (5) with $a = 1$ and $b = 0$, that is, we normalize the von Neumann-Morgenstern utility function $v$ to be such that

$$
\bar{v}_\alpha(c) = \begin{cases} -\frac{1}{\alpha}e^{-\alpha c} & \text{if } \alpha \neq 0 \\ c & \text{if } \alpha = 0 \end{cases} . \tag{11}
$$

In this case, for each $w \in \mathbb{R}$ and for each lottery $x \in \Delta_0(\mathbb{R})$, either $u(x^w) = e^{-\alpha w}u(x)$ or $u(x^w) = u(x) + w$.

**Lemma 2** *If $\succsim$ is a CARA rational preference with representation $(u, I)$, then $\succsim^w$ is a rational preference with representation $(u, I_w)$. Moreover, if we choose $v = \bar{v}_\alpha$ as in (11), then $I_w$ is such that*

$$I_w(\varphi) = \begin{cases} I(\varphi + w) - w & \text{if } \succsim \text{ is risk neutral} \\ e^{\alpha w} I(e^{-\alpha w}\varphi) & \text{otherwise} \end{cases} \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

**Proof.** By Proposition 2, both preferences $\succsim^w$ and $\succsim$ are rational for all $w \in \mathbb{R}$. By assumption, $\succsim$ is CARA. Thus, $\succsim^w$ coincides with $\succsim$ on $\Delta_0(\mathbb{R})$ and it has a canonical representation $(u_w, I_w)$ where $v_w$ of $u_w$ is either exponential or affine as in (5). Wlog, we can thus set $u = u_w$ and choose $v$ as in (11). By [6, Proposition 1], we have that

$$I(\varphi) = u(x_g) \text{ where } x_g \sim g \text{ and } u(g) = \varphi$$

and

$$I_w(\varphi) = u(x_{f,w}) \text{ where } x_{f,w} \sim^w f \text{ and } u(f) = \varphi.$$

a) Assume that $v = \bar{v}_\alpha$ is exponential (risk nonneutral case), that is, $\bar{v}_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$ for all $c \in \mathbb{R}$. This implies that either $\operatorname{Im} u = (0, \infty)$ or $\operatorname{Im} u = (-\infty, 0)$, in particular, for each $w \in \mathbb{R}$ and $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $e^{-\alpha w}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ a certainty equivalent of $f$ for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = e^{-\alpha w}u(f) = e^{-\alpha w}\varphi$ and $u\left(x^w_{f,w}\right) = e^{-\alpha w}u(x_{f,w})$. If we define $g = f^w$, then we also have that $x_g$ can be chosen to be $x^w_{f,w}$, that is,

$$I\left(e^{-\alpha w}\varphi\right) = I(u(g)) = u(x_g) = e^{-\alpha w}u(x_{f,w}) = e^{-\alpha w}I_w(\varphi),$$

and so $I_w(\varphi) = e^{\alpha w}I(e^{-\alpha w}\varphi)$.

b) Assume that $v = \bar{v}_\alpha$ is the identity (risk neutral case). This implies that $\operatorname{Im} u = \mathbb{R}$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ a certainty equivalent of $f$ for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = u(f) + w = \varphi + w$ and $u\left(x^w_{f,w}\right) = u(x_{f,w}) + w$. If we define $g = f^w$, then we also have that $x_g$ can be chosen to be $x^w_{f,w}$, that is,

$$I(\varphi + w) = I(u(g)) = u(x_g) = u(x_{f,w}) + w = I_w(\varphi) + w,$$

and so $I_w(\varphi) = I(\varphi + w) - w$. ■

**Proof of Proposition 4.** Let $w, w' \in \mathbb{R}$ be such that $w \neq w'$. If $\succsim$ is wealth decreasing or constant absolute ambiguity averse, wlog, we can assume that $w' > w$. If $\succsim$ is wealth increasing absolute ambiguity averse, wlog, we can assume that $w > w'$. By Proposition 3 and since $\succsim$ is wealth classifiable, we have that $u_w$ is a positive affine transformation of $u_{w'}$ and this holds for all $w, w' \in \mathbb{R}$, proving that $\succsim$ is CARA. ■

**Proof of Theorem 2.** Let $\succsim$ be a rational preference with canonical representation $(u, I)$ where $u$ is such that $u(x) = \sum_{c \in \mathbb{R}} v(c) x(c)$ for every $x \in \Delta_0(\mathbb{R})$, with $v$ strictly increasing and continuous. Before starting the proof, we add few extra points.

(iv) $\succsim$ is CARA and $I_w \leq I_{w'}$, provided $w' > w$ and $u_w = u_{w'} = u$ and $v = \bar{v}_\alpha$ is as in (11);

(v) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (11), for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $w, w' \in \mathbb{R}$ such that $w' > w$, either

$$e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \text{ if } \bar{v}_\alpha \text{ is exponential} \tag{12}$$

or

$$I(\varphi + w) - w \leq I\left(\varphi + w'\right) - w' \text{ if } \bar{v}_\alpha \text{ is the identity.} \tag{13}$$

(vi) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (11), $I$ is:

(a) superhomogeneous (subhomogeneous) provided $\succsim$ is risk averse (loving);

(b) constant superadditive provided $\succsim$ is risk neutral.

(iii) implies (ii). By Proposition 4, we have that $\succsim$ is CARA. The implication trivially follows.

(ii) implies (vi). By assumption, $\succsim$ is CARA. We can thus choose a canonical representation $\left(\bar{u}, \bar{I}\right)$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} = (-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 16, the implication follows.

(vi) implies (v). $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ is as in (11), we have three cases:

a. $\succsim$ is risk averse, that is, $\alpha > 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w - w')} \in (0, 1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w-w')}\left(e^{-\alpha w}\varphi\right)\right) \geq e^{\alpha(w-w')} I\left(e^{-\alpha w}\varphi\right) \implies e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \geq e^{\alpha w} I\left(e^{-\alpha w}\varphi\right),$$

since $\varphi$ was arbitrarily chosen the statement follows.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w' - w)} \in (0, 1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w'-w)}\left(e^{-\alpha w'}\varphi\right)\right) \leq e^{\alpha(w'-w)} I\left(e^{-\alpha w'}\varphi\right) \implies e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right),$$

since $\varphi$ was arbitrarily chosen the statement follows.

c. $\succsim$ is risk neutral, that is, $\alpha = 0$ and $\bar{v}_\alpha$ is the identity. Consider $w' > w$. It follows that $k = (w' - w) > 0$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $\varphi + w, \varphi + w' \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(\varphi + w + \left(w' - w\right)\right) \geq I(\varphi + w) + \left(w' - w\right) \implies I\left(\varphi + w'\right) - w' \geq I(\varphi + w) - w,$$

since $\varphi$ was arbitrarily chosen the statement follows.

(v) is equivalent to (iv). By assumption, $\succsim$ is CARA. We consider two cases. For each $w, w' \in \mathbb{R}$:

a. $v = \bar{v}_\alpha$ is exponential. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

b. $v = \bar{v}_\alpha$ is the identity. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff I(\varphi + w) - w \leq I\left(\varphi + w'\right) - w' \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

Subpoints a. and b. prove the equivalence between (iv) and (v).

(iv) implies (i). Let $w' > w$. By Lemma 2 and since $\succsim$ is CARA, we have that both preferences, $\succsim^w$ and $\succsim^{w'}$, admit a representation $(u_w, I_w)$ and $(u_{w'}, I_{w'})$. Since $\succsim$ is CARA, we can choose $u_w = u_{w'} = u$ with $v = \bar{v}_\alpha$ for all $w, w' \in \mathbb{R}$. By Lemma 1 and since $I_w \leq I_{w'}$, we can conclude that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$.

(i) implies (iv). By Proposition 4, since $\succsim$ is wealth decreasing absolute ambiguity averse, $\succsim$ is CARA. By Lemma 2, we have that for each $w \in \mathbb{R}$ the preference $\succsim^w$ admits a canonical representation $(u_w, I_w)$. Thus, we can choose $u_w = u$ for all $w \in \mathbb{R}$ with $v = \bar{v}_\alpha$. By Lemma 1 and since $u_w = u_{w'}$ for all $w, w' \in \mathbb{R}$, note that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$ only if $I_w \leq I_{w'}$.

(iv) implies (vi). By the previous part of the proof, we know that (iv) is equivalent to (v). We thus assume (v) and prove (vi). We have three cases.

a. $\succsim$ is risk averse, that is, $\alpha > 0$. In (12) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since $\alpha$ is positive, it follows that $e^{\alpha w'} > 1$ and $\left\{e^{\alpha w'} : w' > 0\right\} = (1, \infty)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma > 1$. In other words, $\lambda I(\varphi) \leq I(\lambda\varphi)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\lambda \in (0, 1)$, proving superhomogeneity.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. In (12) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since $\alpha$ is negative, it follows that $\left\{e^{\alpha w'} : w' > 0\right\} = (0,1)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma \in (0,1)$. If $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, then $\lambda\varphi \in B_0(\Sigma, \operatorname{Im} u)$ for all $\lambda \in (0,1)$. We have that

$$I(\lambda\varphi) \leq \lambda I\left(\frac{1}{\lambda}(\lambda\varphi)\right) = \lambda I(\varphi) \qquad \forall\varphi \in B_0(\Sigma, \operatorname{Im} u), \forall\lambda \in (0,1),$$

proving subhomogeneity.

c. $\succsim$ is risk neutral, that is, $\bar{v}_\alpha$ is the identity. In (13) set $w = 0$ and $k = w'$, so that

$$I(\varphi) \leq I(\varphi + k) - k \qquad \forall\varphi \in B_0(\Sigma, \operatorname{Im} u), \forall k > 0.$$

In other words, $I(\varphi) + k \leq I(\varphi + k)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $k > 0$, proving superadditivity.

(vi) implies (ii). By assumption, $\succsim$ is CARA and represented by $(u, I)$. We can thus choose a canonical representation $(\bar{u}, \bar{I})$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} = (-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 16, the implication follows.

We thus proved that (iii) implies (ii) and (ii) is equivalent to (i), (iv), (v), and (vi). In particular, it follows that (ii) implies (i), thus $\succsim$ is wealth classifiable, and $I$ satisfies condition (a) or (b), that is, (ii) implies (iii). ■

### B.3 Other proofs

**Proof of Corollary 2.** Call $(u, I)$ the rational representation of $\succsim$ on $\mathcal{F}$. Since $\succsim$ is risk neutral, it follows that $\operatorname{Im} u = \mathbb{R}$ and $I : B_0(\Sigma) \to \mathbb{R}$.

"Only if." By point 1 of Corollary 1, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \geq 0$. It is immediate to show that the equality holds for all $k \in \mathbb{R}$. By [24, Lemma 25], it follows that $I$ is a normalized niveloid (see, e.g., [9]). By [24, Lemma 28], we can conclude that $\succsim$ satisfies Weak C-Independence.

"If." By [24, Lemma 28], it follows that $I$ is a normalized niveloid. By [24, Lemma 25] and since $\operatorname{Im} u = \mathbb{R}$, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \in \mathbb{R}$. By point 1 of Corollary 1 (recall that it holds by only assuming CARA in place of wealth classifiable), the statement follows. ■

**Proof of Corollary 3.** Call $(u, I)$ the rational representation of $\succsim$. Note that in all three points (i)–(iii), $\succsim$ is necessarily CARA. Thus, wlog, choose $v$ to be such that $a = 1$ and $b = 0$. By [15], there also exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\Sigma) \to \mathbb{R}$ such that for each $\varphi \in B_0(\Sigma)$

$$\hat{I}(\lambda\varphi + k) = \lambda\hat{I}(\varphi) + k \qquad \forall\lambda > 0, \forall k \in \mathbb{R}$$

and $f \succsim g$ if and only if $\hat{I}(u(f)) \geq \hat{I}(u(g))$. It follows that $\hat{I}$ and $I$ coincide on $B_0(\Sigma, \operatorname{Im} u)$.

(i) implies (iii). By Proposition 4, the implication follows.

(iii) implies (ii). By Corollary 1 (recall that it holds by only assuming CARA in place of wealth classifiable) and since $\hat{I}$ and $I$ coincide on $B_0(\Sigma, \operatorname{Im} u)$, the implication follows.

(ii) implies (i). Trivially, $\succsim$ is wealth classifiable. ■

**Proof of Proposition 5.** Let $(u, I)$ be the canonical representation of $\succsim$. Wlog, if $\succsim$ is CARA, we choose $v$ to be such that $a = 1$ and $b = 0$ (see equation (5)). In this case, by the definition of $c : \mathcal{F} \to \mathbb{R}$, we have that

$$c(f) = \begin{cases} -\frac{1}{\alpha}\log(-\alpha I(u(f))) & \alpha \neq 0 \\ I(u(f)) & \alpha = 0 \end{cases} \qquad \forall f \in \mathcal{F}.$$

Recall that for each $f \in \mathcal{F}$ and for each $w \in \mathbb{R}$

$$u(f^w) = \begin{cases} e^{-\alpha w}u(f) & \alpha \neq 0 \\ u(f) + w & \alpha = 0 \end{cases}.$$

(i). "Only if". By Proposition 4, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c(f^w) = I(u(f^w)) = I(u(f) + w)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) = I(u(f) + w) \geq I(u(f)) + w = c(f) + w,$$

proving that $c$ is wealth superadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in (0, 1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f^w) &= -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \geq -\frac{1}{\alpha}\log\left(-\alpha e^{-\alpha w} I(u(f))\right) \\ &= -\frac{1}{\alpha}\log\left(e^{-\alpha w}(-\alpha I(u(f)))\right) = -\frac{1}{\alpha}\log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha}\log(-\alpha I(u(f))) \\ &= c(f) + w, \end{aligned}$$

proving that $c$ is wealth superadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in (0, 1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f) &= -\frac{1}{\alpha}\log(-\alpha I(u(f))) = -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w}u(f)\right)\right)\right) \\ &\leq -\frac{1}{\alpha}\log\left(-\alpha e^{\alpha w} I\left(e^{-\alpha w}u(f)\right)\right) = -\frac{1}{\alpha}\log\left(e^{\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \\ &= -w + c(f^w), \end{aligned}$$

proving that $c$ is wealth superadditive.

"If". First, observe that

$$f \succsim g \iff I(u(f)) \geq I(u(g)) \iff v^{-1}(I(u(f))) \geq v^{-1}(I(u(g))) \iff c(f) \geq c(g).$$

Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and $c$ is wealth superadditive, it follows that

$$c\left(f^{w'}\right) = c\left((f^{w})^{w'-w}\right) \geq c(f^{w}) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \geq c(f^{w}) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned} f \succsim^{w} x &\implies f^{w} \succsim x^{w} \implies c(f^{w}) \geq c(x^{w}) \implies c(f^{w}) \geq c(x) + w \\ &\implies c(f^{w}) - w \geq c(x) \implies c\left(f^{w'}\right) - w' \geq c(x) \implies c\left(f^{w'}\right) \geq c(x) + w' \\ &\implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies f^{w'} \succsim x^{w'} \implies f \succsim^{w'} x. \end{aligned}$$

Since $f$, $x$, $w$, and $w'$ were arbitrarily chosen, we have that $\succsim^{w}$ is more ambiguity averse than $\succsim^{w'}$, proving the statement.

(ii). "Only if". By Proposition 4, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c(f^{w}) = I(u(f^{w})) = I(u(f) + w)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By what follows right after Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^{w}) = I(u(f) + w) \leq I(u(f)) + w = c(f) + w,$$

proving that $c$ is wealth subadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c(f^{w}) = v^{-1}(I(u(f^{w}))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in (0, 1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f^{w}) &= -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \leq -\frac{1}{\alpha}\log\left(-\alpha e^{-\alpha w} I(u(f))\right) \\ &= -\frac{1}{\alpha}\log\left(e^{-\alpha w}(-\alpha I(u(f)))\right) = -\frac{1}{\alpha}\log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha}\log(-\alpha I(u(f))) \\ &= c(f) + w, \end{aligned}$$

proving that $c$ is wealth subadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c(f^{w}) = v^{-1}(I(u(f^{w}))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in (0, 1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f) &= -\frac{1}{\alpha}\log(-\alpha I(u(f))) = -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w}u(f)\right)\right)\right) \\ &\geq -\frac{1}{\alpha}\log\left(-\alpha e^{\alpha w} I\left(e^{-\alpha w}u(f)\right)\right) = -\frac{1}{\alpha}\log\left(e^{\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \\ &= -w + c(f^{w}), \end{aligned}$$

proving that $c$ is wealth subadditive.

"If". First, recall that $f \succsim g$ if and only if $c(f) \geq c(g)$. Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and $c$ is wealth subadditive, it follows that

$$c\left(f^{w'}\right) = c\left((f^{w})^{w'-w}\right) \leq c(f^{w}) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \leq c(f^{w}) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned} f \succsim^{w'} x &\implies f^{w'} \succsim x^{w'} \implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies c\left(f^{w'}\right) \geq c(x) + w' \\ &\implies c\left(f^{w'}\right) - w' \geq c(x) \implies c(f^{w}) - w \geq c(x) \implies c(f^{w}) \geq c(x) + w \\ &\implies c(f^{w}) \geq c(x^{w}) \implies f^{w} \succsim x^{w} \implies f \succsim^{w} x. \end{aligned}$$

Since $f$, $x$, $w$, and $w'$ were arbitrarily chosen, we have that $\succsim^{w'}$ is more ambiguity averse than $\succsim^{w}$, proving the statement.

(iii). It is an easy consequence of points (i) and (ii). ■

**Proof of Theorem 3.** Recall that an uncertainty averse preference is a rational preference. In particular, given a canonical representation $(u, I)$, we have that

$$G(t,p) = \sup_{\varphi \in B_0(\Sigma, \operatorname{Im} u)} \left\{ I(\varphi) : \int \varphi dp \leq t \right\} \qquad \forall (t,p) \in \operatorname{Im} u \times \Delta.$$

(i) implies (ii). By Theorem 2, it follows that $\succsim$ is CARA and $I$ is either concave at $b$, or convex at $b$, or constant superadditive, depending on $\succsim$ being, respectively, either risk averse, or risk loving, or risk neutral. We consider the three different cases separately:

- $\succsim$ is risk averse. Thus, $\operatorname{Im} u = (-\infty, b)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\lambda\varphi_n + (1-\lambda) b)\, dp \leq \lambda t + (1-\lambda) b \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since $I$ is concave at $b$, we have that for each $n \in \mathbb{N}$

$$G(\lambda t + (1-\lambda) b, p) \geq I(\lambda \varphi_n + (1-\lambda) b) \geq \lambda I(\varphi_n) + (1-\lambda) b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t,p) + (1-\lambda) b$.

- $\succsim$ is risk loving. Thus, $\operatorname{Im} u = (b, \infty)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(\lambda t + (1-\lambda) b, p)$ and $\int \varphi_n dp \leq \lambda t + (1-\lambda) b$ for all $n \in \mathbb{N}$. Define $\{\psi_n\}_{n\in\mathbb{N}}$ to be such that

$$\psi_n = \frac{\varphi_n - (1-\lambda) b}{\lambda} \qquad \forall n \in \mathbb{N}.$$

Note also that

$$\psi_n(s) > b \quad \forall s \in S, \int \psi_n dp \leq t, \text{ and } \varphi_n = \lambda \psi_n + (1-\lambda) b \qquad \forall n \in \mathbb{N}.$$

Since $I$ is convex at $b$, this implies that for each $n \in \mathbb{N}$

$$I(\varphi_n) = I(\lambda \psi_n + (1-\lambda) b) \leq \lambda I(\psi_n) + (1-\lambda) b \leq \lambda G(t,p) + (1-\lambda) b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda) b, p) \leq \lambda G(t,p) + (1-\lambda) b$.

- $\succsim$ is risk neutral. Thus, $\operatorname{Im} u = \mathbb{R}$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $k \geq 0$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\varphi_n + k) dp \leq t + k \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since $I$ is constant superadditive, we have that for each $n \in \mathbb{N}$

$$G(t+k,p) \geq I(\varphi_n + k) \geq I(\varphi_n) + k.$$

By passing to the limit, it follows that $G(t+k,p) \geq G(t,p) + k$.

(ii) implies (iii) and (i). Recall that

$$I(\psi) = \inf_{p\in\Delta} G\left(\int \psi dp, p\right) \qquad \forall \psi \in B_0(\Sigma, \operatorname{Im} u).$$

Observe also that $\succsim$ is CARA by assumption and $G$ satisfies (a) or (b). As before, we consider three cases:

- $\succsim$ is risk averse. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned}
I(\lambda\varphi + (1-\lambda) b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda) b)\, dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda) b, p\right) \\
&\geq \inf_{p\in\Delta} \left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda) b\right) \\
&\geq \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda) b = \lambda I(\varphi) + (1-\lambda) b,
\end{aligned}$$

that is, $I$ is concave at $b$.

- $\succsim$ is risk loving. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned}
I(\lambda\varphi + (1-\lambda) b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda) b)\, dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda) b, p\right) \\
&\leq \inf_{p\in\Delta} \left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda) b\right) \\
&= \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda) b = \lambda I(\varphi) + (1-\lambda) b,
\end{aligned}$$

that is, $I$ is convex at $b$.

- $\succsim$ is risk neutral. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $k \geq 0$. We have that

$$\begin{aligned} I(\varphi + k) &= \inf_{p \in \Delta} G\left(\int (\varphi + k)\, dp, p\right) = \inf_{p \in \Delta} G\left(\int \varphi dp + k, p\right) \\ &\geq \inf_{p \in \Delta} \left(G\left(\int \varphi dp, p\right) + k\right) \geq \inf_{p \in \Delta} G\left(\int \varphi dp, p\right) + k = I(\varphi) + k, \end{aligned}$$

that is, $I$ is constant superadditive.

It follows that $\succsim$ is CARA and $I$ either satisfies (a) or (b) of point (ii) of Theorem 2. By Theorem 2, we can conclude that $\succsim$ is wealth decreasing absolute ambiguity averse and, in particular, is wealth classifiable.

(iii) implies (ii). By Proposition 4 and since $\succsim$ is wealth classifiable, we have that $\succsim$ is also CARA.

We thus have proved that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), proving the statement. ■

**Proof of Corollary 4.** Recall that an uncertainty averse preference is a rational preference. By Corollary 2, we can conclude that a risk neutral uncertainty averse preference is wealth constant absolute ambiguity averse if and only if it satisfies Weak C-Independence. At the same time, by definition, uncertainty averse preferences that satisfy Weak C-Independence are exactly variational preferences.

**Proof of Corollary 5.** Since $\succsim$ is CARA and risk averse, we have that $\operatorname{Im} u = (-\infty, b)$. Recall that $G(t,p) \geq t$ for all $(t,p) \in \operatorname{Im} u \times \Delta$. At the same time, note that for each $(t,p) \in \operatorname{Im} u \times \Delta$ and for each $\lambda \in (0,1)$

$$\begin{aligned} G(\lambda t + (1-\lambda) b, p) &\geq G(\lambda t + (1-\lambda) b_n, p) \geq \lambda G(t,p) + (1-\lambda) G(b_n, p) \\ &\geq \lambda G(t,p) + (1-\lambda) b_n \end{aligned}$$

where $b_n = b - \frac{1}{n}$ for all $n \in \mathbb{N}$. By passing to the limit and since $(t,p)$ and $\lambda$ were arbitrarily chosen, we have that $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t,p) + (1-\lambda) b$. By Theorem 3, the statement follows. ■

**Proof of Corollary 6.** Observe that a variational preference is a rational preference where the canonical representation $(u, I)$ has the extra property of $I$ being quasiconcave and constant additive. In particular, $I$ is normalized and concave.

(i). By Theorem 2 and since $\succsim$ is not risk neutral, if $\succsim$ is either wealth decreasing absolute ambiguity averse or CARA and risk averse, then $v$ is a positive affine transformation of $-\frac{1}{\alpha} e^{-\alpha c}$ where $\alpha \neq 0$. Without loss of generality, we assume that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. The first case holds under risk aversion, the second one under risk love. In the first case, since $I$ is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right) \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right)\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(-\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) - (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that $I$ is concave at 0, that is, $I$ is superhomogeneous. In the second case, since $I$ is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\frac{1}{n} \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\frac{1}{n}\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) + (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that $I$ is concave at 0, that is, $I$ is again superhomogeneous.

"If". By Theorem 2 and since $I$ is concave at 0, if $\succsim$ is CARA and risk averse, it follows that $\succsim$ is wealth decreasing absolute ambiguity averse. "Only if". By Theorem 2, if $\succsim$ is wealth decreasing absolute ambiguity averse, then $\succsim$ is CARA. Since $\succsim$ cannot be risk neutral, it can either be risk averse or risk loving. By contradiction, assume it is risk loving. By Theorem 2, it follows that $I$ is convex at 0, that is, $I$ is subhomogeneous. From the previous part of the proof, we can conclude that $I$ is homogeneous. To sum up, we would have that $I$ is normalized, monotone, continuous, concave, constant additive, and homogeneous, that is, $\succsim$ is maxmin, a contradiction.

(ii). It follows from analogous arguments. ■

**Proof of Corollary 7.** "If". Since $\succsim$ is risk nonneutral, if $\succsim$ is CARA, then either $\succsim$ is risk averse or it is risk loving. If $\succsim$ is homothetic uncertainty averse, then, in both cases, $I$ is positively homogeneous, proving the statement.

"Only if". By Proposition 4 and since $\succsim$ is wealth constant absolute ambiguity averse and uncertainty averse, we have that $\succsim$ is CARA. Since $\succsim$ is uncertainty averse and risk nonneutral, we can consider a canonical representation $(u, I)$ such that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. Since $\succsim$ is wealth constant absolute ambiguity averse, we also have that $I$ is positively homogeneous. Define $\bar{I} : B_0(\Sigma) \to [-\infty, \infty)$ by

$$\bar{I}(\varphi) = \sup\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \qquad \forall \varphi \in B_0(\Sigma).$$

By [7, Theorem 36], it follows that $\bar{I}$ is monotone, lower semicontinuous, quasiconcave, and such that $\bar{I}_{|B_0(\Sigma, \operatorname{Im} u)} = I$. We next show that also $\bar{I}$ is positively homogeneous. Consider $\varphi \in B_0(\Sigma)$ and $\lambda > 0$. We have two cases:

1. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} = \emptyset$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \lambda\varphi\} = \emptyset$, which yields that $\bar{I}(\lambda\varphi) = -\infty = \bar{I}(\varphi) = \lambda\bar{I}(\varphi)$.

2. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \neq \emptyset$. Let $\{\psi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ be such that $\psi_n \leq \varphi$ for all $n \in \mathbb{N}$ and $I(\psi_n) \uparrow \bar{I}(\varphi)$. Let now $\lambda > 0$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, it follows that $\{\lambda\psi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ and it is such that $\lambda\psi_n \leq \lambda\varphi$ for all $n \in \mathbb{N}$. In particular, by the definition of $\bar{I}$, we have that $\bar{I}(\lambda\varphi) \geq I(\lambda\psi_n) = \lambda I(\psi_n) \to \lambda\bar{I}(\varphi)$. We just proved that $\bar{I}(\lambda\varphi) \geq \lambda\bar{I}(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $\lambda > 0$. By choosing $1/\lambda$ with $\lambda > 0$, it follows that

$$\bar{I}(\varphi) = \bar{I}\left(\frac{1}{\lambda}(\lambda\varphi)\right) \geq \frac{1}{\lambda}\bar{I}(\lambda\varphi),$$

that is, $\lambda\bar{I}(\varphi) \geq \bar{I}(\lambda\varphi)$, proving positive homogeneity.

Consider $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ defined by

$$G(t, p) = \sup\left\{\bar{I}(\varphi) : \int \varphi dp \leq t\right\} \qquad \forall (t, p) \in \mathbb{R} \times \Delta.$$

By [8], we have that $G$ is lower semicontinuous, quasiconvex, and such that

$$\bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma) \tag{14}$$

and $G(\lambda t, p) = \lambda G(t, p)$ for all $\lambda > 0$, for all $t \in \mathbb{R}$, and for all $p \in \Delta$. Define $c_1, c_2 : \Delta \to [0, \infty]$ to be such that

$$c_1(p) = \frac{1}{G(1, p)} \text{ and } c_2(p) = -G(-1, p) \qquad \forall p \in \Delta.$$

We now consider two cases:

*Risk averse case.* $\operatorname{Im} u = (-\infty, 0)$. Since $\bar{I} \leq 0$ and $\bar{I}(-1) = I(-1) = -1$, observe that $G(-1, p) \leq 0$ and $G(-1, p) \geq -1$, that is, $c_2(p) \geq 0$ and $c_2(p) \leq 1$ for all $p \in \Delta$. Next, we have that for each $\alpha \in \mathbb{R}$

$$\{p \in \Delta : c_2(p) \geq \alpha\} = \{p \in \Delta : -G(-1, p) \geq \alpha\} = \{p \in \Delta : G(-1, p) \leq -\alpha\}.$$

Since $G$ is quasiconvex and lower semicontinuous, the set is convex and closed, proving that $c_2$ is quasiconcave and upper semicontinuous. By (14), we can conclude that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) = \min_{p\in\Delta}\left(-\int \varphi dp\right) G(-1, p) = \min_{p\in\Delta} c_2(p) \int \varphi dp.$$

Since $-1 = \bar{I}(-1) = \min_{p\in\Delta} -c_2(p)$, we have that $c_2$ is normalized. The statement follows by setting $c = c_2$.

*Risk loving case.* $\operatorname{Im} u = (0, \infty)$. Since $\bar{I}(1) = I(1) = 1$, observe that $G(1, p) \geq 1$, that is, $0 \leq c_1(p) \leq 1$. Next, we have that for each $\alpha \in (0, \infty)$

$$\{p \in \Delta : c_1(p) \geq \alpha\} = \left\{p \in \Delta : \frac{1}{G(1,p)} \geq \alpha\right\} = \left\{p \in \Delta : G(1,p) \leq \frac{1}{\alpha}\right\}.$$

Since $G$ is quasiconvex and lower semicontinuous, for each $\alpha \in (0, \infty)$ the set is convex and closed. Since $\{p \in \Delta : c_1(p) \geq \alpha\} = \Delta$ for all $\alpha \leq 0$, it follows that $c_1$ is quasiconcave and upper semicontinuous. By (14), we can conclude for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p \in \Delta} \left(\int \varphi dp\right) G(1,p) = \min_{p \in \Delta} \frac{\int \varphi dp}{c_1(p)}.$$

Since $1 = \bar{I}(1) = \min_{p \in \Delta} \frac{1}{c_1(p)}$, we have that $c_1$ is normalized. The statement follows by setting $c = c_1$. ■

**Proof of Proposition 6.** Since there exists $\gamma > 0$ such that $\phi(t) = -e^{-\gamma t}$ for all $t \in \mathbb{R}$, we have that $I$, defined as in (6), can be defined over the entire space $B_0(\Sigma)$. Moreover, by [7, Proposition 54], $I$ is normalized, concave and constant additive. In particular, it is concave at $b$, in case $\succsim$ is either risk averse or risk loving.

(i). By Corollary 1 (recall that it holds by only assuming CARA in place of wealth classifiable) and since $I$ is constant additive, if $\succsim$ is risk neutral, then $\succsim$ is wealth constant absolute ambiguity averse.

(ii). By Corollary 5 and since $\succsim$ is CARA, if $\succsim$ is risk averse, then $\succsim$ is wealth decreasing absolute ambiguity averse. ■

**Proof of Proposition 7.** We only prove point (ii). Point (iii) follows from a completely specular argument. Point (i) instead follows from similar techniques (see also Marinacci and Montrucchio [26, Theorem 12]).

(ii). Fix $\mu$. By Theorem 2 and since $\succsim$ is risk averse and $b = 0$, we have that $\succsim$ is wealth decreasing absolute ambiguity averse if and only if $I$ is positive superhomogeneous. Thus, to prove point (ii), we only need to show that $I$ is positive superhomogeneous for all $\mu$ if and only if $\phi$ is IRRA. Since $\succsim$ is risk averse and $b = 0$, we also have that $\operatorname{Im} u = (-\infty, 0)$ and $\phi : (-\infty, 0) \to \mathbb{R}$. For each $\nu > 0$, define $\phi_\nu : (-\infty, 0) \to \mathbb{R}$ to be such that $\phi_\nu(t) = \phi(\nu t)$ for all $t \in (-\infty, 0)$. Note that $\phi_1 = \phi$. Finally, we have that $\operatorname{Im} \phi = \operatorname{Im} \phi_\nu$ for all $\nu > 0$. "If" Let $\mu$ be generic. Consider $\nu > \eta > 0$. It follows that $\phi_\nu = f \circ \phi_\eta$ where $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ is strictly increasing and concave. By the Jensen's inequality, it follows that if $\nu > \eta > 0$, then

$$\phi_\nu^{-1}\left(\int \phi_\nu \left(\int \varphi dp\right) d\mu\right) \leq \phi_\eta^{-1}\left(\int \phi_\eta \left(\int \varphi dp\right) d\mu\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

If we let $\eta \in (0,1)$ and $\nu = 1$, we have that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$\begin{aligned}\phi\left(\eta\phi^{-1}\left(\int\phi\left(\int\varphi dp\right)d\mu\right)\right) &= \phi_\eta\left(\phi^{-1}\left(\int\phi\left(\int\varphi dp\right)d\mu\right)\right)\\ &\leq \int\phi\left(\eta\int\varphi dp\right)d\mu.\end{aligned}$$

We can conclude that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\eta\varphi) = \phi^{-1}\left(\int\phi\left(\eta\int\varphi dp\right)d\mu\right) \geq \eta\phi^{-1}\left(\int\phi\left(\int\varphi dp\right)d\mu\right) = \eta I(\varphi),$$

proving that $I$ is positive superhomogeneous. "Only if" Let $\nu > \eta > 0$. Consider $\frac{\eta}{\nu} \in (0,1)$. Fix $\mu$. Since $I$ is positive superhomogeneous, it follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$\begin{aligned}\phi^{-1}\left(\int\phi_\eta\left(\int\varphi dp\right)d\mu\right) = \phi^{-1}\left(\int\phi\left(\eta\int\varphi dp\right)d\mu\right) &= \phi^{-1}\left(\int\phi\left(\int\frac{\eta}{\nu}\nu\varphi dp\right)d\mu\right)\\ &\geq \frac{\eta}{\nu}\phi^{-1}\left(\int\phi\left(\int\nu\varphi dp\right)d\mu\right)\\ &= \frac{\eta}{\nu}\phi^{-1}\left(\int\phi_\nu\left(\int\varphi dp\right)d\mu\right),\end{aligned}$$

yielding that

$$\begin{aligned}\phi_\eta^{-1}\left(\int\phi_\eta\left(\int\varphi dp\right)d\mu\right) &= \frac{1}{\eta}\phi^{-1}\left(\int\phi_\eta\left(\int\varphi dp\right)d\mu\right)\\ &\geq \frac{1}{\nu}\phi^{-1}\left(\int\phi_\nu\left(\int\varphi dp\right)d\mu\right)\\ &= \phi_\nu^{-1}\left(\int\phi_\nu\left(\int\varphi dp\right)d\mu\right) \qquad \forall\varphi \in B_0(\Sigma, \operatorname{Im} u).\end{aligned}$$

Since both $\phi_\nu$ and $\phi_\eta$ are both strictly increasing and continuous, there exists a strictly increasing function $h : \operatorname{Im}\phi \to \operatorname{Im}\phi$ such that $\phi_\nu = h \circ \phi_\eta$. It follows that

$$h\left(\int\phi_\eta\left(\int\varphi dp\right)d\mu\right) \geq \int h\left(\phi_\eta\left(\int\varphi dp\right)\right)d\mu \qquad \forall\varphi \in B_0(\Sigma, \operatorname{Im} u). \tag{15}$$

Since $\mu$ was arbitrarily chosen, (15) holds for all $\mu$. Since $\Sigma$ is nontrivial, there exists $E \in \Sigma$ such that $E \neq \emptyset, S$. Consider $s_1, s_2 \in S$ such that $s_1 \in E$ and $s_2 \in E^c$. Note that $\delta_{s_i} \in \Delta$ and $\{\delta_{s_i}\} \in \mathcal{B}$ for $i \in \{1,2\}$.[47] Let $\mu = \lambda\delta_{\delta_{s_1}} + (1-\lambda)\delta_{\delta_{s_2}}$ with $\lambda \in (0,1)$. Consider also $k_1, k_2 \in \operatorname{Im}\phi$. It follows that there exist $t_1, t_2 \in (-\infty, 0)$ such that $\phi_\eta(t_i) = k_i$ for $i \in \{1,2\}$.

[47] If $s \in S$, then we denote by $\delta_s$ the Dirac at $s$. We denote by $\mathcal{B}$ the Borel $\sigma$-algebra over $\Delta$.

Define $\varphi = t_1 1_E + t_2 1_{E^c} \in B_0(\Sigma, \operatorname{Im} u)$. By (15), we have that

$$\begin{aligned}
h(\lambda k_1 + (1-\lambda) k_2) &= h(\lambda \phi_\eta(t_1) + (1-\lambda)\phi_\eta(t_2)) \\
&= h\left(\lambda \phi_\eta\left(\int \varphi d\delta_{s_1}\right) + (1-\lambda)\phi_\eta\left(\int \varphi d\delta_{s_2}\right)\right) \\
&= h\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) \geq \int h\left(\phi_\eta\left(\int \varphi dp\right)\right) d\mu \\
&= \lambda h\left(\phi_\eta\left(\int \varphi d\delta_{s_1}\right)\right) + (1-\lambda) h\left(\phi_\eta\left(\int \varphi d\delta_{s_2}\right)\right) \\
&= \lambda h(\phi_\eta(t_1)) + (1-\lambda) h(\phi_\eta(t_2)) = \lambda h(k_1) + (1-\lambda) h(k_2),
\end{aligned}$$

proving that $h$ is concave and $\phi$ is IRRA. ■

**Proof of Proposition 8.** Since $\succsim$ is a smooth ambiguity preference, it admits a canonical representation $(u, I)$ where $I$ is as in (6). Since $\succsim$ is CARA and risk averse and $b \leq 0$, we also have that $I$ is defined over $B_0(\Sigma, (-\infty, 0)) \supseteq B_0(\Sigma, \operatorname{Im} u)$. The functional $\hat{I}$ : $B_0(\Sigma, (0, \infty)) \to (0, \infty)$ defined by

$$\hat{I}(\varphi) = \left(\int \left(\int \varphi dp\right)^\gamma d\mu\right)^{\frac{1}{\gamma}} \qquad \forall \varphi \in B_0(\Sigma, (0, \infty)).$$

is normalized, monotone, continuous, positively homogeneous, and quasiconvex. It follows that $I : B_0(\Sigma, (-\infty, 0)) \to \mathbb{R}$, which is such that $I(\varphi) = -\hat{I}(-\varphi)$, is normalized, monotone, continuous, positively homogeneous, and quasiconcave. In particular, by [8, Proposition 7 and its proof, WP version, Carlo Alberto Notebook 80], it is concave. By Corollary 5 and since $\succsim$ is CARA and risk averse, the statement easily follows. ■

**Proof of Proposition 9.** Since $\succsim$ is wealth constant absolute ambiguity averse, then $\succsim$ is CARA and $I$ is either constant additive or affine at $b$, depending on $\succsim$ being risk neutral or not. As usual, without loss of generality we can normalize $a = 1$ and $b = 0$ (see equation (5)). In both cases, it follows that

$$\begin{aligned}
\beta &\mapsto v^{-1}(I(v(\beta r + (w-\beta) r_f))) = v^{-1}(I(v(w r_f + \beta(r - r_f)))) \\
&= v^{-1}(I(v(\beta(r - r_f)))) + w r_f.
\end{aligned}$$

Thus, for each $w \in (0, \infty)$, maximizing $\beta \mapsto I(v(\beta r + (w-\beta) r_f))$ subject to $\beta \in [0, w]$ is equivalent to maximize $\beta \mapsto v^{-1}(I(v(\beta(r - r_f))))$ subject to $\beta \in [0, w]$. Define $f : [0, \infty) \to \mathbb{R}$ by $f(\beta) = v^{-1}(I(v(\beta(r - r_f))))$ for all $\beta \geq 0$. Let $w' > w$. We have two cases:

1. $\beta^*(w') \geq w$. This implies that $\beta^*(w') \geq w \geq \beta^*(w)$.
2. $\beta^*(w') < w$. Since $\beta^*(w')$ maximizes $f$ on $[0, w']$ and $0 \leq \beta^*(w) \leq w \leq w'$, we have that $f(\beta^*(w')) \geq f(\beta^*(w))$. Since $\beta^*(w)$ maximizes $f$ on $[0, w]$ and $0 \leq \beta^*(w') < w$,

we have that $f(\beta^*(w)) \geq f(\beta^*(w'))$. This implies that $\beta^*(w')$ is a maximizer of $f$ on $[0,w]$. Since the solution of (7) is unique for all $w>0$, we can conclude that $\beta^*(w') = \beta^*(w)$.

Points 1 and 2 yield the main statement.

Note that if $\succsim$ is risk averse and uncertainty averse, it follows that $f(\beta) = v^{-1}(I(v(\beta(r-r_f))))$ is quasiconcave on $[0,\infty)$. Let $w'>w$. By contradiction, assume that $\beta^*(w') \neq \beta^*(w)$. From the previous part of the proof, it follows that $\beta^*(w') > \beta^*(w)$. Consider $\hat{\beta} \in (\beta^*(w), \min\{w, \beta^*(w')\}) \subseteq (0,w) \subseteq (0,w')$. Since $\beta^*(w), \hat{\beta} \in (0,w)$ and the former is the unique maximizer of $f$ on $[0,w]$, it follows that $f(\beta^*(w)) > f\left(\hat{\beta}\right)$. Similarly, since $\beta^*(w'), \hat{\beta} \in [0,w']$ and the former is the unique maximizer of $f$ on $[0,w']$, it follows that $f(\beta^*(w')) > f\left(\hat{\beta}\right)$. On the one hand, we can conclude that $\min\{f(\beta^*(w)), f(\beta^*(w'))\} > f\left(\hat{\beta}\right)$. On the other hand, by construction of $\hat{\beta}$, we also have that there exists $\lambda \in (0,1)$ such that

$$\hat{\beta} = \lambda\beta^*(w) + (1-\lambda)\beta^*\left(w'\right).$$

Since $f$ is quasiconcave, this implies that $f\left(\hat{\beta}\right) \geq \min\{f(\beta^*(w)), f(\beta^*(w'))\}$, a contradiction. ■

**Proof of Proposition 10.** Since $\succsim$ is risk neutral, without loss of generality, let $v$ be the identity. Note that

$$\beta \mapsto I(v(\beta r + (w-\beta) r_f)) = \phi^{-1}\left(\int \phi\left(\beta \int r dp + (w-\beta) r_f\right) d\mu\right).$$

Define $\hat{r} : \Delta \to \mathbb{R}$ by $\hat{r}(p) = \int r dp$ for all $p \in \Delta$. We have that $\hat{r} \geq 0$ is a bounded, real-valued, Borel measurable function. It follows that the problem in (7) is equivalent to solve

$$\max\left(\int \phi(\beta\hat{r} + (w-\beta) r_f) d\mu\right) \text{ subject to } \beta \in [0,w],$$

which is mathematically equivalent to the usual expected utility portfolio choice problem. Since $\phi$ is concave and DARA, it is twice continuously differentiable and such that $\phi' > 0$, and $\beta^*(w) \in (0,w)$ with $w>0$, we have that (8) holds. ■

**Proof of Proposition 11.** Since $\succsim$ is a risk averse multiplier preference, note that

$$\beta \mapsto I(v(\beta r + (w-\beta) r_f)) = \phi^{-1}\left(\int \phi(v(\beta r + (w-\beta) r_f)) dq\right)$$

where $v(c) = -a\frac{1}{\alpha}e^{-\alpha c} + b$ for all $c \in \mathbb{R}$, with $\alpha, a > 0$ and $b \in \mathbb{R}$, and $\phi(t) = -e^{-\theta t}$ for all $t \in \mathbb{R}$, with $\theta > 0$. Define $\hat{v} = \phi \circ v : \mathbb{R} \to \mathbb{R}$. It follows that the problem in (7) is equivalent to solve

$$\max\left(\int \hat{v}(\beta r + (w-\beta) r_f) dq\right) \text{ subject to } \beta \in [0,w],$$

which is mathematically equivalent to the usual expected utility portfolio choice problem. Since $\hat{v}$ is concave and DARA, it is twice continuously differentiable and such that $\hat{v}' > 0$, and $\beta^* (w) \in (0, w)$ with $w > 0$, we have that (8) holds. ■

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