Rationalizable Bidding in General First-Price Auctions

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Abstract

We wish to analyze the consequences of strategically sophisticated bidding without assuming equilibrium behavior. As a first step, we characterize interim rationalizable bids in first-price auctions with interdependent values and affiliated signals. We show that (1) every non-zero bid below the equilibrium is rationalizable, (2) some bids above the equilibrium are rationalizable, (3) the upper bound on rationalizable bids of a given player is a continuous, non-decreasing function of her signal/valuation. In the special case of symmetric bidders with independent signals and quasi-linear valuation functions, (i) the least upper bound on rationalizable bids is increasing and concave; hence (ii) rationalizability is consistent with substantial shading for high valuations, but only little shading for low valuations.

Our main technical contribution is to show that the set of rationalizable bids is essentially determined by iteratively solving a simple one-dimensional optimization problem.

We argue that our theoretical analysis may shed some light on experimental findings about deviations from the risk-neutral Nash equilibrium.

1. Introduction

The analysis of simultaneous bidding games generally builds upon the notion of (Bayesian) Nash equilibrium. Implicit in the latter solution concept are the assumptions that players are rational and hold *correct* beliefs about the play of their opponents.

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This paper represents a first step toward the analysis of simultaneous bidding games under the assumption that bidders' beliefs are *strategically sophisticated*, but not necessarily correct; the rationality hypothesis is maintained.

Strategic sophistication is defined as the conjunction of the following assumptions about beliefs: (1) Bidders expect positive bids to win with positive probability; (2) Bidders are certain that their opponents are rational and are certain of (1); (3) Bidders are certain that their opponents are certain of (2); and so on.

We focus on first-price sealed-bid auctions with private or interdependent valuations, and independent or correlated signals, and adopt the notion of (*interim*) rationalizability to capture strategic sphistication.

Our approach is motivated by the following considerations. In our opinion, the equilibrium assumption that beliefs are correct should be justified in terms of more fundamental hypotheses about the bidders' belief formation process. In particular, one may attempt to find a justification based on either introspection or learning in the *specific* context of auction games.

This paper provides an analysis based on beliefs that are strategically sophisticated, and hence consistent with a careful introspective analysis of the game. We show that, in first-price auctions, although strategic sophistication has non-trivial implications for bidding behavior, it is consistent with a wide range of non-equilibrium beliefs. Thus, introspection alone does not provide a justification for equilibrium analysis.

One may then argue that, even if bidders initially hold heterogeneous non-equilibrium beliefs, a learning process should nevertheless lead to an equilibrium.[1] This argument, however, is subject to important qualifications. First, it applies only to situations where bidders repeatedly play similar auction games with different competitors (a fixed set of bidders could give rise to collusion). Second, whether convergence to an equilibrium occurs at all, as well as the speed of convergence, crucially depend on how much feedback each player obtains about the decision rules adopted by his competitors in previous plays. In auctions games, this feedback is typically very poor: only the actual bids, and not the private information that induced such bids, can typically be observed.[2]

Therefore, we find no compelling reasons to expect approximate equilibrium behavior in the short run. Not surprisingly, experimental evidence shows significant and persistent deviations from the risk-neutral Nash equilibrium in first-price auctions (cf. Kagel [13]).

These considerations suggest that it may be interesting to ascertain the extent to which the predictions of "textbook" auction theory (cf. Milgrom and Weber [16], Myerson [18], Riley and Samuelson [19], Vickrey [20]) are dependent on the assumption that bidders' beliefs are correct.

[1] On learning in games see, for example, Fudenberg and Levine [9].

[2] In a Dutch auction, whose reduced normal form is like a first-price auction, only the winning bid is observed.

Our analysis addresses this issue. We find that *bid shading* (bidding below the expected value of the good conditional on private information) is a robust phenomenon in first-price auctions. In settings with common values, shading is a consequence of rationality if a bidder believes that higher types of his opponents bid higher; hence, the robustness of bid shading in this context is not surprising. However, we also find substantial shading in settings with private values. Moreover, our results are qualitatively consistent with the empirical finding (cf. Kagel and Roth [14]) that higher types tend to shade proportionally more than low types.

On the other hand, *revenue equivalence* appears to be dependent upon the equilibrium assumption. In particular, our analysis of first-price auctions (with either private or interdependent values) shows that (i) for every type, every positive bid below the corresponding equilibrium bid is rationalizable, and (ii) for almost every type, the highest rationalizable bid is above the equilibrium bid.

Note that our assumptions about beliefs and behavior imply that players do not use weakly dominated bids. In a second-price auction with private values, analogous assumptions imply that each player bids its valuation, as in the dominant-strategy equilibrium. Therefore, in light of the standard (i.e. equilibrium) revenue equivalence results, we conclude that the expected revenue of a seller with rationalizable beliefs in a second-price auction may be lower or higher than the rationalizable expected revenue in a first-price auction.

However, when the number of (ex ante symmetric) bidders is large, the expected revenue in a second-price auction is close to the expected valuation of the good, which is an upper bound to the rationalizable expected revenue in a first-price auction. Therefore, when the number of (risk-neutral) bidders is large, a seller with rationalizable but not extremely optimistic beliefs should expect a higher revenue in a second-price auction than in a first-price auction.

A further motivation for our work does not directly apply to this paper, but rather to the general approach we are attempting to develop. In recent years, many novel auction designs have been implemented in practice. When faced with such "novelties", bidders cannot be expected to have learned to play equilibrium strategies—even if, say, they may be reasonably expected to have learned the shape of each other's valuation functions, each other's signal distribution, and so on.

In such situations, we find the case for an analysis based on strategic sophistication alone particularly compelling. We hope that the methodology of this paper can be extended to more complex bidding games.

This paper employs an interim notion of rationality: different types of the same player are allowed to hold different beliefs about the bidding behavior of his opponents. Correspondingly, our results characterize interim rationalizability.

The latter solution concept involves the iterative deletion, for each possible type, of bids that cannot be justified by beliefs consistent with progressively higher degrees of strategic sophistication. A direct application of this procedure to bidding games would be analytically cumbersome and numerically intractable.

Our main technical contribution is to provide a more efficient implementation of interim rationalizability in the setting under consideration. The methodology we propose entails constructing bounds on the set of rational(izable) bids for a given type (valuation, signal), and then proving that every bid within these bounds is rational(izable).

A stronger solution concept, *ex ante rationalizability*, may alternatively be adopted to study bidding games. In particular, under the independent-types hypothesis, the beliefs of a player about his competitors' bidding functions are independent of his valuation (more generally, his signal), while interim rationalizability allows for type-dependent beliefs. Therefore, our analysis does provide bounds on ex ante rationalizable bidding strategies, but these bounds need not be tight.

Which solution concept (interim or ex ante) is more appropriate depends on our interpretation of the formal asymmetric-information model. If it represents a situation with genuine incomplete information without an ex ante stage (as in some auctions with independent, private values), then interim rationalizability is appropriate. If it represents a situation where the bidders obtain information about the outcome of a random experiment (as in most common-value auctions), then ex ante rationalizability is appropriate.[3]

The remainder of the paper is organized as follows. Section 2 introduces our basic methodology and characterization result in the simplified setting of symmetric IPV auctions. In particular, for each possible valuation, we characterize the highest bid of a rational player (for each valuation) who believes that his opponents do not bid above a given type-dependent upper bound (e.g., the identity function). Section 3 extends the result to asymmetric auctions with interdependent values and affiliated signals. These results are used in Section 4 to obtain an iterative characterization of interim rationalizable bids in auctions. Section 5 discusses the relationship with experimental evidence, and indicates a number of extensions of our results. The Appendix contains some proofs and ancillary results.

2. The Symmetric IPV Model

Consider the following game with asymmetric information representing a single-object, first-price auction with private values and risk-neutral bidders. There are n players, or bidders. Each bidder i knows her valuation for the object $v_i \in [0,1]$, but does not know the valuations of her competitors. Beliefs about the competitors valuations are derived from a common prior cumulative distribution function (c.d.f.) $F : [0,1]^n \to [0,1]$. Each player chooses a bid $b \geq 0$

[3] The difference between ex ante and interim rationalizability is related to the difference between ex ante and interim dominance (see Fudenberg and Tirole [10], pp 226-228). Morris and Skiadas [17] compares ex ante and interim rationalizability in trading games.

which in general depends on her valuation. The object is assigned to one of the high bidders, breaking ties at random. The winner pays her bid, losers do not pay anything.

2.1. Assumptions and Notation

We assume that valuations are identically and independently distributed according to a continuous strictly positive density.

Assumption 2.1. *For all* $(x_1, ..., x_n) \in [0,1]^n$, $F(x_1, .., x_n) = \prod_{i=1}^n G(x_i)$, *where* G *is an absolutely continuous c.d.f. with continuous density* g *bounded away from zero.*

We refer to an auction game satisfying Assumption 2.1 as the *symmetric IPV model*, where "IPV" stands for Independent Private Values.

Random variables and beliefs. From the point of view of a bidder, her competitors' bids are random variables. We use **boldface** letters to denote random variables. A function (random variable) $\mathbf{b}_j : [0,1] \to \mathbb{R}_+$ can be interpreted as a conjecture of Player i about the bidding behavior of Player j—a description of how Player j would bid for any possible valuation v_j.[4] To allow for the possibility that a player is uncertain about the bidding behavior of her competitors, we model beliefs as probability distributions over $(n-1)$-*tuples* of bidding functions (random variables). Let $\mathcal{B}_j$ denote the jth copy of the set of bounded functions with domain $[0,1]$ and range $\mathbb{R}_+$, interpreted as the set of conjectures about j. The set of possible conjectures for Bidder i about her competitors is $\mathcal{B}_{-i} = \prod_{j \neq i} \mathcal{B}_j$. A *belief* for Player i is a probability measure on $\mathcal{B}_{-i}$, that is, an element μ of the set $\Delta(\mathcal{B}_{-i})$.[56] With a slight abuse of notation we identify a belief μ assigning probability one to a *tuple* $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ with $\mathbf{b}_{-i}$ itself. As a matter of terminology, we refer to elements $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ as *conjectures*. Thus, in our setting conjectures are degenerate beliefs.

Inequalities and probabilities of events. Inequalities between random variables are interpreted as pointwise inequalities which hold almost everywhere. For example, $\mathbf{b}_j < \mathbf{B}_j$ if and only if the set of s_j such that $\mathbf{b}_j(s_j) \geq \mathbf{B}_j(s_j)$ has (Lebesgue) measure zero. Similarly, inequalities between tuples of random variables are interpreted as coordinate-wise inequalities: $\mathbf{b}_{-i} < \mathbf{B}_{-i}$ if and only if $\mathbf{b}_j < \mathbf{B}_j$ for all $j \neq i$. Degenerate random variables and tuples of identical degenerate random variables are represented by the corresponding real numbers.

[4] We need *not* interpret $\mathbf{b}_j$ as a bidding strategy chosen *ex ante*.

[5] Our results do not depend on the choice of a specific sigma-algebra of measurable subsets of $\mathcal{B}_{-i}$. We only require that singletons be measurable, so that degenerate beliefs belong to $\Delta(\mathcal{B}_{-i})$.

[6] Note that we allow for *correlated* choices of bidding functions, and hence spurious correlation among opponents' bids. However, the formulation in the text does entail a mild restriction: Player i cannot believe that Player j's bid is a function of the valuation of competitor k. This (plausible) restriction does play a role in the analysis of the general interdependent values model, but not it in the present setting.

The probability of subsets of $[0,1]$ or $[0,1]^{n-1}$ obtained from the c.d.f. G is denoted by $\Pr[\cdot]$. Thus, for example, $\Pr[\mathbf{b}_{-i} \leq b]$ is the probability that b is the high bid given conjecture $\mathbf{b}_{-i}$.

Expected payoff. For any belief $\mu \in \Delta(\mathcal{B}_{-i})$, let $P(b;\mu)$ denote the probability of winning the object with bid b given belief μ. Note that $P(\cdot;\mu)$ is a nondecreasing[7] function with (an at most countable set of) discontinuity points at bids that yield ties with positive probability according to belief μ. The expected payoff for a bidder with valuation v and belief μ who bids b is

$$\pi(b,v;\mu) = (v-b)P(b;\mu);$$

also let $\pi^*(v;\mu) = \sup_{b \geq 0} \pi(b,v;\mu)$.

Observe that, for any pair of real-valued functions π' and π'' with the same domain, if $\pi'(x) \leq \pi''(x)$ for all x, then $\sup_x \pi'(x) \leq \sup_x \pi''(x)$. Therefore we obtain the following obvious remark:

Remark 1. *For any two beliefs $\mu', \mu'' \in \Delta(\mathcal{B}_{-i})$, suppose that*

$$\forall b \geq 0,\ P(b;\mu') \leq P(b;\mu'').$$

Then

$$\forall v \in [0,1],\ \pi^*(v;\mu') \leq \pi^*(v;\mu'').$$

In particular, this is true when μ' and μ'' are given by symmetric conjectures $\mathbf{B}'_{-i} = \{\mathbf{B}', \mathbf{B}', ...\}$ and $\mathbf{B}''_{-i} = \{\mathbf{B}'', \mathbf{B}'', ...\}$ where $\mathbf{B}'$ and $\mathbf{B}''$ are functions such that $\mathbf{B}' \geq \mathbf{B}''$.

Positive probability of winning. We assume that beliefs μ satisfy the following restriction: *for all bids $b > 0$, the probability of winning the object is positive.* We let

$$\Delta^+(\mathcal{B}_{-i}) = \{\mu \in \Delta(\mathcal{B}_{-i}) : \forall b > 0, P(b;\mu) > 0\}$$

denote this restricted set of beliefs. Clearly, a rational bidder with belief $\mu \in \Delta^+(\mathcal{B}_{-i})$ would not choose a bid above her valuation, nor the zero bid (unless her valuation is zero). Thus, restricting attention to beliefs $\mu \in \Delta^+(\mathcal{B}_{-i})$ entails a form of *cautiousness*.

2.2. Upper Bounds on Best Responses

We now turn to the main result of this section. Consider a nondecreasing, bounded and measurable bid function $\mathbf{B}$. The intended interpretation is as follows: for every competitor $j \neq i$ and valuation $v \in [0,1]$, Bidder i is certain that, if Bidder j's valuation is v, then he bids less than $\mathbf{B}(v)$.

[7] Throughout the paper we call a function h *increasing* if $x' > x''$ implies $h(x') > h(x'')$, and we call h *nondecreasing* if $x' > x''$ implies $h(x') \geq h(x'')$.

For instance, one may wish to assume that Bidder i is certain that her opponents never bid above their valuations: to reflect this, let $\mathbf{B}(v) = v$ for every $v \in [0, 1]$.

Let $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B} \ldots\}$. We wish to characterize the set of interim best replies to beliefs in $\Delta^+(\mathcal{B}_{-i})$ assigning probability one to bids below this common upper bound. The set of such beliefs for Bidder i is

$$\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i}) = \left\{\mu \in \Delta^+(\mathcal{B}_{-i}) : \mu\left(\{\mathbf{b}_{-i} : \mathbf{b}_{-i} < \ \mathbf{B}_{-i}\}\right) = 1\right\}.$$

Theorem 2.2. *Let $\mathbf{B} : [0, 1] \to \mathbb{R}_+$ be a nondecreasing function such that $\mathbf{B} > 0$, and define $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B} \ldots\}$. Then, for every bid $b^* > 0$ and valuation $v \in [0, 1]$,*

(1) if $b^* > v - \pi^*(v; \mathbf{B}_{-i})$, then b^* is not *a best reply for v to any belief $\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$;*
(2) if $b^* < v - \pi^*(v; \mathbf{B}_{-i})$, then b^* is a strict best reply for v to some belief $\mu^* \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$;
(3) furthermore,

$$\pi^*(v; \mathbf{B}_{-i}) = \sup_{b \geq 0} \left\{(v - b) \cdot \Pr[\mathbf{B}_{-i} \leq b]\right\} = \inf_{\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})} \pi^*(v; \mu)$$

and the supremum is attained.

Proof. We prove (1) using (3). The proof of (2) is sketched below. See the Appendix for a proof of (3) and a complete proof of (2).

(1) Fix a belief $\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$. If $v = 0$, then $\pi^*(v; \mu) = 0$; since $b^* > 0 = v$ and $P(b, \mu) > 0$ for all $b > 0$, (1) follows. For the same reason, if $v > 0$, we can focus on bids $b^* \in (0, v)$. Notice that, for such bids, $0 < \pi(b^*, v; \mu) \leq v - b^*$; therefore, if $v - b^* < \pi^*(v; \mathbf{B}_{-i})$, we get

$$\begin{aligned}
\pi(b^*, v; \mu) &\leq v - b^* < \\
&< \pi^*(v; \mathbf{B}_{-i}) = \\
&= \inf_{\mu' \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})} \pi^*(v; \mu') \leq \\
&\leq \pi^*(v; \mu)
\end{aligned}$$

where the equality follows form (3); thus, b^* cannot be a best reply to belief μ. Since the latter was chosen arbitrarily in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$, the proof is complete. ■

We now sketch the argument for (2); see the Appendix for the actual proof. First, note that $b^* < v - \pi^*(v; \mathbf{B}_{-i})$ implies that $b^* < v$, because $\pi^*(v; \mathbf{B}_{-i}) \geq 0$.

The proof is constructive; we exhibit a bid function $\mathbf{g}^{b^*} : [0, 1] \to \mathbb{R}_+$ such that the conjecture $\mathbf{g}^{b^*}_{-i} = \{\mathbf{g}^{b^*}, \mathbf{g}^{b^*} \ldots\}$ is consistent with the assumed restrictions on Bidder i's beliefs, and b^* is the unique best reply to it. In particular, we ensure that $\Pr[\mathbf{g}^{b^*}_{-i} < b^*] = 1$, so that $\pi^*(v; \mathbf{g}^{b^*}_{-i}) = v - b^*$.

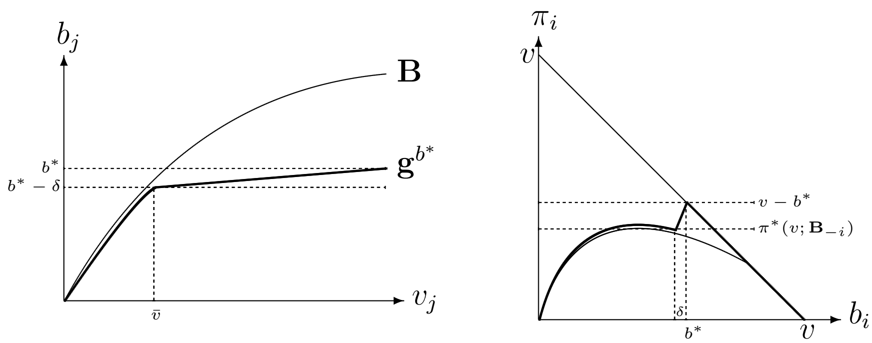

Figure 2.1: The bid function $\mathbf{g}^{b^*}$ (L) and the corresponding payoff function for Bidder i (R).

The main features of the function $\mathbf{g}^{b^*}$ are apparent in Figure 2.1. Assume for simplicity that the upper bound $\mathbf{B}$ is continuous, increasing, and such that $\mathbf{B}(0) = 0$; also, choose $\delta > 0$ small. Then $\mathbf{g}^{b^*}$ approximates $\mathbf{B}$ from below until the point $\bar{v}$ where $\mathbf{B}(\bar{v}) = b^* - \delta$; for $v > \bar{v}$, it is defined as the line segment joining the points $(\bar{v}, b^* - \delta)$ and $(1, b^*)$.

If δ is sufficiently small, then Bidder i loses by decreasing her bid from b^* to $b \in [b^* - \delta, b^*)$ because the payoff reduction corresponding to a decrease in the probability of winning is larger than the payoff increase due to the lower price paid conditional on winning.

On the other hand, $\mathbf{g}^{b^*}$ can be chosen to approximate $\mathbf{B}$ close enough for $v \in [0, \bar{v}]$ that $\max_{b \in [0, b^* - \delta]} \pi(b, v; \mathbf{g}^{b^*}_{-i}) \approx \max_{b \in [0, b^* - \delta]} \pi(b, v; \mathbf{B}_{-i}) \leq \pi^*(v; \mathbf{B}_{-i}) < v - b^*$; hence, b^* is the unique best reply to $\mathbf{b}^*$.

The proof in the Appendix allows for discontinuous and nondecreasing upper bounds, as well as for the possibility that $\mathbf{B} > b$ almost surely for some $b > 0$.

As a consequence of Theorem 2.2, we can characterize our restriction on beliefs in terms of weak dominance and best replies. A bid b is *weakly dominated for valuation* v_i if there is another bid b' such that $\pi(b, v_i; \mathbf{b}_{-i}) \leq \pi(b', v_i; \mathbf{b}_{-i})$ for all $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ and the inequality is strict for at least one $\mathbf{b}_{-i}$.

Proposition 2.3. *For all $v > 0$ and b^*, the following conditions are equivalent:*
(i) there exists some $\mu \in \Delta^+(\mathcal{B}_{-i})$ such that $\pi^*(v; \mu) = \pi(b^*, v; \mu)$,
(ii) $0 < b^* < v$,
(iii) b^* is not weakly dominated for valuation v.

Proof. The result is obvious for $b^* = 0$ and $b^* \geq v$.

Suppose that $0 < b^* < v$ and consider a high upper bound $\mathbf{B}_{-i} = \bar{b} > 1$. Clearly, $\pi^*(v; \mathbf{B}_{-i}) = 0$ and the set of best replies to beliefs in $\Delta^+(\mathcal{B}_{-i})$ and $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$ coincide. Therefore, Theorem 2.2 implies that (i) and (ii) are equivalent. Furthermore, Theorem 2.2 (2) implies that b^* is the *unique* best reply to some belief $\mathbf{b}_{-i} \in \Delta^+(\mathcal{B}_{-i})$ for v. Therefore b^* cannot be weakly dominated for v. ∎

Theorem 2.2 shows that for every (common) least upper bound $\mathbf{B} : [0,1] \to \mathbb{R}_+$ on the bids of the opponents we can derive a least upper bound on the interim best replies of Player i. We denote by $\phi^{\mathbf{B}}$ the new upper bound, that is,

$$\forall v \in [0,1],\ \phi^{\mathbf{B}}(v) = v - \pi^*(v; \mathbf{B}_{-i}),$$

where $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, ...\}$. The following proposition lists some useful properties of the map $\phi^{\mathbf{B}}$.

Proposition 2.4. *For every continuous, increasing function* $\mathbf{B}$ *such that* $\mathbf{B}(0) = 0$*, the function* $\phi^{\mathbf{B}}$ *satisfies the following properties:*
(1) $\phi^{\mathbf{B}}(0) = 0$ *and* $0 < \phi^{\mathbf{B}}(v) \le v$ *for all* $v \in (0,1]$.
(2) $\max\{\arg\max_{b\ge 0} \pi(b, v; \mathbf{B}_{-i})\} \le \phi^{\mathbf{B}}(v) \le \mathbf{B}(1)$ *(where* $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, ...\}$*).*
(3) $\phi^{\mathbf{B}}$ *is concave (hence continuous).*
(4) There exists a valuation $v^{\mathbf{B}}$ *such that* $\phi^{\mathbf{B}}$ *is increasing on* $[0, v^{\mathbf{B}})$ *and* $\phi^{\mathbf{B}}(v) = \mathbf{B}(1)$ *for all* $v \in [v^{\mathbf{B}}, 1]$.

Proof. First note that, since $\mathbf{B}$ is increasing, ties have probability zero and $P(\cdot; \mathbf{B}_{-i})$ is continuous. Therefore, $\pi^*(v; \mathbf{B}_{-i}) = \max_{0\le b\le 1} \pi(b, v; \mathbf{B}_{-i})$ and $\arg\max_{0\le b\le 1} \pi(b, v; \mathbf{B}_{-i})$ is nonempty and compact.

Let $b^*(v) = \max\{\arg\max_{0\le b\le 1} \pi(b, v; \mathbf{B}_{-i})\}$ and $b_*(v) = \min\{\arg\max_{0\le b\le 1} \pi(b, v; \mathbf{B}_{-i})\}$.

(1) Since $\pi^*(0; \mathbf{B}_{-i}) = 0$, $\phi^{\mathbf{B}}(0) = 0$. Since $\mathbf{B}$ is continuous and $\mathbf{B}(0) = 0$, $P(b; \mathbf{B}_{-i}) > 0$ for all $b > 0$, which implies $\mathbf{B}_{-i} \in \Delta^+(\mathcal{B}_{-i})$ (recall that we let $\mathbf{B}_{-i}$ denote also the degenerate belief assigning probability one to conjecture $\mathbf{B}_{-i}$). By Proposition 2.3, for all $v > 0$ and $b' \in \arg\max_{0\le b\le v} \pi(b, v; \mathbf{B}_{-i})$, $0 < b' < v$. Thus, $\pi^*(v; \mathbf{B}_{-i}) = (v - b')P(b'; \mathbf{B}_{-i}) \in (0, v)$, which implies $\phi^{\mathbf{B}}(v) = v - \pi^*(v; \mathbf{B}_{-i}) \in (0, v)$.

(2) For all $b < \mathbf{B}(1) = \max_w \mathbf{B}(w)$, $P(b; \mathbf{B}_{-i}) < 1$ and $\pi(b, v; \mathbf{B}_{-i}) = (v - b)P(b; \mathbf{B}_{-i}) < v - b$; while $\pi(b, v; \mathbf{B}_{-i}) = v - b$ if $b \ge \mathbf{B}(1)$. Therefore, either $b^*(v) = \phi^{\mathbf{B}}(v) = \mathbf{B}(1)$ or $b^*(v) < \phi^{\mathbf{B}}(v) < \mathbf{B}(1)$.

(3) The payoff function $\pi(b, v; \mathbf{B}_{-i}) = (v - b)P(b; \mathbf{B}_{-i})$ is linear in v. Thus, by a standard argument, the value function $\pi^*(v; \mathbf{B}_{-i})$ is convex in v, which implies that $\phi^{\mathbf{B}}(v) = v - \pi^*(v; \mathbf{B}_{-i})$ is concave.

(4) By the envelope theorem, for any $\Delta v > 0$

$$\begin{aligned}
\lim_{\alpha\downarrow 0} \frac{\pi^*(v + \alpha\Delta v; \mathbf{B}_{-i}) - \pi^*(v; \mathbf{B}_{-i})}{\alpha\Delta v} &= P(b^*(v); \mathbf{B}_{-i}),\\
\lim_{\alpha\downarrow 0} \frac{\pi^*(v; \mathbf{B}_{-i}) - \pi^*(v - \alpha\Delta; \mathbf{B}_{-i})}{\alpha\Delta v} &= P(b_*(v); \mathbf{B}_{-i}),
\end{aligned}$$

where the first (second) limit exists because, by concavity of π^*, the right (left) incremental ratio is non-increasing (non-decresing) in α (cf. [1], p. 133). Therefore

$$\lim_{\alpha\downarrow 0} \frac{\phi^{\mathbf{B}}(v + \alpha\Delta v) - \phi^{\mathbf{B}}(v)}{\alpha\Delta v} = [1 - P(b^*(v); \mathbf{B}_{-i})] \in [0,1],$$

$$\lim_{a\downarrow 0}\frac{\phi^{\mathbf{B}}(v)-\phi^{\mathbf{B}}(v-\Delta v)}{\alpha\Delta v} \quad = \quad [1-P(b_*(v);\mathbf{B}_{-i})]\in[0,1].$$

The same holds for the right derivative in $v=0$ and the left derivative in $v=1$. Define $v^{\mathbf{B}} = \inf\{v : P(b^*(v);\mathbf{B}_{-i})=1\}$ (let $\inf\emptyset = 1$ for notational convenience). Since $P(b;\mathbf{B}_{-i})<1$ for all $b<\mathbf{B}(1)$ and $P(b;\mathbf{B}_{-i})=1$ for all $b\geq\mathbf{B}(1)$, $\phi^{\mathbf{B}}$ is increasing on $[0,v^{\mathbf{B}})$ and constant on $[v^{\mathbf{B}},1]$. ■

3. The General Interdependent-Values Model

Consider the following game with asymmetric information representing a single-object, first-price auction with (possibly) interdependent values and risk-neutral bidders. There are n players, or bidders. Each bidder i observes a random signal $\mathbf{s}_i$ with realizations s_i in the compact interval $S_i=[0,1]$. Signals are distributed according to the joint c.d.f. $F: S\rightarrow[0,1]$, where $S=\prod_{i=1}^n S_i$.

After observing her signal, each player chooses a bid $b\geq 0$. The object is assigned to one of the high bidders, breaking ties at random. The winner pays her bid, losers do not pay anything.

Bidder $i's$ value for the object is given by a function (random variable) $\mathbf{v}_i : S\rightarrow\mathbb{R}$. For example, in an auction with private values $\mathbf{v}_i(s_1,...,s_i,...,s_n)=s_i$. In an auction with common values $\mathbf{v}_i(s_1,...,s_n)=\mathbf{v}(s_1,...,s_n)$ is the expected value of the object conditional on the realization $(s_1,...,s_n)$.

Conditional expectations and probabilities. The expected value of a random variable $\mathbf{x}: S\rightarrow\mathbb{R}$ conditional on realization s_i is denoted $\mathrm{E}[\mathbf{x}|s_i]$ and the expected value of $\mathbf{x}$ conditional on s_i and event $C_{-i}\subset S_{-i}$ is denoted $\mathrm{E}[\mathbf{x}|s_i,C_{-i}]$. For example,

$$\mathrm{E}[\mathbf{v}_i|s_i,\mathbf{b}_{-i}\leq b]=\int_{[\mathbf{b}_{-i}\leq b]}\mathbf{v}_i(s_i,s_{-i})dF_{-i|i}(s_{-i}|s_i)$$

is the expected valuation for Bidder i conditional on the signal and the event that b is the high bid.

A similar notation is used for conditional probabilities: the probability of event $A_{-i}\subset S_{-i}$ given s_i and $C_{-i}\subset S_{-i}$ is denoted $\Pr[A_{-i}|s_i,C_{-i}]$ and C_{-i} is omitted if $C_{-i}=S_{-i}$. For example,

$$\Pr[\mathbf{b}_{-i}\leq b|s_i]=\int_{[\mathbf{b}_{-i}\leq b]}dF_{-i|i}(s_{-i}|s_i)$$

is the conditional probability that b is the high bid given conjecture $\mathbf{b}_{-i}$.

We consider beliefs assigning zero probability to ties. Given such a belief $\mu\in\Delta(\mathcal{B}_{-i})$, the expected payoff of bidding b conditional on signal s_i is

$$\pi_i(b,s_i;\mu)=\int_{\mathcal{B}_{-i}}\mathrm{E}[\mathbf{v}_i-b|s_i,\mathbf{b}_{-i}\leq b]\Pr[\mathbf{b}_{-i}<b|s_i]\mu(d\mathbf{b}_{-i}).$$

As in Section 2, let

$$\pi_i^*(s_i;\mu) = \sup_{b\geq 0} \pi_i(b, s_i; \mu).$$

Bid b is a best reply to belief μ for type s_i if $\pi(b, s_i; \mu) = \pi^*(s_i; \mu)$.

3.1. Assumptions

Assumption 3.1. *The cumulative distribution function F is absolutely continuous, with continuous density f bounded away from zero.*

Assumption 3.2. *For each bidder i, the value function $\mathbf{v}_i : S \to \mathbb{R}$ is continuous, non-negative, increasing in the i th argument and nondecreasing in all the other arguments.*

Non-negativity and monotonicity of the value function imply that the expected valuation conditional on a player's signal is positive:

Remark 2. *For each player i and each signal $s_i > 0$,*

$$\mathrm{E}[\mathbf{v}_i|s_i] > 0. \tag{3.1}$$

We also assume that *signals are affiliated*. As is well-known from Milgrom and Weber [16] (MW henceforth), this is equivalent to the supermodularity of $\log f$. For any pair of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ let $x \vee y$ and $x \wedge y$ denote the componentwise maximum and minimum respectively, i.e., $x \vee y = (\max(x_1, y_1), ..., \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), ..., \min(x_n, y_n))$.

Assumption 3.3. *For all $s, s' \in S$,*

$$f(s \vee s')f(s \wedge s') \geq f(s)f(s'), \tag{3.2}$$

Two key properties of affiliated random variables will be employed here:

Remark 3. *(cf MW, Theorem 4) For every nonempty $J \subset N$, and for every $K \subset N$ disjoint from J, the random variables $\{\mathbf{s}_j\}_{j\in J}$ are affiliated conditional upon the realizations of the (possibly empty) collection of random variables $\{\mathbf{s}_k\}_{k\in K}$.*

Remark 4. *(MW, Theorem 5) For every random variable $\mathbf{H} : S \to \mathbb{R}$, if $\mathbf{H}$ is nondecreasing in each argument, then the conditional expectation function*

$$h(x_1, y_1, \ldots, x_n, y_n) = \mathrm{E}[\mathbf{H}|x_1 \leq \mathbf{s}_1 \leq y_1, \ldots, x_n \leq \mathbf{s}_n \leq y_n]$$

is nondecreasing in each argument.

We refer to an auction game satisfying Assumptions 3.1, 3.2 and 3.3 as the *interdependent-values model*.

3.2. Upper Bounds on Best Responses

It is often said that a player bidding close to her expected valuation $\mathrm{E}[\mathbf{v}_i|s_i]$ is subject to the *winner's curse*, because she is not taking into account that if she wins the object it must be the case the competitors have observed low signals. This argument relies on the assumption that Player i thinks that her competitors are using *increasing* bidding functions. To see this, note that if Bidder i has conjecture $\mathbf{b}_{-i}$ and $\mathbf{b}_{-i}$ is increasing (in each component), then the expected valuation *conditional on* (the signal and) *the event of winning the object* is $\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{b}_{-i} \leq b] \leq \mathrm{E}[\mathbf{v}_i|s_i]$, where the inequality is strict in non-degenerate cases. But if conjecture $\mathbf{b}_{-i}$ is not increasing, then it may be the case that $\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{b}_{-i} \leq b] > \mathrm{E}[\mathbf{v}_i|s_i]$ and also the best response to $\mathbf{b}_{-i}$ is above $\mathrm{E}[\ \mathbf{v}_i|s_i]$.

To analyze the behavior of strategically sophisticated bidders who are not subject to the winner's curse, we restrict our attention to beliefs that assing positive weight only to increasing bidding functions (including in the support of beliefs nondecreasing bidding functions with flat segments involves technical complications that are briefly discussed in the Appendix). Furthermore, we continue to assume that a player believes that every positive bid yields a positive probability of winning the object. Formally, let $\mathcal{M}_j$ denote the set of monotone increasing bidding functions for Player j and let $\mathcal{M}_{-i} = \prod_{j\neq i} \mathcal{M}_j$. Then the set of Bidder i's beliefs we restrict our attention to is

$$\Delta^+(\mathcal{M}_{-i}) = \left\{ \mu \in \Delta(\mathcal{M}_{-i}) : \forall b > 0, \int_{\mathcal{M}_{-i}} \Pr[\mathbf{b}_{-i} \leq b|s_i]\mu(d\mathbf{b}_{-i}) > 0 \right\},$$

where s_i on the right hand side is arbitrary.[8]

Let $\mathbf{B}_{-i} = \{\mathbf{B}_j\}_{j\neq i}$ be an arbitrary upper bound on the bids of Player i's competitor. The main result of this section characterizes the set of interim best replies to "monotonic" beliefs assigning probability one to bids below this upper bound. The set of such beliefs is

$$\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}) = \left\{ \mu \in \Delta^+(\mathcal{M}_{-i}) : \mu\left(\{\mathbf{b}_{-i} : \mathbf{b}_{-i} < \ \mathbf{B}_{-i}\}\right) = 1 \right\}.$$

Theorem 3.4. *Let $\mathbf{B}_j : [0,1] \to \mathbb{R}_+$, $j \in N\backslash\{i\}$, be nondecreasing functions such that $\mathbf{B}_j > 0$. For every bid $b^* > 0$ and signal $s_i \in [0,1]$,*

(1) if $b^* > \mathrm{E}[\mathbf{v}_i|s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu)$, then b^* is not a best reply to any belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$ for s_i;
(2) if $b^* < \mathrm{E}[\mathbf{v}_i|s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu)$, then b^* is a strict best reply to some belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$ for s_i.
(3) Furthermore,

$$\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu) = \sup_{b\geq 0} \quad \left\{ \quad \Big(\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i} < b] - b\Big) \Pr[\mathbf{B}_{-i} < b|s_i] + \right.$$

[8]By Assumption 3.1, for every s_i, the conditional density $f_{-i|i}(\cdot|s_i)$ is bounded away from zero; hence the expression on the right hand side is independent of s_i.

$$+ \max\left\{0, \left(\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} = b] - b\right) \Pr[\mathbf{B}_{-i}^{\max} = b|s_i]\right\}\Big\}$$

and the supremum is attained.

Note that if $\mathbf{B}_{-i}$ is increasing, then the right hand side of the above equality is simply $\pi^*(s_i; \mathbf{B}_{-i})$ and the expression for the new upper bound on i's bids is very similar to the one found for the IPV model. Note also that the given bound $\mathbf{B}_{-i}$ can be chosen so high that $\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i; \mu) = 0$ (e.g. let $\min_{s_j} \mathbf{B}_j^G(s_j) > \max_s \mathbf{v}_i(s)$). Therefore we obtain:

Corollary 3.5. *The set of best replies for $s_i > 0$ to beliefs in $\Delta^+(\mathcal{M}_{-i})$ is the interval $(0, \mathrm{E}[\mathbf{v}_i|s_i])$.*

For any *tuple* of nondecreasing functions $\mathbf{B}_{-i} = \{\mathbf{B}_j\}_{j\neq i}$, define $\phi_i^{\mathbf{B}_{-i}}(s_i) = \mathrm{E}[\mathbf{v}_i|s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi_i^*(s_i; \mu)$. The following result generalizes Proposition 2.4:

Proposition 3.6. *For every tuple of non-decreasing functions $\mathbf{B}_{-i} \in \mathcal{B}_{-i}$ such that $\mathbf{B}_{-i} > 0$, the function $\phi_i^{\mathbf{B}_{-i}}$ satisfies the following properties:*
(1) $0 < \phi_i^{\mathbf{B}_{-i}}(s_i) \leq \min\left(\mathrm{E}[\mathbf{v}_i|s_i], \max_{j\neq i} \lim_{s_j\uparrow 1} \mathbf{B}_j(s_j)\right)$ for all $s_i \in (0, 1]$.
(2) $\phi_i^{\mathbf{B}_{-i}}$ is continuous.
(3) There exists a signal $s_i^{\mathbf{B}_{-i}}$ such that $\phi_i^{\mathbf{B}_{-i}}$ is increasing on $[0, s_i^{\mathbf{B}_{-i}})$ and $\phi_i^{\mathbf{B}_{-i}}(s_i) = \max_{j\neq i} \lim_{s_j\uparrow 1} \mathbf{B}_j(s_j)$ for all $s_i \in [s_i^{\mathbf{B}_{-i}}, 1]$.

4. Rationalizable Bids

The standard definition of (interim) rationalizability captures the implications of the assumption that bidders are (interim) rational and there is common certainty of this fact. We analyze a strengthening of this definition because we also assume that bidders' beliefs satisfy some restrictions, and that there is common certainty of this fact (see, e.g., Battigalli [2]).

Let $\Delta_i \subset \Delta(\mathcal{B}_{-i})$ be a restricted set of beliefs and let $\Delta = (\Delta_1, ..., \Delta_n)$. In particular, we consider the case $\Delta_i = \Delta^+(\mathcal{B}_{-i})$ for the symmetric IPV model (see Section 2), and the case $\Delta_i = \Delta_i^+(\mathcal{M}_{-i})$ for the interdependent values model (see Section 3). The following definition of Δ-interim rationalizability captures the implications of the assumption that (a) the bidders are expected payoff maximizers, (b) for each bidder $i = 1, ..., n$, i's beliefs belong to Δ_i (but different bidders and different types of the same bidder may have different beliefs), and (c) there is common certainty of (a) and (b).

Definition 4.1. *For all $i = 1, ..., n$, $s_i \in [0, 1]$ and $k = 0, 1, ...$ let $\mathcal{R}_i^\Delta(v; 0) = \mathbb{R}_+$, $\Delta_i(0) = \Delta_i$,*

$$\mathcal{R}_i^\Delta(s_i; k + 1) = \{b \geq 0 : \exists\mu \in \Delta_i(k), \pi_i(b, v; \mu) = \pi^*(v; \mu)\},$$

$$\Delta_i(k+1) = \left\{\mu \in \Delta_i : \mu\left(\left\{\mathbf{b}_{-i} : \forall j \neq i, \forall s_j \in [0,1], \mathbf{b}_j(s_j) \in \mathcal{R}_j^{\Delta}(s_j; k+1)\right\}\right) = 1\right\}.$$

Finally, for all $i = 1, ..., n$ and $s_i \in [0,1]$, let $\mathcal{R}_i^{\Delta}(s_i; \infty) = \bigcap_{k>0} \mathcal{R}_i^{\Delta}(s_i; k)$. A bid b is interim Δ-rationalizable *for type s_i of Bidder i if $b \in \mathcal{R}_i^{\Delta}(v, \infty)$.*

For $k = 0, 1, \ldots, \infty$, we denote by $\mathcal{R}_i^{\Delta}(k)$ the set of selections from $\mathcal{R}_i^{\Delta}(\cdot; k)$; that is, for any conjecture $\mathbf{b}_i \in \mathcal{B}_i$, we write $\mathbf{b}_i \in \mathcal{R}_i^{\Delta}(k)$ if and only if, for all $s_i \in [0,1]$, $\mathbf{b}_i(s_i) \in \mathcal{R}_i^{\Delta}(s_i; k)$.

The following remarks are immediate consequences of the preceding definition.

Remark 5. *For all $i \in N$, $s_i \in (0,1]$ and $b \geq 0$, $b \in \mathcal{R}_i^{\Delta}(s_i; \infty)$ if and only if there exists some belief $\mu \in \Delta_i$ such that $\mu\left(\mathcal{R}_{-i}^{\Delta}(\infty)\right) = 1$ and $\pi_i(b, s_i; \mu) = \pi_i^*(s_j; \mu)$.*

Remark 6. *Let $(\mathbf{b}_1^{eq}, ..., \mathbf{b}_n^{eq})$ be a Bayesian Nash equilibrium such that $\mathbf{b}_{-i}^{eq} \in \Delta_i$, $i \in N$. Then, for all $i \in N$ and $s_i \in [0,1]$, the equilibrium bid $\mathbf{b}_i^{eq}(s_i)$ is interim Δ-rationalizable for s_i.*

4.1. Rationalizability in the Symmetric IPV Model

Consider the symmetric IPV model of Section 2 and let $\Delta_i = \Delta^+(\mathcal{B}_{-i})$ ($i \in N$), that is, consider beliefs such that every positive bid yields a positive probability of winning the object. To simplify the notation omit the superscript Δ from the set of interim Δ-rationalizable strategies. Since the model is symmetric, we also drop the bidder's subscript whenever this causes no confusion.

By Proposition 2.3, the procedure given by in Definition 4.1 is equivalent to performing one round of elimination of all weakly dominated bids for each type, followed by the iterated elimination of strictly dominated bids for each type.[9]

Remark 7. *By Proposition 2.3, $\mathcal{R}(v; 1) = (0, v)$ for all $v \in (0,1]$ and $\mathcal{R}(0; k) = \{0\}$ for all k. This implies that for all μ and all $k = 1, 2, ...$, $\mu \in \Delta(k)$ if and only if*

$$\mu\left(\{\mathbf{b}_{-i} : \forall j \neq i, \forall s_j \in [0,1], \mathbf{b}_j(s_j) \in \mathcal{R}(s_j; k)\}\right) = 1,$$

that is, the condition $\forall b > 0$, $P(b; \mu) > 0$ is superfluous because it is implied by the condition above.

We are now ready to state the main characterization result of this subsection. We derive a pointwise (weakly) decreasing sequence of upper bounds on rationalizable bids. These bounds exactly characterize each step k of the rationalizability procedure given by Definition 4.1,

[9]The "ex ante version" of this procedure was first put forward and motivated by Dekel and Fudenberg [5]. Since then, several papers provided other epistemic characterizations. See Section 6 in the survey by Dekel and Gul [6] and the references therein.

hence each assumption of the form "(everybody is certain that)$^{k-1}$ everbody is rational and believes that positive bids win with positive probability" ($k = 1, 2, ...$). Define $\mathbf{B}(v; 1) = v$, $\mathbf{B}(v; k+1) = v - \pi^*(v; \mathbf{B}_{-i}(\cdot; k))$ for all v (where $\mathbf{B}_{-i}(\cdot; k) = \{\mathbf{B}(\cdot; \mathbf{k}), \mathbf{B}(\cdot; k), ...\}$). Since $\pi^*(v; \mathbf{B}_{-i}(\cdot; k)) \geq 0$, we have $\mathbf{B}(\cdot; 1) \geq \mathbf{B}(\cdot; 2)$ and, by induction, Remark 1 yields $\mathbf{B}(\cdot; k) \geq \mathbf{B}(\cdot; k+1)$ for all k. Therefore we can define $\mathbf{B}(v; \infty) = \lim_{k\to\infty} \mathbf{B}(v; k)$ for all v. Note also that, for all k, $\pi^*(0; \mathbf{B}_{-i}(\cdot; k)) = 0$; thus, $\mathbf{B}(0, k) = 0$.

Theorem 4.2.
(1) For all $k = 1, 2, ...$ and $v \in (0, 1]$, the interior of $\mathcal{R}(v; k)$ is the open interval $(0, \mathbf{B}(v; k))$; the upper bound $\mathbf{B}(\cdot; k)$ is increasing and concave.
(2) For all $v \in (0, 1]$, the interior of $\mathcal{R}(v; \infty)$ is the open interval $(0, \mathbf{B}(v; \infty))$, where $\mathbf{B}(v; \infty) = v - \pi^*(v; \mathbf{B}_{-i}(\cdot; \infty))$; the upper bound $\mathbf{B}(\cdot; \infty)$ is nondecreasing and concave.

Proof. (1) The statement is true for $k = 1$. Suppose it is true for some k. Then Theorem 2.2 implies that the interior of $\mathcal{R}(v; k+1)$ is $(0, v - \pi^*(v; \mathbf{B}_{-i}(\cdot; k)) = (0, \mathbf{B}(v; k+1)$ for all $v \in (0, 1]$.

By way of contradiction, suppose that $\mathbf{B}(\cdot; k+1)$ is not increasing. Then Proposition 2.4 (4) implies that there is some $v^k < 1$ such that $\mathbf{B}(v; k+1) = \mathbf{B}(1; k)$ for all $v \in [v^k, 1]$. Since $\mathbf{B}(\cdot; k)$ is increasing (by the inductive hypothesis), we obtain $\mathbf{B}(v; k+1) = \mathbf{B}(1, k) > \mathbf{B}(v; k)$ for some $v < 1$, which contradicts $\mathbf{B}(v; k+1) \leq \mathbf{B}(v; k)$.

Concavity is a consequence of Proposition 2.4 (3).

(2) Clearly, part (1) implies that the interior of $\mathcal{R}(v; \infty)$ is $(0, \mathbf{B}(v; \infty))$ for all $v > 0$ and that $\mathbf{B}(0; \infty) = 0$. Since each $\mathbf{B}(\cdot; k)$ is increasing and concave, $\mathbf{B}(\cdot; \infty)$ must be nondecreasing and concave. Moreover, the symmetric IPV model has a symmetric equilibrium $\mathbf{b}^{eq}$ which satisfies $\mathbf{b}^{eq}(0) = 0$, $\mathbf{b}^{eq} > 0$ and, by Remark 6, $\mathbf{b}^{eq} < \mathbf{B}(\cdot; k)$. Hence $0 < \mathbf{b}^{eq} \leq \mathbf{B}(\cdot; \infty)$, so the function $\mathbf{B}(\cdot; \infty)$ satisfies the hypotheses of Theorem 2.2 and the interior of the set of best responses for type v to beliefs in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i}(\cdot; \infty))$ must be $(0, v - \pi^*(v; \mathbf{B}_{-i}(\cdot; \infty))$. Then, by Remark 5 $(0, v - \pi^*(v; \mathbf{B}_{-i}(\cdot; \infty))$ must coincide with $(0, \mathbf{B}(v; \infty))$, the interior of $\mathcal{R}(v; \infty)$. ■

It is instructive to compare the predictions of equilibrium analysis and interim Δ-rationalizability. Observe that the symmetric IPV model has a unique symmetric equilibrium bidding function, which must be incresing and continuous. Let $\mathbf{b}^{eq}(v)$ denote the equilibrium bid of type v. Theorem 4.2 and Remark 6 imply that every bid below the equilibrium is interim rationalizable. Thus, we have:

Corollary 4.3. *For all $v > 0$, all bids $b \in (0, \mathbf{b}^{eq}(v)]$ are interim rationalizable for type v.*

Moreover, we can show that this implies that some bids above the equilibrium are Δ-rationalizable. Recall that $\phi^{\mathbf{B}}(v) = v - \pi^*(v; \mathbf{B}_{-i})$ is the new bound on best replies obtained by a given upper bound $\mathbf{B}$ on beliefs.

Lemma 4.4. *$\phi^{\mathbf{b}^{eq}}(1) = \mathbf{b}^{eq}(1)$, $\mathbf{b}^{eq}(v) < \phi^{\mathbf{b}^{eq}}(v) < \mathbf{b}^{eq}(1)$ for all $v \in (0,1)$, and $\phi^{\mathbf{b}^{eq}}$ is increasing. Every bid $b \in (0, \phi^{\mathbf{b}^{eq}}(v))$ is interim Δ-rationalizable for v, for all $v \in (0,1]$.*

Proof. The equilibrium condition and Proposition 2.4 (2) imply that $\pi^*(v; \mathbf{b}^{eq}_{-i}) = (v - \mathbf{b}^{eq}(v))P(\mathbf{b}^{eq}(v); \mathbf{b}^{eq}_{-i}) \le v - \mathbf{b}^{eq}(v)$ and $\mathbf{b}^{eq}(v) \le \phi^{\mathbf{b}^{eq}}(v) \le \mathbf{b}^{eq}(1)$. Since $\mathbf{b}^{eq}$ is increasing, $\mathbf{b}^{eq}(v) \le \mathbf{b}^{eq}(1)$, $P(\mathbf{b}^{eq}(v); \mathbf{b}^{eq}_{-i}) \le P(\mathbf{b}^{eq}(1); \mathbf{b}^{eq}_{-i})$ and all the previous inequalities are strict, if $v < 1$, while they hold as equalities if $v = 1$. Then, Proposition 2.4 (4) implies that $\phi^{\mathbf{b}^{eq}}$ is increasing. The second part of the statement follows from Corollary 4.3 and Theorem 2.2. ■

Corollary 4.5. *For all $v \in (0,1)$ there is some above-equilibrium bid $b > \mathbf{b}^{eq}(v)$ such that b is interim Δ-rationalizable for v.*

It is well known that equilibrium bid increase with the number of players. A similar result is true for the upper bound on Δ-rationalizable bids. Let $\mathbf{B}^n(\cdot; k)$ ($k = 1, 2, ..., \infty$) denote the least upper bound at the k-th step of the rationalizability procedure in the n-person auction game.

Proposition 4.6. *For each step $k = 2, 3, ...$, the upper bound on k-rationalizable bids is increasing in the number of bidders, that is, $\mathbf{B}^n(\cdot; k) < \mathbf{B}^{n+1}(\cdot; k)$ for all $n = 2, 3, ...$.*

Proof. For any fixed n, the proof is by induction on k. We use superscript n to denote that a given mathematical object is defined for the n-person game. Note that $\mathbf{B}^n(v; 1) = v = \mathbf{B}^{n+1}(v; 1)$.

Now suppose that, for some $k \ge 1$, $\mathbf{B}^{n+1}(\cdot; k) \ge \mathbf{B}^n(\cdot; k)$. We show that this implies $\mathbf{B}^{n+1}(v; k+1) > \mathbf{B}^n(v; k+1)$ for all $v \in (0,1)$.

By Remark 1 and Theorem 2.2 we have

$$\mathbf{B}^{n+1}(v; k+1) = v - \pi^{n+1,*}(v; \mathbf{B}^{n+1}_{-i}(\cdot; k)) \ge v - \pi^{n+1,*}(v; \mathbf{B}^{n}_{-i}(\cdot; k)).$$

Now observe that for any continuous and increasing bidding function $\mathbf{B}$ with $\mathbf{B}(0) = 0$ (such as $\mathbf{B}^n(\cdot; k)$), and for any $b \in (0, \mathbf{B}(1))$,

$$\begin{aligned} & 0 < G(\mathbf{B}^{-1}(b)) < 1 \ \text{ and} \\ P^{n+1}(b; \mathbf{B}_{-i}) &= \left[G\left(\mathbf{B}^{-1}(b)\right)\right]^n < \left[G\left(\mathbf{B}^{-1}(b)\right)\right]^{n-1} = P^n(b; \mathbf{B}_{-i}), \end{aligned} \tag{4.1}$$

which implies $\pi^{n+1,*}(v; \mathbf{B}_{-i}) \le \pi^{n,*}(v; \mathbf{B}_{-i})$ for all v.

Now we show that $\pi^{n+1,*}(v; \mathbf{B}^n_{-i}(\cdot; k)) < \pi^{n,*}(v; \mathbf{B}^n_{-i}(\cdot; k))$ for all $v \in (0,1)$. By way of contradiction, suppose that $\pi^{n+1,*}(v; \mathbf{B}^n_{-i}(\cdot; k)) \ge \pi^{n,*}(v; \mathbf{B}^n_{-i}(\cdot; k))$. Then, for some $b^* \in (0, v)$,

$$\begin{aligned} (v - b^*)P^{n+1}(b^*; \mathbf{B}^n_{-i}(\cdot; k)) &= \pi^{n+1,*}(v; \mathbf{B}^n_{-i}(\cdot; k)) \ge \\ \pi^{n,*}(v; \mathbf{B}^n_{-i}(\cdot; k)) &\ge (v - b^*)P^n(b; \mathbf{B}^n_{-i}(\cdot; k)) \end{aligned}$$

(recall that 0 cannot be a best reply). By (4.1), this can happen only if $b^* = \mathbf{B}^n(1;k)$, $P^{n+1}(b^*, \mathbf{B}^n_{-i}(\cdot;k)) = 1 = P^{n+1}(b^*, \mathbf{B}^n_{-i}(\cdot;k))$ and b^* maximizes also $\pi^n(\cdot, v; \mathbf{B}^n_{-i}(\cdot;k))$. But, by Proposition 2.4 (2) and Theorem 4.2 (1), this would imply $\mathbf{B}^n(v;k+1) = \mathbf{B}^n(1;k) > \mathbf{B}^n(v;k)$, which contradicts $\mathbf{B}^n(v;k+1) \leq \mathbf{B}^n(v;k)$.

Therefore, using Theorem 2.2,

$$v - \pi^{n+1,*}(v; \mathbf{B}^n_{-i}(\cdot;k)) > v - \pi^{n,*}(v; \mathbf{B}^n_{-i}(\cdot;k)) = \mathbf{B}^n(\cdot; k+1).$$

We conclude that $\mathbf{B}^{n+1}(v;k+1) > \mathbf{B}^{n+1}(v;k)$ for all $v \in (0,1)$. ■

We now turn to *bid shading*. The following proposition shows that interim Δ-rationalizability *implies* substantial shading for high valuations, but only little shading for low valuations (of course, interim Δ-rationalizability is consistent with extreme shading for all valuations because $\inf \mathcal{R}(v;\infty) = 0$).

Proposition 4.7. *For all $k = 2, 3, ..., \infty$, the "minimum-shading" function $\mathbf{S}(w;k) = w - \mathbf{B}(w;k)$ is increasing and convex, with $\mathbf{S}(0;k) = 0$ and $\mathbf{S}(v;k) > 0$ for all $v \in (0,1]$. Therefore the "minimum-shading-ratio" function $\frac{\mathbf{S}(v,k)}{v}$ is positive and nondecreasing. However,*

$$\lim_{v \downarrow 0} \frac{\mathbf{S}(v,k)}{v} = 0.$$

Proof. $\mathbf{S}(v;k) = \pi^*(v; \mathbf{B}_{-i}(\cdot;k-1))$ and $\mathbf{B}_{-i}(\cdot;k-1) \in \Delta^+(\mathcal{B}_{-i})$ for all $k \geq 2$. Therefore, by the same argument as in the proof of Proposition 2.4 (3)-(4), $\mathbf{S}(.;k)$ is increasing and convex, and $\mathbf{S}(v;k) > 0$.

Next note that, by Theorem 4.2 4.2 and Remark 6 6, $\mathbf{b}^{eq} \leq \mathbf{B}(v;k-1)$. Therefore,

$$\mathbf{S}(v;k) = \pi^*(v; \mathbf{B}_{-i}(\cdot;k-1)) \leq \pi^*(v; \mathbf{b}^{eq}_{-i}) \leq vP(v; \mathbf{b}^{eq}_{-i}) = v\,[G(v)]^{n-1},$$

where the first inequality follows from Remark 1 1. Thus

$$0 \leq \lim_{v \downarrow 0} \frac{\mathbf{S}(v;k)}{v} \leq \lim_{v \downarrow 0} [G(v)]^{n-1} = 0.$$

■

4.2. Rationalizability in the Interdependent-Values Model

Consider the interdependent values model of Section 3 and let $\Delta_i = \Delta^+(\mathcal{M}_{-i})$ $(i \in N)$. To simplify the notation, continue to omit the superscript Δ from the set of interim Δ-rationalizable strategies. Since the model need not be symmetric, we have to allow for different upper bounds on the bids of different players and different sets of Δ-rationalizable bids. Asymmetries may also cause the upper bounds to have a flat segment at the top.

Let $\mathbf{B}_i(s_i;1) = \mathrm{E}[\mathbf{v}_i|s_i]$, $\mathbf{B}_i(s_i;k+1) = \mathrm{E}[\mathbf{v}_i|s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k))}\pi_i^*(s_i;\mu)$ for all s_i. Since $\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k))}\pi_i^*(s_i;\mu) \geq 0$, we have $\mathbf{B}_i(\cdot;1) \geq \mathbf{B}_i(\cdot;2)$; by induction, if $\mathbf{B}_i(\cdot;k-1) \geq \mathbf{B}_i(\cdot;k)$ for some $k > 1$,

$$\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k-1)) \supset \Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k)),$$

so that, for every $s_i \in [0,1]$,

$$\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k-1))}\pi_i^*(s_i;\mu) \leq \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i}(\cdot;k))}\pi_i^*(s_i;\mu),$$

and hence $\mathbf{B}_i(\cdot;k) \geq \mathbf{B}_i(\cdot;k+1)$ for all k. Thus, we can define $\mathbf{B}_i(s_i;\infty) = \lim_{k\to\infty}\mathbf{B}_i(s_i;k)$ for all s_i, and we obtain the following counterpart to Theorem 4.2 (we omit the proof because it is similar to the proof of Theorem 4.2).

Theorem 4.8.
(1) For all $k = 1,2,...$ and $s_i \in (0,1]$, the interior of $\mathcal{R}_i(s_i;k)$ is the open interval $(0,\mathbf{B}_i(s_i;k))$; the upper bound $\mathbf{B}_i(s_i;k))$ is non-decreasing and continuous.
(2) For all $s_i \in (0,1]$, the interior of $\mathcal{R}_i(s_i;\infty)$ is the open interval $(0,\mathbf{B}_i(s_i;\infty))$; the upper bound $\mathbf{B}_i(s_i;k))$ is non-decreasing.

Theorem 4.8 provides weaker results on the upper bounds on rationalizable bids, because we have generalized the model along several dimensions. First, when bidders are asymmetric, the upper bounds can be flat at the top. Second, signals may be correlated and the valuation function of Bidder i need not be linear in her own signal. This implies that the value function $\pi_i^*(s_i;\mathbf{B}_{-i})$ need not be convex in the signal, and the derived bound $\phi_i^{\mathbf{B}_{-i}}(s_i) = \mathrm{E}(\mathbf{v}_i|s_i) - \pi_i^*(s_i;\mathbf{B}_{-i})$ need not be concave.

However, there are interesting examples of interdependent-values models where (a) bidders are symmetric[10], (b) signals are independent, and (c) valuations functions are *quasi-linear*: that is, for each bidder i, the valuation function has the form $\mathbf{v}_i(s_i,s_{-i}) = \boldsymbol{\upsilon}_i(s_{-i})s_i + \boldsymbol{\kappa}_i(s_{-i})$, where $\boldsymbol{\upsilon}_i$ and $\boldsymbol{\kappa}_i$ are nondecreasing and $\boldsymbol{\upsilon}_i > 0$. A well-know example is the "wallet game" (see Klemperer [15]). For such models we can replicate all qualitative results of the symmetric IPV-model.

Proposition 4.9. *If the signals are independent, the valuation functions quasi-linear and bidders are symmetric, then the upper bounds $\mathbf{B}_i(\cdot;k)$ are (identical for all bidders) increasing and concave for all $k = 1,2,...,\infty$ and i.*

[10] That is, the c.d.f. F is symmetric and there exists a function $\mathbf{v} : [0,1]\times[0,1]^{n-1} \to \mathbb{R}$ such that: (i) for every $s_1 \in [0,1]$, $s_{-1} \in [0,1]^{n-1}$, and permutation $\{\pi(2),\ldots,\pi(n)\}$ of $\{2,\ldots,n\}$, $\mathbf{v}(s_1,s_{-1}) = \mathbf{v}(s_1,(s_{\pi(j)})_{j\neq 1})$. (ii) For every $i \in N$, and $s = (s_i,s_{-i}) \in S$, $\mathbf{v}_i(s) = \mathbf{v}(s_i,s_{-i})$. See also Milgrom and Weber [16].

Sketch of proof. Fix a common increasing bound $\mathbf{B}$. Since $\mathbf{v}_i$ is quasi-linear, $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} \leq b]$ and $\mathrm{E}[\mathbf{v}_i|s_i]$ are also linear in s_i. By independence, $\Pr[\mathbf{B}_{-i} \leq b|s_i] = \Pr[\mathbf{B}_{-i} \leq b]$. Thus, the payoff function $\pi(b, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} \leq b] \Pr[\mathbf{B}_{-i} \leq b]$ is linear in s_i and, by a standard argument, the value function $\pi^*(s_i; \mathbf{B}_{-i})$ is convex in s_i, which implies that $\phi^{\mathbf{B}}(s_i) = \mathrm{E}[\mathbf{v}_i|s_i] - \pi^*(s_i; \mathbf{B}_{-i})$ is concave.

The proof that each bound $\mathbf{B}(\cdot; k)$ is increasing relies on symmetry and is similar to the proof for that symmetric IPV-model. ■

5. Discussion

This section discusses the relationship between our findings and the experimental evidence on first-price auctions, and indicates a number of extensions. To fix ideas, we begin with an overview of our results so far.

Our key analytical tool is a characterization of the best responses for a type to beliefs satisfying certain restrictions. Specifically, if a (risk neutral) rational player i of type s_i believes that his opponents are not going to bid above a given (type-dependent) upper bound $\mathbf{B}_{-i}$, then she can choose any bid in the interval $(s_i, \phi_i^{\mathbf{B}_{-i}}(s_i))$, where the derived upper bound $\phi_i^{\mathbf{B}_{-i}}(s_i)$ is not higher than $\mathrm{E}(\mathbf{v}_i|s_i)$, the expected valuation of the good for type s_i.

We then use this result to obtain an iterative characterization of interim rationalizable bids. In the first step of the algorithm, we eliminate all the bids $b \geq \mathbf{B}_i(s_i, 1) = \mathrm{E}(\mathbf{v}_i|s_i)$. This reflects a form of cautiousness: positive bids are expected to yield a positive probability of winning the object, and hence weakly dominated choices cannot be best responses. In subsequent steps $k = 2, 3...$, we apply our characterization result and derive an upper bound $\mathbf{B}_i(s_i; k) = \phi_i^{\mathbf{B}_{-i}(\cdot;k-1)}(s_i)$ from the previous upper bound $\mathbf{B}_{-i}(\cdot; k-1)$; all bids in the interval $(0, \mathbf{B}_i(s_i; k))$ are k-rationalizable. This provides a relatively simple implementation of interim rationalizability.

Finally, we show that, in the symmetric IPV model, the upper bound on rationalizable bid is increasing, *concave* and *strictly above the Nash equilibrium*. Therefore, rationalizability is consistent with bids above and below the equilibrium, and it implies substantial proportional shading (bidding below the valuation) for high types. On the other hand, we show that, for low types, the highest rationalizable bid is close to the valuation. Some of these conclusions also hold for specific classes of interdependent-values models, such as "generalized wallet games" (cf. [15]) and the "general symmetric model" of Milgrom and Weber [16].

Our analysis has implications for revenue comparisons in auctions with private values. The "cautiousness" assumption rules out weakly dominated bids. In *second*-price auctions, this form of cautiousness implies that players will bid their valuation. On the other hand, without the full strength of Bayesian Nash equilibrium analysis, *expected revenues in a first-price auction can be lower or higher than in a second-price auction.*

Perhaps more interestingly, *expected revenues in a first-price auction can be arbitrarily close to zero, and will always be lower than they would be under truthful bidding.*

Observe that, as the number of bidders grow, the equilibrium bid function approaches the identity function. Thus, in rough, intuitive terms, in the limit revenue equivalence obtains only in the "best scenario" (from the point of view of the seller) where players bid close to their upper bound.

5.1. Experiments and Deviations From the Risk-Neutral Nash Equilibrium

There are (at least) three "stylized facts" emerging from the experimental studies on first price auctions with private values, which we find relevant in relation to our theoretical analysis:[11]

Overbidding. A large majority of subjects show a persistent tendency to bid above the risk-neutral Nash equilibrium (RNNE).

Decreasing Proportional Deviations. Deviations from RNNE are proportionally smaller for subjects with smaller valuations; in other words, the ratio

$$\frac{|\text{actual bid-RNNE bid}|}{\text{valuation}}$$

is negatively correlated with subjects' private valuations.[12]

Heterogeneity. Bidding behavior is heterogeneous across subjects. [13]

In a series of papers, Cox, Smith and Walker try to explain the data with a family of models featuring bidders with heterogeneous degrees of (constant relative) risk aversion. In such models, equilibrium bidding functions are linear (like the RNNE function) except for the largest valuations, but have heterogenous slopes and are steeper than the RNNE (e.g., Cox *et al.* [3, 4]). The risk-aversion explanation of Overbidding is controversial. In particular, it leaves Decreasing Proportional Deviations largely unexplained. Furthermore, it is at odds with experimental findings concerning different auction settings.[14]

We think that our paper sheds light on a different explanation of these experimental findings: different subjects have different beliefs about the bidding behavior of their competitors

[11] See, for example, Kagel [13] for a survey about experiments on auctions.

[12] See, e.g., Kagel and Roth [14], p 1381.

[13] For example, Cox *et al* [3] reject the null hypothesis of a common bidding function in symmetric IPV auctions.

[14] Other partial explanations of Overbidding involve (i) psychological biases related to frame effects and the complexity of the decsion problem (e.g., Kagel [13], Section I.B), and (ii) lack of experimental control on subjects incentives due to a small expected cost of deviations from the optimal bid (Harrison [12]). Section I.G in Kagel [13] provides a discussion of the debate about the risk-aversion explanation. Kagel and Roth [14] presents Decreasing Proportional Deviations as one of the empirical findings at odds with the constant relative risk aversion model. A recent experimental paper by Goeree *et al.* [11] provides support for the risk-aversion explanation.

and the limited feedback they get from the outcomes of previous auctions prevents them from approaching the equilibrium sufficiently fast (e.g., Friedman [8]). But even if subjects do not hold equilibrium beliefs, they may be sophisticated enough to take into account that their competitors' behavior satisfies some rationality restrictions and, possibly, that also their opponents' beliefs conform to analogous assumptions. Our paper identifies the least upper bound on bids of strategically sophisticated, risk-neutral bidders with heterogenous beliefs. Since the upper bound is above the (linear) RNNE and concave, Overbidding and Decreasing Proportional Deviations are qualitatively consistent with risk-neutrality and (a degree of) strategic sophistication.

We regard non-equilibrium (but strategically sophisticated) bidding as a complementary explanation of experimental findings, which can be integrated with risk-aversion. We show in the Appendix how risk-aversion can be incorporated into our analysis.

Experimental evidence suggests a number of extensions to our results. First, our analysis so far does not offer an explanation of the asymmetry in subjects' deviations from RNNE (i.e. the tendency to bid *above* the RNNE), nor does it explain why very small bids are so rare for subjects with intermediate or high valuations. Second, it may be argued that rational bidders should form their beliefs about the competitors and make plans before they are told their valuation, and therefore *ex ante* rationalizability is a more appropriate solution concept in this context. Third, in most experimental settings there is an (explicit or implicit) minimum bid increment. We discuss these issues in the following subsections.

5.2. Lower Bounds and Upper Bounds

Theorems 4.2 and 4.8 show that imposing successively higher-order mutual belief in the assumptions that players are rational, and that positive bids win with positive probability, yields a decreasing sequence of tight upper bounds $\mathbf{B}(\cdot; k)$ on the set of best replies for every bidder type. However, arbitrarily small but positive bids are interim rationalizable.

In view of the experimental findings mentioned above, it may be interesting to *exogenously* specify a lower bound $\mathbf{L} \in \mathcal{B}_{-i}$ to players' bids, and investigate the consequences of the further assumption that (it is mutual belief that) players do not expect their opponents to bid below $\mathbf{L}$.

It turns out that Propositions 2.2 and 3.4 may be extended to characterize the set of best replies to beliefs consistent with the assumption that positive bids win with positive probability, and that opponents j's bids always lie between a given upper bound $\mathbf{B}_j$ and a given lower bound $\mathbf{L}_j$. The Appendix provides some details for the symmetric IPV case and in particular indicates necessary and sufficient conditions for the extension to be feasible.

It can be shown that the introduction of a lower bound only affects the derived upper bound for low types. Moreover, calculations for IPV models commonly used in experiments show that Proposition 2.2 and Theorem 4.2 do extend as required in a number of interesting cases; however, the differences between the derived upper bounds with and without a given lower bound are numerically small.

Finally, it is interesting to note that, even in a symmetric IPV setting, derived lower bounds may lie *below* the exogenously specified lower bound $\mathbf{L}$, at least for some types. For instance, this is always the case for type $v = 1$, provided the exogenous lower bound lies below the Nash equilibrium bidding function.

We again refer the reader to the Appendix for a discussion of these issues.

5.3. Ex Ante Rationalizability

Our analysis of *interim* rationalizability provides an upper bound on *ex-ante* rationalizable bidding functions, but we are not able to show the the upper bound is tight.[15]

However, some of our qualitative results also hold for ex ante rationalizability. In particular, we can show that the analog of Corollary 4.5 continues to hold.

Say that bid b is *ex-ante* rationalizable for type v_i if there is an *ex-ante* rationalizable bidding function $\mathbf{b}_i$ such that $\mathbf{b}_i(v_i) = b$. Then, in the symmetric IPV model, *for every type* v, *every bid* $b \in (0, \phi^{\mathbf{b}^{eq}}(v)]$ *is ex-ante rationalizable*, where $\mathbf{b}^{eq}$ is the symmetric equilibrium bidding function and $\phi^{\mathbf{b}^{eq}}(v) = v - \pi^*(v, \mathbf{b}^{eq}) > \mathbf{b}^{eq}(v)$ for all $v \in (0, 1)$.

Here is a sketch of the argument. In order to simplify the analysis, we assume that the bidders have arbitrary subjective beliefs about the tie-breaking rule. In particular, a player may "optimistically" believe that whenever he is one the high bidders he gets the object with probability one.[16] Consider the conjecture that every competitor j is using the following bidding function:

$$\mathbf{g}(v_j, \overline{v}) = \begin{cases} \mathbf{b}^{eq}(v_j), & \text{if } v_j < \overline{v}, \\ \phi^{\mathbf{b}^{eq}}(\overline{v}), & \text{if } v_j \geq \overline{v}. \end{cases}$$

It can be checked that, for each type v, $b = \mathbf{g}(v, \overline{v})$ is a best reply given the "optimistic" beliefs about tie breaking.[17] Therefore, for every "threshold" $\overline{v}$, $\mathbf{g}(\cdot, \overline{v})$ is an *ex-ante* ratio-

[15] As mentioned in the introduction, *interim* rationalizability allows bidders to hold different beliefs about their opponents' behavior depending on their own signal; *ex-ante* rationalizability, on the other hand, precludes this (assuming independent types). Hence, not every selection from the set of *interim* rationalizable bids for each type is an *ex-ante* rationalizable bid *function*—even if one restricts one's attention to increasing selections. Furthermore, we cannot rule out the possibility that this results in additional restrictions on the *bids* a given bidder type may place, if that bidder employs only *ex-ante* rationalizable bid functions.

[16] The argument can be adapted to the usual beliefs about tie-breaking.

[17] Clearly, no bid $b \in (\mathbf{b}^{eq}(\overline{v}), \phi^{\mathbf{b}^{eq}}(\overline{v})) \cup (\phi^{\mathbf{b}^{eq}}(\overline{v}), +\infty)$ can be a best reply for any type.

If $v < \overline{v}$, then bid $b = \mathbf{g}(v, \overline{v}) = \mathbf{b}^{eq}(v)$ is maximal in the set $[0, \mathbf{b}^{eq}(\overline{v})]$ and we must compare it with bid $\phi^{\mathbf{b}^{eq}}(\overline{v})$, which "optimistically" wins with probability one. Since $\phi^{\mathbf{b}^{eq}}$ is non-decreasing (Proposition 2.4 (4)), $(v - \phi^{\mathbf{b}^{eq}}(\overline{v})) \leq (v - \phi^{\mathbf{b}^{eq}}(v)) = \pi^*(v; \mathbf{b}^{eq}) = (v - \mathbf{b}^{eq}(v))P(\mathbf{b}^{eq}(v); \mathbf{b}^{eq})$. Thus $\mathbf{g}(v, \overline{v})$ is a best reply for v.

If $v \geq \overline{v}$, then $(v - \phi^{\mathbf{b}^{eq}}(\overline{v})) \geq (v - \phi^{\mathbf{b}^{eq}}(v)) = \pi^*(v; \mathbf{b}^{eq}) \geq \pi(b, v; \mathbf{b}^{eq}) = \pi(b, v; \mathbf{g}(\cdot, \overline{v}))$ for all $b < \mathbf{b}^{eq}(\overline{v}) \leq \mathbf{b}^{eq}(v)$.

nalizable bidding function. Now, fix any type v_i and bid $b \in (0, \phi^{\mathbf{b}^{eq}}(v)]$. Then the *ex-ante* rationalizable bidding function $\mathbf{b}_i$ such that $\mathbf{b}_i(v_i) = b$ is $\mathbf{g}(\cdot; (\phi^{\mathbf{b}^{eq}})^{-1}(b))$.

Also observe that in an IPV setting an *ex-ante* best reply must be a nondecreasing function. Thus, *ex-ante* rationalizability implies the restriction that the support of beliefs is included in the the set of nondecreasing functions. But our analysis of interim rationalizability shows that this restriction has no bite, because all the rationalizing conjectures used in our proofs are nondecreasing.

5.4. Discrete Bids

In most experimental settings, bids are discrete because there is a (possibly only implicit) minimum increment δ (e.g., a cent), so that the set of bids is $\{0, \delta, 2\delta, ..., k\delta, (k+1)\delta), ...\}$. Our analysis of rationalizable bids provides an acceptable approximation if the number of players is not very large and δ is small. This appears to be the case in most experiments, as well as in many real-life situations.

Dekel and Wolinsky [7] analyze the opposite case (a large population of players and a non-negligible minimum increment) in an IPV setting. They identify a non-decreasing lower bound to the set of rationalizable bids and show that, for any *fixed* minimum increment δ, as the number of bidders n gets large, this lower bound approches the equilibrium bidding function. Thus, rationalizability and equilibrium roughly coincide in large IPV auctions with a non-negligible minimum increment on bids.

6. Appendix

In the interest of conciseness, we omit an explicit proof of (parts (2)-(3) of) Theorem 2.2. We provide complete proofs for the interdependent-values model. To see how this relates to the IPV case, recall that the IPV auction *game* is a special case of the auction game with interdependent values. The only reason why Theorem 2.2 does not follow from Theorem 3.4 is that we exploit the private values assumption to strengthen parts (1) and (3) of the theorem. In particular, (1) and (3) hold for a larger set of beliefs, which includes non-monotonic ones. On the other hand, part (2) of Theorem 3.4 is implied by part (2) of Theorem 3.4. Thus, we only need to show how to adapt the proofs of part (1) and (3) to the IPV-large beliefs space case.

We conclude with some extensions of the basic setting.

6.1. Interdependent-Values Model

We begin by introducing additional notation used in the proofs.

6.1.1. Notation

Sets of bidders and signals. Let $N = \{1, ..., n\}$ denote the set of players. For any subset of players $J \subset N$, let $S_J = \prod_{j \in J} S_j$, and for simplicity define $S_{N \setminus \{i\}} = S_{-i}$. A generic element of S_J is denoted s_J. For any partition $\{K, L\}$ of the set of players J and any $s_K \in S_K$, $s_L \in S_L$, (s_K, s_L) is the element of S_J obtained from s_K and s_L.

For any nonempty subset $J \subset N$ of players, denote by F_J the marginal of c.d.f. F on S_J, and by F_{-J} its marginal on $S_{N \setminus J}$; finally, given a subset K of players such that $K \cap J = \emptyset$, denote by $F_{K|J}$ the conditional c.d.f on S_K given the signals of players $j \in J$.

Random variables and events. We continue to denote m-dimensional random vectors, defined as (Borel measurable) functions with domain S_J and range $\mathbb{R}^m$, with boldface letters. In particular, for any set of players J and random variables $\mathbf{b}_j$, $j \in J$, we let $\mathbf{b}_J$ denote the joint function (random vector) defined by $\mathbf{b}_J(s_J) = \{\mathbf{b}_j(s_j)\}_{j \in J}$. Similarly, we write $\mathbf{s}_j$ to denote the random signal observed by Player j (formally, $\mathbf{s}_j$ is the identity function on the probability space $([0, 1], F_j)$). As in the IPV case, degenerate random variables and tuples of identical degenerate random variables are represented by the corresponding real numbers. Also, for any set of players J and tuple of random variables $\mathbf{b}_J$, we denote by $\mathbf{b}_J^{\max}$ the scalar-valued random variable defined by the map $s_J \mapsto \max_{j \in J} \mathbf{b}_j(s_j)$.

Inequalities between random variables continue to be interpreted as almost sure inequalities. For example, $\mathbf{b}_j < \mathbf{B}_j$ if and only $\mathbf{b}_j(s_j) \geq \mathbf{B}_j(s_j)$ for a set of signals s_j of (Lebesgue) measure zero. Inequalities between tuples of random variables continue to be interpreted as coordinate-wise inequalities: $\mathbf{b}_{-i} < \mathbf{B}_{-i}$ if and only if $\mathbf{b}_j < \mathbf{B}_j$ for all $j \neq i$.

Events related to the signals of players in set J are represented by means of square brackets with an index J specifying that we refer to a subset of S_J. For example, let $K \subset J$; then $[\mathbf{b}_K < b]_J \subset S_J$ is the set of vectors of signals s_J such that $\mathbf{b}_j(s_j) < b$ for all $j \in K$. When $J = N \setminus \{i\}$ we suppress the index, as S_{-i} is the basic space of uncertainty from the point of view of Bidder i. Thus, for example, $[\forall j \in K, \mathbf{b}_j < b_j] \subset S_{-i}$.

Taking the possibility of ties into account, the expected payoff of bidding b given signal s_i and belief $\mu \in \Delta(\mathcal{B}_{-i})$ is

$$\pi_i(b, s_i; \mu) = \int_{\mathcal{B}_{-i}} \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{b}_{-i} < b] \Pr[\mathbf{b}_{-i} < b | s_i] \mu(d\mathbf{b}_{-i}) +$$

$$+ \sum_{\emptyset \neq J \subset N \setminus \{i\}} \frac{1}{1 + |J|} \int_{\mathcal{B}_{-i}} \mathrm{E}\left[\mathbf{v}_i - b | s_i, T(J, b; \mathbf{b}_{-i})\right] \Pr[T(J, b, \mathbf{b}_{-i}) | s_i] \mu(d\mathbf{b}_{-i}), \tag{6.1}$$

where $T(J, b; \mathbf{b}_{-i}) = [\mathbf{b}_{-J \cup \{i\}} < b] \cap [\forall j \in J, \mathbf{b}_j = b]$.

It is convenient to define a modfied version of the expected payoff function for a bidder with nondecreasong conjectures, which may assign positive probability to ties. For any tuple $\{\mathbf{B}_j\}_{j \neq i}$ of nondecreasing real-valued functions on $[0, 1]$, define

$$\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) = \Big(\mathrm{E}[\mathbf{v}_i | s_i, \mathbf{B}_{-i} < b] - b\Big) \Pr[\mathbf{B}_{-i} < b | s_i] + \tag{6.2}$$
$$+ \max\Big\{0, \Big(\mathrm{E}[\mathbf{v}_i | s_i, \mathbf{B}_{-i}^{\max} = b] - b\Big) \Pr[\mathbf{B}_{-i}^{\max} = b | s_i]\Big\}.$$

To avoid repeating tedious qualifications, for every random variable $\mathbf{h}$ and event F, if $\Pr[F] = 0$, we assume that $\mathrm{E}[\mathbf{h}|F]$ is such that $\mathrm{E}[\mathbf{h}|F] \Pr[F] = 0$.

Observe that, in the symmetric IPV model, for any valuation $v \in [0, 1]$ and bid $b \in [0, v]$, $\bar{\pi}_i(b, v; \mathbf{B}_{-i}) = (v - b) \Pr[\mathbf{B}_{-i} \leq b]$. We emphasize that this is *not* the case for $b > v$ (cf. the paragraph following the proof of Lemma 6.3).

6.1.2. Preliminary Results

Lemma 6.1. *For every tuple $\mathbf{B}_{-i}$ of nondecreasing bid functions and every signal $s_i \in [0, 1]$, the function $\bar{\pi}_i(\cdot, s_i; \mathbf{B}_{-i}) : \mathbb{R}_+ \to \mathbb{R}$ is upper semicontinuous. Therefore, $\sup_{b \geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ is attained. Finally, if $\mathbf{B}_{-i} \in \mathcal{M}_{-i}$, then $\bar{\pi}_i(\cdot, s_i; \mathbf{B}_{-i}) = \pi_i(\cdot, s_i; \mathbf{B}_{-i})$, and both functions are continuous.*

Proof. Fix $b \geq 0$ and consider a sequence $b^k \to b$ such that $\lim_{k \to \infty} \bar{\pi}_i(b^k, s_i; \mathbf{B}_{-i})$ exists; denote the latter by L. We must show that $L \leq \bar{\pi}_1(b, s_i; \mathbf{B}_{-i})$. Observe that, since $\sum_{k \geq 1} \Pr[\mathbf{B}_{-i}^{\max} = b^k | s_i] \leq 1$, it must be the case that $\Pr[\mathbf{B}_{-i}^{\max} = b^k | s_i] \to 0$.

Assume first that $b^k \uparrow b$. Note that the indicator functions of the events $[\mathbf{B}_{-i} < b^k]$ converge pointwise to the indicator function of $[\mathbf{B}_{-i} < b]$. Therefore, since $\mathbf{v}_i$ is bounded and so is the convergent sequence $\{b^k\}$, by Dominated Convergence (cf. Aliprantis and Border [1], p. 323) and the observation that the probability of ties vanishes as $k \to \infty$ we conclude that $L = (\mathrm{E}[\mathbf{v}_i | s_i, \mathbf{B}_{-i} < b] - b) \Pr[\mathbf{B}_{-i} < b | s_i]$, so $L \leq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$. Observe that the inequality can only be strict if $\Pr[\mathbf{B}_{-i}^{\max} = b | s_i] > 0$.

Assume next that $b^k \downarrow b$. Now the indicator functions of the events $[\mathbf{B}_{-i} < b^k]$ converge pointwise to the indicator function of $[\mathbf{B}_{-i} \leq b]$, so $L = (\mathrm{E}[\mathbf{v}_i | s_i, \mathbf{B}_{-i} \leq b] - b) \Pr[\mathbf{B}_{-i} \leq b | s_i] \leq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$. Again, the inequality can only be strict if $\Pr[\mathbf{B}_{-i}^{\max} = b | s_i] > 0$.

To complete the proof, consider an arbitrary convergent sequence $b^k \to b$ and an arbitrary subsequence $\{b^{k_n}\}$ such that $\lim_{n \to \infty} \bar{\pi}_i(b^{k_n}, s_i; \mathbf{B}_{-i}) \equiv L$ exists. If the subsequence is itself monotonic, then $L \leq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ by the above arguments. Otherwise, it must contain a monotonic sub-subsequence $\{b^{k_{n_m}}\}$ such that $b^{k_{n_m}} \to b$ and $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) \geq \lim_{m \to \infty} \bar{\pi}_i(b^{k_{n_m}}, s_i; \mathbf{B}_{-i}) = L$.

Finally, to prove the last claim note that if $\mathbf{B}_{-i} \in \mathcal{M}_{-i}$, then $\Pr[\mathbf{B}_{-i}^{\max} = b | s_i] = 0$ for all b. ■

<u>IPV Model:</u> The claims and their proofs hold *verbatim*.

We state another preliminary lemma, which states that the conjecture $\mathbf{B}_{-i}$ is "more pessimistic" than any belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$.

Lemma 6.2. *For every tuple $\mathbf{B}_{-i} > 0$ of nondecreasing bid functions, signal $s_i \in [0,1]$, and bid $b \geq 0$:*
(1) if $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) > 0$ and $\Pr[\mathbf{B}_{-i}^{\max} = b] = 0$, then $\pi_i(b, s_i; \mu) \geq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ for any $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$;
(2) if $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) \geq 0$, $b > 0$ and $\Pr[\mathbf{B}_{-i}^{\max} = b] > 0$, then $\mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B}_{-i}^{\max} = b] \geq 0$.
Therefore:
(3) $\inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})} \pi_i^*(s_i; \mu) \geq \max_{b \geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$.

To clarify the argument, we first prove Claim (1) for the case of a single competitor $-i = j$ and continuous, increasing conjectures $\mathbf{B}_j, \mathbf{b}_j$; in this case, $\bar{\pi}_i(\cdot, s_i, \mathbf{b}_j) = \pi_i(\cdot, s_i, \mathbf{b}_j)$, and similarly for $\mathbf{B}_j$. Note that

$$\begin{aligned} \pi(b, s_i; \mathbf{b}_j) &= \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B} < b] \Pr[\mathbf{B} < b | s_i] + \\ &+ \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{b}_j < b \leq \mathbf{B}] \cdot \Pr[\mathbf{b}_j < b \leq \mathbf{B} | s_i]. \end{aligned}$$

Since $\Pr[\mathbf{B} = b | s_i] = 0$, the first term in the r.h.s. equals $\pi(b, s_i; \mathbf{B})$. If $\Pr[\mathbf{b}_j < b \leq \mathbf{B} | s_i] = 0$ we are done. If $\Pr[\mathbf{b}_j < b \leq \mathbf{B} | s_i] > 0$, we must show that $\mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{b}_j < b \leq \mathbf{B}] \geq 0$. Since $\pi(b, s_i; \mathbf{B}_{-i}) > 0$, $b > \mathbf{B}^{-1}(0)$. Therefore $[\mathbf{B} < b]_j = [0, \mathbf{B}^{-1}(b))$ and $[\mathbf{b}_j < b \leq \mathbf{B}]_j = [\mathbf{B}^{-1}(b), \mathbf{b}_j^{-1}(b))$ (let $\mathbf{b}_j^{-1}(b) = 1$ if $b > \mathbf{b}_j(1)$). Hence, by Remark 4, $\mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{b}_j < b \leq \mathbf{B}] \geq \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B} < b] > 0$, where the latter inequality follows from $\pi(b, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B} < b] \Pr[\mathbf{B} < b | s_i] > 0$. This establishes the claim in the simple case.

Proof. (1) It is enough to prove the claim for a belief μ concentrated on a single profile $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$ such that $\mathbf{b}_{-i} < \mathbf{B}_{-i}$. Note that

$$\begin{aligned} \pi_i(b, s_i; \mathbf{b}_{-i}) &= \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B}_{-i} < b] \Pr[\mathbf{B}_{-i} < b | s_i] + \\ &+ \sum_{\emptyset \neq J \subset N \setminus \{i\}} \mathrm{E}[\mathbf{v}_i - b | s_i, \mathbf{B}_{N \setminus (J \cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J] \cdot \\ &\quad \cdot \Pr[\mathbf{B}_{N \setminus (J \cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J | s_i]. \end{aligned}$$

Since $\Pr[\mathbf{B}_{-i}^{\max} = b | s_i] = 0$, the first term in the r.h.s. equals $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$. If $\Pr[\mathbf{B}_{N \setminus (J \cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J | s_i] = 0$ for all nonempty $J \subset N \setminus \{i\}$, the proof of the claim is complete. Otherwise, for any nonempty $J \subset N \setminus \{i\}$ such that $\Pr[\mathbf{B}_{N \setminus (J \cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J | s_i] > 0$, we must show that $\mathrm{E}[\mathbf{v}_i - b | s_i, \ \mathbf{B}_{N \setminus (J \cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J] \geq 0$. Fix one such J and note that, for every $j \neq i$, the greatest lower bound of the set $[\mathbf{B}_j < b]_j$ is 0 (recall that $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) > 0$), and its least upper bound is $\sup\{s_j : \mathbf{B}_j(s_j) < b\}$; also, the g.l.b of $[\mathbf{b}_j < b \leq \mathbf{B}_j]_j$ is $\inf\{s_j : \mathbf{B}_j(s_j) \geq b\} \geq 0$ and its l.u.b. is $\sup\{s_j : \mathbf{b}_j(s_j) < b\} \geq \sup\{s_j : \mathbf{B}_j(s_j) < b\}$. [18]

[18] The above, detailed argument will be omitted henceforth.

Hence, by Remark 4, $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leq \mathbf{B}_J] \geq \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} < b]$; since $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) > 0$ and $\Pr[\mathbf{B}^{\max}_{-i} = b|s_i] = 0$ by assumption, we must have $\Pr[\mathbf{B}_{-i} < b|s_i] > 0$ and $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} < b] > 0$, which establishes the claim.

(2) If $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} < b] < 0$ the claim follows immediately because $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) \geq 0$. If instead $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} < b] \geq 0$, note that

$$\begin{aligned}\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}^{\max}_{-i} = b] &= \sum_{\emptyset \neq J \subset N\setminus\{i\}} \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b] \cdot \\ &\quad \cdot \Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b|s_i, \mathbf{B}^{\max}_{-i} = b].\end{aligned}$$

As above, Remark 4 implies that, for any nonempty $J \subset N \setminus \{i\}$ such that $\Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b|s_i, \mathbf{B}^{\max}_{-i} = b] > 0$, $\mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b] \geq \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i} < b] \geq 0$, which proves the claim.[19]

(3) Choose $b^* \in \arg\max_{b\geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$. The required inequality holds trivially if $\bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) = 0$, so assume $\bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) > 0$. In particular, $\mathbf{B}_{-i} > 0$ implies that $\Pr[\mathbf{B}^{\max}_{-i} = 0] = 0$, so we can assume that $b^* > 0$. Choose $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$. Assume that $\Pr[\mathbf{B}^{\max}_{-i} = b^*|s_i] = 0$; then, by Claim (1), $\pi^*_i(s_i; \mu) \geq \pi_i(b^*, s_i; \mu) \geq \bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i})$, and we are done. If instead $\Pr[\mathbf{B}^{\max}_{-i} = b^*|s_i] > 0$, consider a sequence $b^k \downarrow b^*$ such that $\Pr[\mathbf{B}^{\max}_{-i} = b^k|s_i] = 0$ for all k: this is possible because there can be at most countably many positive-probability ties. Since, for any $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$, $\pi_i(\cdot, s_i; \mathbf{b}_{-i})$ is continuous, $\pi_i(b^*, s_i; \mathbf{b}_{-i}) = \lim_{k\to\infty} \pi_i(b^k, s_i; \mathbf{b}_{-i}) \geq \lim_{k\to\infty} \bar{\pi}_i(b^k, s_i; \mathbf{B}_{-i}) = \bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i})$, where the inequality follows from Claim (1), and the second equality follows from Claim (2). Integrating with respect to μ, this implies that $\pi^*_i(s_i; \mu) \geq \pi_i(b^*, s_i; \mu) \geq \bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i})$. ■

IPV Model: (1) holds with $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$ in lieu of $\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$; moreover, the proof is simplified by observing that $\bar{\pi}_i(b, v; \mathbf{B}_{-i}) > 0$ implies $b \in (0, v)$. (2) merely states that $b \leq v$, which is an immediate consequence of $\bar{\pi}_i(b, v; \mathbf{B}_{-i}) \geq 0$ and $\Pr[\mathbf{B}_{-i} = b] > 0$. Finally, (3) holds with $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$ in lieu of $\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$: the proof requires no modification.

We next develop the machinery required to approximate $\mathbf{B}_{-i}$ with beliefs belonging to $\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$. Define an increasing and continuous map $\sigma : [0,1] \times (0,1) \to [0,1]$ by

$$\forall (x, \alpha) \in [0,1] \times (0,1], \quad \sigma(x, \alpha) = \begin{cases} \frac{1-\alpha}{\alpha} x & x \in [0, \alpha] \\ \left(1 - \frac{\alpha}{1-\alpha}\right) + \frac{\alpha}{1-\alpha} x & x \in (\alpha, 1] \end{cases}. \tag{6.3}$$

That is, for every $\alpha \in (0,1]$, the graph of $\sigma(\cdot, \alpha)$ is the piecewise linear function joining the origin with the point $(\alpha, 1-\alpha)$, and the latter with the point $(1,1)$. Each function $\sigma(\cdot, \alpha)$ is continuous and differentiable everywhere except at $x = \alpha$. Its inverse $\tau : [0,1] \times (0,1) \to [0,1]$ is given by

$$\forall (y, \alpha) \in [0,1] \times (0,1], \quad \tau(y, \alpha) = \begin{cases} \frac{\alpha}{1-\alpha} y & y \in [0, 1-\alpha] \\ \left(1 - \frac{1-\alpha}{\alpha}\right) + \frac{1-\alpha}{\alpha} y & y \in (1-\alpha, 1] \end{cases}, \tag{6.4}$$

[19] The claim may be false for $b = 0$, if $\Pr[\mathbf{B}^{\max}_{-i} = 0|s_i] > 0$.

i.e. the piecewise linear function joining the origin with the point $(1-\alpha,\alpha)$, and the latter with the point $(1,1)$. Each function τ is continuous and differentiable everywhere except at $y=1-\alpha$.

Note that, as $\alpha \downarrow 0$, $\sigma(\cdot,\alpha)$ converges pointwise on $(0,1]$ to the constant function 1; for notational convenience, we let $\sigma(x,0)=1$ for all $x\in[0,1]$. Similarly, τ converges pointwise to the constant function $0\equiv\tau(y,0)$.

Now, for any $j\in N$, non-negative real number $\alpha\geq 0$, and bid function $\mathbf{b}_j\in\mathcal{B}_j$, define the function $\mathbf{b}_j^{(\alpha)}\in\mathcal{B}_j$ by

$$\forall s_j\in[0,1],\quad \mathbf{b}_j^{(\alpha)}(s_j)=\sigma(s_j,\alpha)\mathbf{b}_j(s_j). \tag{6.5}$$

In conjunction with Lemma 6.2, the following Lemma implies that $\sup_{b\geq 0}\bar{\pi}_i(b,s_i;\mathbf{B}_{-i})$ is the least upper bound Bidder i may obtain by best-responding to beliefs in $\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})$.

Lemma 6.3. *Fix arbitrarily a tuple $\mathbf{B}_{-i}>0$ of nondecreasing bid functions and a signal $s_i\in[0,1]$.*
(1) For all $\alpha>0$, $\mathbf{B}_{-i}^{(\alpha)}\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})$.
Moreover, for every sequence $\alpha_k\downarrow 0$:
(2) $\mathbf{B}_j^{(\alpha_k)}\to\mathbf{B}_j$ a.s. pointwise;
(3) for every $b\geq 0$ and every sequence $b^k\to b$, $\lim_{k\to\infty}\pi_i(b^k,s_i;\mathbf{B}_{-i}^{(\alpha_k)})\leq\bar{\pi}_i(b,s_i;\mathbf{B}_{-i})$;
(4) $\pi_i^*(s_i;\mathbf{B}_{-i}^{(\alpha_k)})\to\max_{b\geq 0}\bar{\pi}_i(b,s_i;\mathbf{B}_{-i})$.

Proof. (1) Fix $j\neq i$. Note that, for any $\alpha>0$, $\mathbf{B}_j^{(\alpha)}<\mathbf{B}_j$ (with equality for $s_j=1$). Moreover, $s_j'>s_j$ implies $\sigma(s_j,\alpha)\mathbf{B}_j(s_j)<\sigma(s_j',\alpha)\mathbf{B}_j(s_j)\leq\sigma(s_j',\alpha)\mathbf{B}_j(s_j')$, so $\mathbf{B}_j^{(\alpha)}\in\mathcal{M}_j$. Finally, for any $b>0$ there exists $s_j>0$ such that $\sigma(s_j,\alpha)\mathbf{B}_j(1)<b$, and thus $\mathbf{B}_j^{(\alpha)}(s_j)<b$. Therefore, $\Pr[\mathbf{B}_{-i}^{(\alpha)}<b|s_i]>0$, and the first assertion of the Lemma is proved.

Now fix $b\geq 0$ and consider sequences $\alpha_k\downarrow 0$ and $b^k\to b$.

(2) Pointwise convergence on $(0,1]$ follows from the properties of the function σ.

(3) We begin by computing the a.s. pointwise limit of the sequence of indicator functions corresponding to the events $[\mathbf{B}_{-i}^{(\alpha_k)}\leq b]$.

Choose any $s_j>0$. If $\mathbf{B}_j(s_j)<b$, then for k large $\mathbf{B}_j^{(\alpha_k)}(s_i)<b^k$; similarly, if $\mathbf{B}_j(s_j)>b$, then for k large $\mathbf{B}_j^{(\alpha_k)}(s_i)>b^k$. Also note that, if $\mathbf{B}_j(0)>b$, then, for every $\epsilon>0$, there exists K such that $k\geq K$ implies $\mathbf{B}_j^{(\alpha_k)}(\epsilon)>b^k$, so $\bigcap_{k\geq 1}[\mathbf{B}_j^{(\alpha_k)}\leq b^k]_j=\{0\}$.

Finally, suppose $\mathbf{B}_j(s_j)=b$. Since $\mathbf{B}_j^{(\alpha_k)}$ is increasing for every k, it follows that, for every k, the set $L_j(k)=\{s_j\in[0,1]:\mathbf{B}_j(s_j)=b,\ \mathbf{B}_j^{(\alpha_k)}(s_j)>b^k\}$ satisfies

$$s_j\in L_j(k),\ \mathbf{B}_j(s_j')=b,\ s_j'>s_j\quad\Rightarrow\quad s_j'\in L_j(k).$$

Now define $\ell_j(k) = \min(1, \inf L_j(k))$ (where $\inf \emptyset = \infty$). Thus, if $\mathbf{B}_j(s_j) = b$, then $s_j > \ell_j(k)$ implies $\mathbf{B}_j^{(\alpha_k)}(s_j) > b^k$ (otherwise s_j would be a greater lower bound to $L_j(k)$ than $\ell_j(k)$), and $s_j < \ell_j(k)$ implies $\mathbf{B}_j^{(\alpha_k)}(s_j) \leq b^k$ (either because $L_j(k) = \emptyset$, or because otherwise there would be some other $s'_j \in (s_j, \ell_j(k))$ such that $\mathbf{B}_j(s'_j) = b$ and $\mathbf{B}_j^{(\alpha_k)}(s'_j) > b^k$).

Next, w.l.o.g. assume that $\ell_j(k) \to \ell_j$ for all $j \neq i$. Then the indicator functions of the events $[\mathbf{B}_{-i}^{(\alpha_k)} \leq b^k]$ converge a.s. pointwise to the indicator function of the event

$$[\mathbf{B}_{-i} < b] \cup \Big([\mathbf{B}_{-i}^{\max} = b] \cap [\mathbf{s}_{-i} \leq \ell_{-i}]\Big) \equiv [\mathbf{B}_{-i} < b] \cup T_{-i}(b).$$

By the Dominated Convergence theorem, this yields

$$\pi_i(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \to \int_{[\mathbf{B}_{-i} < b]} (\mathbf{v}_i - b) F_{-i|i}(ds_{-i}|s_i) + \int_{T_{-i}(b)} (\mathbf{v}_i - b) F_{-i|i}(ds_{-i}|s_i).$$

If $\Pr[T_{-i}(b)|s_i] = 0$, then $\lim_{k\to\infty} \pi_i(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \leq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ follows immediately. Otherwise, by Remark 4, $\mathrm{E}[\mathbf{v}_i - b|s_i, T_{-i}(b)] \leq \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i}^{\max} = b] \leq \max(0, \mathrm{E}[\mathbf{v}_i - b|s_i, \mathbf{B}_{-i}^{\max} = b])$, and Claim (3) follows.

(4) Consider a sequence $\alpha_k \downarrow 0$ and, for every k, choose $b^k \in \arg\max_{b \geq 0} \pi_i(b, s_i; \mathbf{B}_{-i}^{(\alpha_k)})$; note that the maximum is achieved because $\pi_i(\cdot, s_i; \mathbf{B}_{-i}^{(\alpha_k)})$ is continuous. Assume w.l.o.g. that the sequence of maximizers converges, and let $b^* = \lim_{k\to\infty} b^k$. Claim (3) implies that $\lim_{k\to\infty} \pi_i^*(s_i; \mathbf{B}_{-i}^{(\alpha_k)}) = \lim_{k\to\infty} \pi_i(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \leq \bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) \leq \max_{b\geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$. Since the reverse inequality follows from Lemma 6.2, (3), the proof is complete. ■

IPV Model: the proof of (1), (2) and (4) holds *verbatim*. Moreover, (3) holds in general, but for $b \in [0, v]$ the argument is simplified by noting that $\Pr[\mathbf{B}_{-i}^{(\alpha_k)} \leq b^k] \to \Pr[\mathbf{B}_{-i} < b] + \Pr[T_{-i}(b)] \leq \Pr[\mathbf{B}_{-i} \leq b]$.

6.1.3. Theorem 3.4

We are finally able to prove the main result of Section 3, Theorem 3.4. The key step is the proof of Claim (2). We sketch the main argument here (see also the discussion following Theorem 2.2).

For any bid $b^* < \phi_i^{\mathbf{B}_{-i}}(s_i)$, the justifying belief $\mathbf{g}_{-i}^{(\alpha)}$ has the qualitative features illustrated in Figure 2.1. Specifically, for every opponent $j \neq i$, $\mathbf{g}_j^{(\alpha)}$ is increasing and lies below $\mathbf{B}_j$; moreover, it approximates the upper bound $\mathbf{B}_j$ up to the point $\bar{s}_j$ where the latter crosses the bid b^*, and approximates b^* thereafter.

To verify the optimality of b^* given $\mathbf{g}_{-i}^{(\alpha)}$, we proceed in two steps. First, we argue that Bidder i's payoff function $\pi_i(b, s_i; \mathbf{g}_{-i}^{(\alpha)})$ is pointwise dominated by the "two-bidder, private-values" objective function $(\mathrm{E}[\mathbf{v}_i|s_i] - b) \Pr[\mathbf{g}_{-i}^{(\alpha)} \leq b|s_i]$. Moreover, the two functions share the same value $\mathrm{E}[\mathbf{v}_i|s_i] - b^*$ for $b = b^*$. We then prove that $b = b^*$ maximizes this auxiliary objective function among all bids b chosen by at least one opponent j with type $s_j \geq \bar{s}_j$—that is, a type for which $\mathbf{g}_j^{(\alpha)}(s_j)$ approximates b^*.

The second and concluding step entails verifying that, for all remaining bids b, Bidder i's payoff given $\mathbf{g}_{-i}^{(\alpha)}$ is close to her payoff given $\mathbf{B}_{-i}$, her "pessimistic" conjecture. By the definition of $\phi_i^{\mathbf{B}_{-i}}(s_i)$, this implies that no such bids can be profitable deviations from b^*.

Proof of Theorem 3.4. Note first that Lemmata 6.2 and 6.3 imply that part (3) of the Theorem is true.

To see that (1) holds, observe that, by Remark 3, for any $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$, $\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{b}_{-i} < b^*] \leq \mathrm{E}[\mathbf{v}_i|s_i]$. Hence, if $b^* > \mathrm{E}[\mathbf{v}_i|s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi_i^*(s_i;\mu)$ and $b^* \leq \mathrm{E}[\mathbf{v}_i|s_i]$, then, for any $\mu^* \in \Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})$,

$$\begin{aligned}
\pi_i(b^*, s_i; \mu^*) &= \int_{\mathcal{M}_{-i}} (\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{b}_{-i} < b^*] - b^*) \Pr[\mathbf{b}_{-i} < b^*|s_i]\mu^*(d\mathbf{b}_{-i}) \leq \\
&\leq \mathrm{E}[\mathbf{v}_i|s_i] - b^* < \\
&< \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi_i^*(s_i;\mu) \leq \\
&\leq \pi_i^*(s_i;\mu^*),
\end{aligned}$$

so b^* cannot be a best reply to $\mathbf{b}_{-i}$. On the other hand, if $b^* > \mathrm{E}[\mathbf{v}_i|s_i]$, then $\pi_i(b^*, s_i; \mu^*) < 0$ (recall that $b^* > 0$ and positive bids win with positive probability), so again b^* cannot be a best reply to μ^*.

We now prove (2).

Claim. $b^* < \min(\mathrm{E}[\mathbf{v}_i|s_i], \max_{j\neq i}\lim_{s_j\uparrow 1}\mathbf{B}_j(s_j))$.

To see this, note that $\mathrm{E}[\mathbf{v}_i|s_i] - b^* > \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi_i^*(s_i;\mu) = \sup_{b\geq 0}\bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ implies that $b^* < \mathrm{E}[\ \mathbf{v}_i|s_i]$. Moreover, suppose that $b^* \geq \max_{j\neq i}\lim_{s_j\uparrow 1}\mathbf{B}_j(s_j)$. If $\Pr[\mathbf{B}_{-i}^{\max} = b^*|s_i] = 0$, then $\bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i|s_i] - b^* > \sup_{b\geq 0}\bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$, a contradiction. If instead $\Pr[\mathbf{B}_{-i}^{\max} = b^*|s_i] > 0$ (so $b^* = \max_{j\neq i}\lim_{s_j\uparrow 1}\mathbf{B}_j(s_j)$), note that, by Remark 4, $\mathrm{E}[\mathbf{v}_i - b^*|s_i, \mathbf{B}_{-i}^{\max} = b^*] \geq \mathrm{E}[\mathbf{v}_i - b^*|s_i, \mathbf{B}_{-i} \leq b^*] = \mathrm{E}[\mathbf{v}_i|s_i] - b^* > 0$; hence, we again obtain $\bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i|s_i] - b^*$, which yields the same contradiction.

We now construct the conjecture to which b^* is a unique best response. First, for every $j \neq i$, define a bounded, nondecreasing and measurable function $\mathbf{g}_j : [0,1] \to \mathbb{R}_+$ by

$$\forall s_j \in [0,1], \quad \mathbf{g}_j(s_j) = \min\left(\mathbf{B}_j(s_j), b^*\right). \tag{6.6}$$

Correspondingly, for every $j \neq i$, define the quantities

$$\begin{aligned}
\bar{s}_j &= \min(1, \inf\{s_j \in [0,1] : \mathbf{B}_j(s_j) > b^*\}) \\
\bar{s}_{-i}^{\min} &= \min_{j\neq i}\bar{s}_j \\
\bar{b}_j^{(\alpha)} &= \sigma(\bar{s}_j, \alpha)b^*
\end{aligned} \tag{6.7}$$

For every $\alpha > 0$ and $b \geq 0$, by Remark 4 and the observation that $\mathbf{g}_{-i}^{(\alpha)} \in \mathcal{M}_{-i}$, $\pi_i(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) \leq (\mathrm{E}[\mathbf{v}_i - b|s_i]) \Pr[\mathbf{g}_{-i}^{(\alpha)} \leq b|s_i]$, with equality for $b = b^*$. Also note that $\bar{s}_{-i}^{\min} < 1$, for otherwise we would have $\mathbf{B}_j(s_j) \leq b^*$ for all $j \neq i$ and $s_j \in [0,1)$, and therefore $b^* \geq \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$.

Here and in the remainder of this proof, let j be such that $\bar{s}_j = \bar{s}_{-i}^{\min}$; then, letting $\bar{v}_i = \mathrm{E}[\mathbf{v}_i|s_i]$ for notational convenience,

$$\begin{aligned}
\Pr[\mathbf{g}_{-i}^{(\alpha)} \leq b|s_i] &\leq \Pr[\mathbf{g}_{j}^{(\alpha)} \leq b|s_i] = \\
&= \Pr[s_j : s_j < \bar{s}_j \text{ and } \mathbf{B}_j^{(\alpha)}(s_j) < b \text{ or } s_j > \bar{s}_j \text{ and } \sigma(s_j, \alpha) b^* \leq b|s_i] \leq \\
&\leq \Pr[s_j : s_j < \bar{s}_j \text{ or } s_j > \bar{s}_j \text{ and } \sigma(s_j, \alpha) b^* \leq b|s_i],
\end{aligned}$$

with equality for $b = b^*$. Moreover, for $b \in [\bar{b}_j^{(\alpha)}, b^*]$, $s_j < \bar{s}_j$ implies $\sigma(s_j, \alpha) b^* < \sigma(\bar{s}_j, \alpha) b^* = \bar{b}_j^{(\alpha)} \leq b$, so we also have

$$\Pr[\mathbf{g}_{-i}^{(\alpha)} \leq b|s_i] \leq \Pr[\sigma(s_j, \alpha) b^* \leq b|s_i] = F_{j|i}(\tau(\frac{b}{b^*}, \alpha)|s_i)$$

(recall that $\tau(\cdot, \alpha)$ is the inverse of $\sigma(\cdot, \alpha)$), and therefore $\pi_i(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) \leq (\bar{v}_i - b) F_{j|i}(\tau(\frac{b}{b^*}, \alpha)|s_i)$, with equality for $b = b^*$. Hence, if b^* is the unique maximizer of the r.h.s. in the region $[\bar{b}_j^{(\alpha)}, b^*]$ of bids, then b^* is also the unique maximizer of Player i's payoff in the same region. We now show that this is indeed the case.

It is expedient to represent bids as convex combinations of $\bar{b}_j^{(\alpha)} = \sigma(\bar{s}_j, \alpha) b^*$ and b^*. For every $\lambda \in [0,1]$, define

$$b(\lambda, \alpha) = [(1-\lambda)\sigma(\bar{s}_j, \alpha) + \lambda] b^*; \tag{6.8}$$

note that $\frac{\partial}{\partial \lambda} b(\lambda, \alpha) = [1 - \sigma(\bar{s}_j, \alpha)] b^*$. Also, for every $\lambda \in [0,1]$, define the quantity

$$s_j(\lambda, \alpha) = \tau\left(\frac{b(\lambda, \alpha)}{b^*}, \alpha\right) = \tau((1-\lambda)\sigma(\bar{s}_j, \alpha) + \lambda, \alpha), \tag{6.9}$$

which yields the type of Player j who bids $b(\lambda, \alpha)$, according to the bid function $\mathbf{g}_j^{(\alpha)}$, if $\mathbf{B}_j$ is right-continuous at $\bar{s}_j$.

Claim. The function $H : [0,1] \times (0,1] \to \mathbb{R}_+$ defined by

$$H(\lambda, \alpha) = (\bar{v}_i - b(\lambda)) F_{j|i}(s_j(\lambda, \alpha)|s_i) \tag{6.10}$$

has a unique maximum at $\lambda = 1$ for sufficiently small α.

<u>Case 1:</u> $\bar{s}_j > 0$. Consider $\alpha \in (0, \bar{s}_j)$. Then $\sigma(\bar{s}_j, \alpha) = (1 - \frac{\alpha}{1-\alpha}) + \frac{\alpha}{1-\alpha}\bar{s}_j$, so $(1-\lambda)\sigma(\bar{s}_j, \alpha) + \lambda = 1 - (1-\lambda)\frac{\alpha}{1-\alpha}(1 - \bar{s}_j) > 1 - \alpha$, where the inequality follows from the choice of α. Therefore, $s_j(\lambda, \alpha) = 1 - (1-\lambda)(1-\bar{s}_j) \equiv s_j(\lambda)$, which is independent of α. Also, $\frac{\partial}{\partial \lambda} s_j(\lambda) = (1 - \bar{s}_j)$ for $\lambda \in (0,1)$.

We conclude that

$$\frac{\partial}{\partial\lambda}H(\lambda,\alpha) = -[1-\sigma(\bar{s}_j,\alpha)]b^* F_{j|i}(s_j(\lambda)|s_i) + (\bar{v}_i - b(\lambda,\alpha)) f_{j|i}(s_j(\lambda)|s_i)(1-\bar{s}_j).$$

Now let $f^{\min}_{j|i}(s_i) = \min_{x\in[0,1]} f_{j|i}(x|s_i) > 0$: then, since $\bar{s}_j = \bar{s}^{\min}_{-i} < 1$, $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) \geq -[1-\sigma(\bar{s}_j,\alpha)]b^* + (\bar{v}_i - b^*) f^{\min}_{j|i}(s_i)(1-\bar{s}_j) \equiv h(\alpha)$. Since $\bar{s}_j > 0$, $\lim_{\alpha\to 0}\sigma(\bar{s}_j,\alpha) = 1$; this implies that, for α sufficiently small, $h(\alpha) > 0$, and therefore $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) > 0$. This implies that $\arg\max_{\lambda\in[0,1]} H(\lambda,\alpha) = \{1\}$.

<u>Case 2</u>: $\bar{s}_j = 0$. Then $\sigma(\bar{s}_j,\alpha) = 0$, $b(\lambda,\alpha) = \lambda b^*$, and $s_j(\lambda,\alpha) = \tau(\lambda,\alpha)$. We have two sub-cases.

First, for $0 < \lambda \leq 1-\alpha$, $\tau(\lambda,\alpha) = \frac{\alpha}{1-\alpha}\lambda$. Define $f^{\max}_{j|i}(s_i) = \max_{x\in[0,1]} f_{j|i}(x|s_i) < \infty$; then, since $\frac{\lambda}{1-\alpha} \leq 1$,

$$H(\lambda,\alpha) = (\bar{v}_i - \lambda b^*) F_{j|i}(\frac{\alpha}{1-\alpha}\lambda|s_i) \leq \bar{v}_i f^{\max}_{j|i}(s_i)\alpha.$$

For α sufficiently small, the r.h.s. is smaller than $H(1,\alpha) = \bar{v}_i - b^* > 0$. Hence, for all $\lambda \in (0, 1-\alpha]$, $H(1,\alpha) > H(\lambda,\alpha)$.

For $\lambda \in (1-\alpha, 1)$, $\tau(\lambda,\alpha) = (1 - \frac{1-\alpha}{\alpha}) + \frac{1-\alpha}{\alpha}\lambda$, so

$$\frac{\partial}{\partial\lambda}H(\lambda,\alpha) = -b^* F_{j|i}(\tau(\lambda,\alpha)|s_i) + (\bar{v}_i - \lambda b^*) f_{j|i}(\tau(\lambda,\alpha)|s_i)\frac{1-\alpha}{\alpha}.$$

Thus, with $f^{\min}_{j|i}(s_i)$ as above, $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) \geq -b^* + (\bar{v}_i - b^*) f^{\min}_{j|i}(s_i)\frac{1-\alpha}{\alpha}$, which is positive for α sufficiently small. Thus, $H(1,\alpha) > H(\lambda,\alpha)$ for α small and $\lambda \in (1-\alpha, 1)$.

Since $H(0,\alpha) = 0$, we can again conclude that $\arg\max_{\lambda\in[0,1]} H(\lambda,\alpha) = \{1\}$, and the proof of the claim is complete.

The claim implies that b^* is the unique maximizer of $\pi_i(b, s_i; \mathbf{g}^{(\alpha)}_{-i})$ in the region $[\bar{b}^{(\alpha)}_j, b^*]$. Clearly, every $b > b^*$ is dominated by b^* given $\mathbf{g}^{(\alpha)}_{-i}$.

If $\bar{s}^{\min}_j = 0$, then $\bar{b}^{(\alpha)}_j = 0$, so we are done. Otherwise, notice that $\bar{b}^{(\alpha)}_j = \sigma(\bar{s}_j,\alpha)b^* = \min_{k\neq i}\sigma(\bar{s}_k,\alpha)b^* = \min_{k\neq i}\bar{b}^{(\alpha)}_k$; this implies that $[\mathbf{g}^{(\alpha)}_{-i} \leq b] = [\mathbf{B}^{(\alpha)}_{-i} \leq b]$ for all $b < \bar{b}^{(\alpha)}_j$. To see this, suppose $\mathbf{g}^{(\alpha)}_{-i}(s_{-i}) \leq b$, and fix $k \neq i$. If $\mathbf{g}_k(s_k) = \mathbf{B}_k(s_k)$, then $\mathbf{B}^{(\alpha)}_k(s_k) \leq b$ follows immediately. Suppose instead that $\mathbf{g}_k(s_k) = b^*$. Note that, since we are assuming that $b < \bar{b}^{(\alpha)}_j \leq \bar{b}^{(\alpha)}_k$, $\sigma(s_k,\alpha)\mathbf{g}_k(s_k) = \sigma(s_k,\alpha)b^* \leq b < \bar{b}^{(\alpha)}_k = \sigma(\bar{s}_k,\alpha)b^*$. Then we get $\sigma(s_k,\alpha) < \sigma(\bar{s}_k,\alpha)$, and since $\sigma(\cdot,\alpha)$ is increasing, $s_k < \bar{s}_k$; but then $\mathbf{B}_k(s_k) \leq b^*$, so $\mathbf{g}_k(s_k) = \mathbf{B}_k(s_k)$, and both must be equal to b^*; moreover, $\mathbf{B}^{(\alpha)}_k(s_k) \leq b$. Conversely, if $\mathbf{B}^{(\alpha)}_{-i}(s_{-i}) \leq b$, then *a fortiori* $\mathbf{g}^{(\alpha)}_{-i}(s_{-i}) \leq b$, because, for all $k \neq i$, $\mathbf{g}^{(\alpha)}_k(s_k) = \sigma(s_k,\alpha)\mathbf{g}_k(s_k) \leq \sigma(s_k,\alpha)\mathbf{B}_k(s_k) = \mathbf{B}^{(\alpha)}_k(s_k)$.

Since $\mathbf{B}_{-i}^{(\alpha)} \in \mathcal{M}_{-i}$, this implies that, for $b \in [0, \bar{b}_j^{(\alpha)})$, $\pi_i(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) = \pi_i(b, s_i; \mathbf{B}_{-i}^{(\alpha)})$. For any such b, Lemma 6.3, Claim (3) implies that $\lim_{\alpha \downarrow 0} \pi_i(b, s_i; \mathbf{B}_{-i}^{(\alpha)}) \leq \bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) \leq \sup_{b' \geq 0} \bar{\pi}_i(b', s_i; \mathbf{B}_{-i})$. By assumption, the latter quantity is smaller than $\mathrm{E}[\mathbf{v}_i|s_i] - b^* = \pi_i(b^*, s_i; \mathbf{g}_{-i}^{(\alpha)})$; therefore, for sufficiently small α, and for all $b \in [0, \bar{b}_j^{(\alpha)})$, $\pi_i(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) < \pi_i(b^*, s_i; \mathbf{g}_{-i}^{(\alpha)})$. ■

<u>IPV Model</u>: (1) holds for all beliefs in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$, as shown in Section 2. Moreover, part (3) of the Theorem follows from the IPV versions of Lemmata 6.2 and 6.3. In (2), the assertion that $b^* < \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$ follows immediately from $b^* < v$. Moreover, in the symmetric IPV setting, $\pi_i(b, v; \mathbf{g}_{-i}^{(\alpha)}) = (v - b) P(b; \mathbf{g}_{-i}^{(\alpha)}) \leq (v - b) \Pr[\mathbf{g}_{-i}^{(\alpha)} \leq b]$, where the inequality follows from $b \leq b^* < v$; again, we have equality for $b = b^*$. The remainder of the proof (beginning with the choice of j such that $\bar{s}_j = \bar{s}_j^{\min}$) goes through *verbatim*, invoking the IPV version of Lemma 6.3, Claim (3).

6.1.4. Proof of Proposition 3.6

Note first that, for any $b^* \in \arg\max_{b \geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$, $\bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b^* | s_i, \mathbf{B}_{-i} \leq b^*] \Pr[\mathbf{B}_{-i} \leq b^* | s_i]$.

(1) Recall that, by Lemma 6.3 (1), $\mathbf{B}_{-i}^{(\alpha)} \in \Delta^+(\mathcal{M}_{-i}, \mathbf{B}_{-i})$; hence, by Lemma 6.1, the payoff function π_i is continuous in its first argument, so there exists a best reply $b \in \arg\max_{b' \geq 0} \pi_i(b', s_i; \mathbf{B}_{-i}^{(\alpha)})$; moreover, $b > 0$. Hence, $\phi_i^{\mathbf{B}_{-i}}(s_i) > 0$ for all $s_i \in [0, 1]$. The other inequality follows immediately from the first Claim in the proof of Theorem 3.4, part (2).

(2) Note that, since f is continuous, by Dominated Convergence, $\mathrm{E}[\mathbf{v}_i|s_i]$ and $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ are continuous in s_i; also, the latter function is upper semi-continuous in b by Lemma 6.1. By a version of the Maximum Theorem, the function $s_i \mapsto \max_{b \geq 0} \bar{\pi}_i(b, s_i; \mathbf{B}_{-i})$ is thus continuous, and this implies that $\phi_i^{\mathbf{B}_{-i}}$ is continuous.

(3) **Observation**. By Assumption 3.2 and Remark 4, for all $s_i, s_i' \in [0, 1]$ such that $s_i > s_i'$, $\mathrm{E}[\mathbf{v}_i|s_i] > \mathrm{E}[\mathbf{v}_i|s_i']$. Similarly, for every $b \geq 0$, if $\Pr[\mathbf{B}_{-i} \leq b|s_i] > 0$ (resp. $\Pr[\mathbf{B}_{-i}^{\max} > b|s_i] > 0$), then $\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i} \leq b]$ (resp. $\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} > b]$) is increasing in s_i.

Choose $s_i > 0$ and $b \in \arg\max_{y \geq 0} \bar{\pi}_i(y, s_i; \mathbf{B}_{-i})$.

Claim. Assume that $\Pr[\mathbf{B}_{-i} \leq b|s_i] < 1$, so $b < \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$. Then, for all $s_i' < s_i$, $\phi_i^{\mathbf{B}_{-i}}(s_i') < \phi_i^{\mathbf{B}_{-i}}(s_i)$.

To prove the claim, note first that, if $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) = 0$, then $\phi_i^{\mathbf{B}_{-i}}(s_i) = \mathrm{E}[\mathbf{v}_i|s_i] > \mathrm{E}[\mathbf{v}_i|s_i'] \geq \phi_i^{\mathbf{B}_{-i}}(s_i')$, where the strict inequality follows from the initial Observation, and the weak inequality from Part (1).

Thus, assume $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) > 0$, so in particular $\Pr[\mathbf{B}_{-i} \leq b|s_i] \in (0,1)$ because $\mathbf{B}_{-i} > 0$. Then $\mathrm{E}[\mathbf{v}_i|s_i] = \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i} \leq b] \Pr[\mathbf{B}_{-i} \leq b|s_i] + \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} > b] \Pr[\mathbf{B}_{-i}^{\max} > b|s_i]$ and

$$\phi_i^{\mathbf{B}_{-i}}(s_i) = \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} > b] \Pr[\mathbf{B}_{-i}^{\max} > b|s_i] + b \Pr[\mathbf{B}_{-i} \leq b|s_i].$$

Since $\bar{\pi}_i(b, s_i; \mathbf{B}_{-i}) > 0$, $b < \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i} \leq b] \leq \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} > b]$, where the second inequality follows from Remark 4 by an argument analogous to the one used in the proof of Lemma 6.2, (2). Also note that the indicator function of the event $[\mathbf{B}_{-i}^{\max} > b]$ is nondecreasing, because $\mathbf{B}_{-i}^{\max}$ is nondecreasing; therefore, for $s_i' < s_i$, $\Pr[\mathbf{B}_{-i}^{\max} > b|s_i] \geq \Pr[\mathbf{B}_{-i}^{\max} > b|s_i']$. Note that, since f is bounded away from zero, $\Pr[\mathbf{B}_{-i}^{\max} > b|s_i'] > 0$; therefore,

$$\begin{aligned}
\phi_i^{\mathbf{B}_{-i}}(s_i) &\geq \mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i}^{\max} > b] \Pr[\mathbf{B}_{-i}^{\max} > b|s_i'] + b \Pr[\mathbf{B}_{-i} \leq b|s_i'] > \\
&> \mathrm{E}[\mathbf{v}_i|s_i', \mathbf{B}_{-i}^{\max} > b] \Pr[\mathbf{B}_{-i}^{\max} > b|s_i'] + b \Pr[\mathbf{B}_{-i} \leq b|s_i'] = \\
&= \mathrm{E}[\mathbf{v}_i|s_i'] - \mathrm{E}[\mathbf{v}_i - b|s_i', \mathbf{B}_{-i} \leq b] \Pr[\mathbf{B}_{-i} \leq b|s_i'] \geq \\
&\geq \mathrm{E}[\mathbf{v}_i|s_i'] - \max_{x \geq 0} \bar{\pi}_i(x, s_i'; \mathbf{B}_{-i}) = \phi_i^{\mathbf{B}_{-i}}(s_i'),
\end{aligned}$$

where the strict inequality follows from the initial Observation, and the proof of the claim is complete.

Consider now signals s_i, s_i' such that $s_i' < s_i$ and $\phi_i^{\mathbf{B}_{-i}}(s_i') = \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$. Then it must be the case that $\arg\max_{x \geq 0} \bar{\pi}_i(x, s_i; \mathbf{B}_{-i}) = \{\max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)\}$: otherwise, by the preceding Claim, $s_i' < s_i$, $b \in \arg\max_{y \geq 0} \bar{\pi}_i(y, s_i; \mathbf{B}_{-i})$ and $b < \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$ would imply

$$\begin{aligned}
\phi_i^{\mathbf{B}_{-i}}(s_i') &< \phi_i^{\mathbf{B}_{-i}}(s_i) = \\
&= \mathrm{E}[\mathbf{v}_i|s_i] - \pi_i^*(s_i, \mathbf{B}_{-i}) \leq \\
&\leq \mathrm{E}[\mathbf{v}_i|s_i] - \bar{\pi}_i(\max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j), s_i; \mathbf{B}_{-i}) \leq \\
&\leq \mathrm{E}[\mathbf{v}_i|s_i] - \Big(\mathrm{E}[\mathbf{v}_i|s_i, \mathbf{B}_{-i} \leq \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)] - \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)\Big) = \\
&= \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j),
\end{aligned}$$

a contradiction. Thus, we must have $\phi_i^{\mathbf{B}_{-i}}(s_i) = \phi_i^{\mathbf{B}_{-i}}(s_i') = \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j)$. Let

$$s_i^{\mathbf{B}_{-i}} = \min\left(1, \inf\{s_i \in [0,1] : \max_{j \neq i} \lim_{s_j \uparrow 1} \mathbf{B}_j(s_j) \in \arg\max_{x \geq 0} \bar{\pi}_i(x, s_i; \mathbf{B}_{-i})\}\right).$$

Then $\phi_i^{\mathbf{B}_{-i}}$ is increasing on $[0, s_i^{\mathbf{B}_{-i}})$ and constant on $[s_i^{\mathbf{B}_{-i}}, 1]$. ■

6.2. Extensions

6.2.1. Risk-Aversion

We show how to extend our characterization of interim rationalizable bids to the following more general assumptions: Bidders' preferences over lotteries are characterized by v, their monetary valuation for the object and a vector θ of parameters which determine their risk aversion in terms of monetary gains/losses, so that a rational bidder of type (v, θ) with belief μ maximizes the expected utility $u(v-b;\theta)P(b;\mu)$. Let $\theta \in \Theta$ where Θ is a partially ordered set with a maximal element $\overline{\theta}$ and let u satisfy the following monotonicity condition:

$$\theta' > \theta'' \Rightarrow \forall x, A(x,\theta') > A(x,\theta''),$$

where

$$A(x,\theta) = -\frac{\frac{\partial^2 u(x,\theta)}{(\partial x)^2}}{\frac{\partial u(x,\theta)}{\partial x}},$$

$A(x,\overline{\theta}) > 0$ for all x, and $\frac{\partial u(x,\theta)}{\partial x} > 0$ for all x and θ. Furthermore we make the normalization assumption that $u(x) > 0$ for all $x > 0$.

Bidders beliefs satisfy the assumption that valuations are i.i.d with absolutely continuous marginal c.d.f. G and that the probability of winning with a positive bid is positive. Although bids now depend on both v and θ, we still consider upper bounds $\mathbf{B}$ on the bid of a player as a function of the valuation v. $\Delta^+(\mathcal{B}_{-i}, \mathbf{B}_{-i})$ is defined as in the symmetric IPV-model with risk-neutrality. Thus, for all b and all $\mu \in \Delta^+(\mathcal{B}_{-i}, \mathbf{B}_{-i})$, $P(b;\mu) \geq P(b;\mathbf{B}_{-i})$.

Let $\pi^*(v,\theta;\mu) = \sup_{0\leq b\leq v} u(v-b;\theta)P(b;\mu)$. First observe that *for any* $\theta \in \Theta$ *and any increasing upper bound* $\mathbf{B}$ *such that* $\mathbf{B}(0) = 0$ *the following holds*

$$\inf_{\mu\in\Delta^+(\mathcal{B}_{-i},\mathbf{B}_{-i})} \pi^*(v,\theta;\mu) = \pi^*(v,\theta;\mathbf{B}_{-i})$$

Proof. We show that $\pi^*(v,\theta;\mu) < \pi^*(v,\theta;\mathbf{B}_{-i})$ implies $\mu \notin \Delta^+(\mathcal{B}_{-i}, \mathbf{B}_{-i})$. This, in turn, implies that $\inf_{\mu\in\Delta^+(\mathcal{B}_{-i},\mathbf{B}_{-i})} \pi^*(v,\theta;\mu) \geq \pi^*(v,\theta;\mathbf{B}_{-i})$. The rest of the proof is analogous to the case of risk-neutrality.

Suppose that $\pi^*(v,\theta;\mu) < \pi^*(v,\theta;\mathbf{B}_{-i})$. Since $u(v-b)P(b;\mathbf{B}_{-i})$ is continuous in the bid b, $\pi^*(v,\theta;\mu) = \max_{0\leq b\leq v} u(v-b;\theta)P(b;\mathbf{B}_{-i})$. Let b^* be a maximizer of $u(v-b;\theta)P(b;\mathbf{B}_{-i})$. Then $b^* < v$ and

$$u(v-b^*;\theta)P(b;\mu) \leq \pi^*(v,\theta;\mu) < u(v-b^*;\theta)P(b^*;\mathbf{B}_{-i}).$$

Since $b^* < v$ implies $u(v-b^*,\theta) > 0$, we obtain $P(b;\mu) < P(b;\mathbf{B}_{-i})$, which in turn implies $\mu \notin \Delta^+(\mathcal{B}_{-i}, \mathbf{B}_{-i})$. ■

Now we can use the same methods as in the case of risk neutrality to show that *for all* $(v,\theta) \in [0,1]\times\Theta$ *and* $b \geq 0$, *if* $\pi^*(v,\theta;\mathbf{B}_{-i}) > u(v-b)$*, then* b *is not a best reply for* (v,θ) *to any belief* $\mu \in \Delta^+(\mathcal{B}_{-i},\mathbf{B}_{-i})$;

if $\pi^*(v,\theta;\mathbf{B}_{-i}) < u(v-b)$*, then* b *is a strict best reply for* (v,θ) *to some belief* $\mu \in \Delta^+(\mathcal{B}_{-i},\mathbf{B}_{-i})$.

We can deduce that the upper bound on the bid of type (v,θ) is

$$\varphi^{\mathbf{B}}(v,\theta) = u^{-1}\left(\pi^*(v,\theta;\mathbf{B}_{-i})\right)$$

and any bid in $(0,\varphi^{\mathbf{B}}(v,\theta))$ is rationalizable for (v,θ).

In order to find an upper bound independent of θ, we argue that $\varphi^{\mathbf{B}}(v,\theta)$ is nondecreasing in θ. The intuition is that the more risk-averse is the bidder (the higher θ), the less attractive is bid $b^* \in \arg\max_{0\leq b\leq v} u(v-b;\theta)P(b;\mathbf{B}_{-i})$ compared with a bid b that yields utility $u(v-b;\theta)$ with certainty. Thus, as θ increases, the inequality $\pi^*(v,\theta;\mathbf{B}_{-i}) < u(v-b;\theta)$ is satisfied for a wider set of bids.

We conclude that *for any given increasing upper bound* $\mathbf{B}$*, with* $\mathbf{B}(0)=0$*,the new derived upper bound on the bids of players with valuation vis* $\phi^{\mathbf{B}}(v) = \varphi^{\mathbf{B}}(v,\overline{\theta})$. Since beliefs about the opponents parameter θ are arbitrary, each bidder may believe that all the opponents are characterized by the highest risk-aversion and *the new set of beliefs is* $\Delta^+(\mathcal{B}_{-i},\phi^{\mathbf{B}}_{-i})$.

6.2.2. Lower Bounds

We focus on the symmetric IPV case for simplicity. The objective of this section is thus to extend Proposition 2.2 to allow for an exogenously specified, non-decreasing common lower bound $\mathbf{L}$ on bids.

First of all, since we wish to maintain the assumptions that positive bids win with positive probability, and that players never bid above the non-decreasing common upper bound $\mathbf{B}$, we necessarily require that $\mathbf{L}_{-i} = \{\mathbf{L},\ldots,\mathbf{L}\} \in \Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i})$.

The problem is then to characterize the set of best replies for any valuation v to beliefs $\mu \in \Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i},\mathbf{L}_{-i})$, where

$$\Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i},\mathbf{L}_{-i}) = \left\{\mathbf{b}_{-i} \in \mathcal{B}_{-i} : \forall j \neq i, \forall v_j, \mathbf{L}_j(v_j) < \mathbf{b}_j(v_j) < \mathbf{B}_j(v_j)\right\}.$$

We have the following result.

For every bid b *and valuation* v *:*

If $\pi^*(v,\mathbf{B}_{-i}) > (v-b)P(b;\mathbf{L}_{-i})$ *then* b *cannot be a best reply for* v *to any belief* $\mu \in \Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i},\mathbf{L}_{-i})$.

If $\pi^*(v,\mathbf{B}_{-i}) < (v-b)P(b;\mathbf{L}_{-i})$ *then there is a belief* $\mu \in \Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i},\mathbf{L}_{-i})$ *such that* b *is a best reply to* μ *for valuation* v*.*

In the presence of a lower bound, it is still the case that $\mathbf{B}_{-i}$ is the "most pessimistic" conjecture a player may hold. On the other hand, $\mathbf{L}_{-i}$ is her "most optimistic" conjecture:

it maximizes the probability of winning with any bid. Thus, the interpretation of the above inequalities is as in Proposition 2.2. In particular, a bid cannot be a best reply if it yields a lower expected payoff against the most optimistic conjecture than the bidder's maxmin payoff.

If the equation in b

$$\pi^*(v, \mathbf{B}_{-i}) = (v - b)P(b; \mathbf{L}_{-i})$$

admits only two solutions non-negative solutions, $\mathbf{L}^*(v)$ and $\mathbf{B}^*(v)$, then the latter quantities represent the derived bounds: given the assumed restriction on beliefs, the interior of set of justifiable bids is the interval $(\mathbf{L}^*(v), \mathbf{B}^*(v))$. Otherwise, the above result still characterizes the set of justifiable bids, but the latter is not an interval.

Note that if $v - \pi^*(v, \mathbf{B}_{-i}) = \varphi^{\mathbf{B}_{-i}}(v) > \mathbf{L}(1)$ then we obtain the same derived upper bound as in the standard case where the lower bound is 0. Hence, the introduction of a lower bound only affects the upper bound for low types. [20]

Observe also that if the above condition applies to type $v = 1$, then the derived lower bound for that type at least will be at most equal to the exogenous lower bound. For example, this will be the case if $\mathbf{L} < \mathbf{b}^{eq}$ and $\mathbf{B}(1) = \mathbf{B}(1; \infty)$. If $\mathbf{L}$ is increasing, the derived lower bound for type $v = 1$ will actually lie below $\mathbf{L}(1)$, and the same will be true for an interval of types to the left of $v = 1$. Thus, as claimed in the text, some types may wish to bid below the common lower bound, even if they believe that their opponents don't (and that this is mutual belief).

Sketch of the proof. Suppose that $\pi^*(v, \mathbf{B}_{-i}) > (v - b)P(b; \mathbf{L}_{-i})$ and consider a belief $\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i}, \mathbf{L}_{-i})$. If $b \geq v$ then b is dominated and there is nothing to prove. Suppose that $b < v$. Then

$$\pi(b, v; \mu) \leq (v - b)P(b; \mathbf{L}_{-i}) < \pi^*(v, \mathbf{B}_{-i}) \leq \pi^*(v; \mu)$$

where the first and last inequalities follow from the fact that $\mathbf{L}_{-i}$ is the most optimistic conjecture and $\mathbf{B}_{-i}$ is the most pessimistic conjecture. Thus b is not a best reply to μ.

Now suppose that $\pi^*(v, \mathbf{B}_{-i}) < (v - b)P(b; \mathbf{L}_{-i})$. We show that b is a best reply to a conjecture $\mathbf{g}^\varepsilon$ whose graph is ε-close to the graph of

$$\mathbf{g}(v) = \begin{cases} \mathbf{B}(v), & \text{if } v < \mathbf{B}^{-1}(b), \text{ or } v \geq \mathbf{L}^{-1}(b), \\ b, & \text{if } \mathbf{B}^{-1}(b) \leq v < \mathbf{L}^{-1}(b). \end{cases}$$

The conjecture $\mathbf{g}^\varepsilon$ is a strictly increasing version of $\mathbf{g}$ such that $P(b; \mathbf{g}^\varepsilon) = P(b; \mathbf{g})$ and $\mathbf{L}(\mathbf{v}) < \mathbf{g}^\varepsilon(v) < \mathbf{B}(v)$ for all v. Since $P(b; \mathbf{g}) = P(b; \mathbf{g}) = P(b; \mathbf{L}_{-i})$, bid b yields $(v - b)P(b; \mathbf{L}_{-i})$. By construction, there is a local maximum at b. The best Player i can do

[20] Note that $\pi^*(v; \mathbf{B}_{-i}) \geq 0$; the above inequality implies that $\pi^*(v; \mathbf{B}_{-i}) < (v - b)P(b; \mathbf{L}_{-i})$ for b in a right-hand neighborhood of $\mathbf{L}(1)$. Therefore the r.h.s. is positive for all $b \in (\mathbf{L}(1), \varphi^{\mathbf{B}_{-i}}(v))$.

against $\mathbf{g}^{\varepsilon}$ when he does not choose b is to use bids that make his expected payoff close to $\pi(\cdot, v; \mathbf{B})$. The maximum payoff of such bids is approximately $\pi^*(v; \mathbf{B}_{-i}) < (v-b)P(b; \mathbf{L}_{-i})$. Thus b is a best reply.

Example. Consider the symmetric IPV model with the uniform distribution and two bidders. Let $\mathbf{L}(v) = \mathbf{L}(v; 0) = \alpha v$, $\alpha < \frac{1}{2}$ (we do not want to exclude the Nash conjecture). Then $P(b; \mathbf{L}) = \min\{b/\alpha, 1\}$.

We now attempt to mimic the construction in Theorem 4.2.

Starting from the (high) upper bound $\mathbf{B}(v; 0) = 1$ and the (exogenous) lower bound $\mathbf{L}(v; 0) = \mathbf{L}(v)$, we get $\pi^*(v; \mathbf{B}(\cdot; 0)) = 0$; the inequality $\pi^*(v; \mathbf{B}(\cdot; 0)) < (v-b)P(b; \mathbf{L}(\cdot; 0))$ then yields the new upper bound $\mathbf{B}(v; 1) = v$. The implied lower bound is $\mathbf{L}(v; 1) = 0 \leq \mathbf{L}(v)$: note that it lies entirely below the exogenous lower bound. This exhausts the implications of Step 1 of the iterative procedure.

Now consider Step 2. By the preceding analysis, $\mathbf{L}(v; 1)$ is the lower bound on actual bids if (a) each player is rational and (b) her beliefs belong to $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}(\cdot, 0)\ , \mathbf{L})$. If a player's beliefs are in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}(\cdot, 0), \mathbf{L})$, and moreover she is certain of (a) and (b), then her beliefs must be in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}(\cdot, 1), \max\{\mathbf{L}, \mathbf{L}(\cdot; 1)\})$; if $\mathbf{L} \leq \mathbf{L}(\cdot; 1)$ the latter set is $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}(\cdot, 1)\ , \mathbf{L})$. In other words, the Step-1 lower bound can be disregarded at Step 0.

Now the best reply to $\mathbf{B}(v; 1) = v$ is $\mathbf{b}^*(v) = \frac{v}{2}$ and $\pi^*(; \mathbf{B}(\cdot; 1)) = \frac{v^2}{4}$. Solving the inequality

$$\frac{v^2}{4} < (v-b)\min\{b/\alpha, 1\}$$

we get

$$\frac{v(1-\sqrt{1-\alpha})}{2} < b < \min\left\{\frac{v(1+\sqrt{1-\alpha})}{2}, v - \frac{v^2}{4}\right\}.$$

Observe that the new lower bound on bids is still lower than the conjectured lower bound: $\frac{(1-\sqrt{1-\alpha})}{2}v < \alpha v$ for $\alpha < \frac{3}{4}$, and we are assuming $\alpha < \frac{1}{2}$).

The new upper bound is $\mathbf{B}(v; 2) = \min\left\{\frac{v(1+\sqrt{1-\alpha})}{2}, v - \frac{v^2}{4}\right\}$. This upper bound is different from the one on best replies to beliefs in $\Delta^+(\mathcal{B}_{-i}; \mathbf{B}(\cdot, 1))$ only for low types (e.g., for $\alpha = \frac{1}{4}$ the threshold is $v = 2 - \sqrt{3} \approx 0.27$). Moreover, the difference between the two derived upper bounds is numerically small (for $\alpha = \frac{1}{4}$, it is maximal at $v = \frac{2-\sqrt{3}}{2} \approx 0.13$, where it equals approximately 0.0045, i.e. about 3.4% of the valuation.)

6.2.3. A Note on Non-Decreasing Conjectures

Our results for the general interdependent values model rest on the assumption that bidders' beliefs are concentrated on the set of increasing bid functions. In particular, the proof of Part 1 of Lemma 6.2 exploits the fact that ties occur with probability zero, given a belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$ (note, however, that the upper bound $\mathbf{B}_{-i}$ itself *may* be non-decreasing).

One can allow for non-decreasing conjectures, and hence for the possibility of positive-probability ties. However, one must verify or ensure that the following regularity condition holds: for every non-decreasing conjecture $\mathbf{b}_{-i}$, no bid b such that $\Pr[\mathbf{b}_{-i}^{\max} = b|s_i] > 0$ is a best reply for Bidder i.

The condition is always met in private-values auctions: if the bid b ties with positive probability and Bidder i's valuation is v, then Bidder i will prefer to bid below b if $v < b$, and bid slightly above b if $v > b$. It is also met in general auctions with interdependent values, provided there are only two bidders; the argument is similar.

However, in settings with more than two bidders, the situation is complicated by the fact that the bid b may induce a tie with different *subsets* of opponents with positive probability. Thus, the expected surplus of Bidder i conditional upon a tie is a sum of products of three terms: the surplus conditional upon a tie with a specific subset of opponents, the probability of a tie with those opponents, and the probability of ultimately receiving the object given a tie with those opponents (cf. Equation 6.1).

Invoking affiliation and monotonicity of the valuation functions, we have been able to show that best replies never induce positive-probability ties in general interdependent-values setting with three bidders. However, we have been unable to rule out the possibility that this condition might fail if there are more than three bidders, although we are unaware of examples in which it does fail.

The problem may be circumvented by adopting a different tie-breaking rule; for instance, assume that each high bidder receives the object with probability $\frac{1}{|N|}$, so the seller may end up keeping the object with positive probability. Alternatively, one may impose restrictions on bidders' beliefs; for instance, assume that players expect to tie with *at most* two other opponents. Further details are available from the authors upon request.

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