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Marco Maffezzoli and Federico Trionfetti

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Approximation Methods: an Application to the Core-Periphery Model

Marco Maffezzoli¹

Federico Trionfetti²

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¹Istituto di Economia Politica, Università Bocconi, Via Salasco 3/5, 20136 Milano, Italy. Telephone: +39 0258363314; fax: +39 0258363309. E-mail: marco.maffezzoli@uni-bocconi.it.

²King's College London and CEPPI Paris. Mailing address: The Management Centre, King's College London, 150 Stamford Street, London SE1 9NN. United Kingdom. Telephone and Fax: +44-20-78484252. E-mail: Federico.Trionfetti@kcl.ac.uk.

Abstract

A feature of new economic geography model is their mathematical intractability. This intractability results from the fact that the functional relationship between the indirect utility differential and the state variable cannot be found explicitly. We illustrate three methods that can be utilized to approximate the unknown function. These methods are simple and give a remarkable improvement in the precision of approximation with respect to the commonly utilized Lagrange approximation. Precision of approximation is important in models that feature catastrophic behavior. We apply these methods to the core-periphery model. Naturally, they can be applied to all cases of unknown functional relationships.

Keywords: projection methods, spatial models, economic geography.

JEL Codes: R1, C63.

1 Introduction

The voluminous literature on the geographical distribution of economic activity has identified many potential sources of agglomeration economies (see Fujita and Thisse, 2002). Among the recent contributions, a stream of literature which followed the seminal paper by Krugman (1991a) has highlighted demand externalities as a source of agglomeration forces. Although the structure of these models is simple, the functional relationships between endogenous and exogenous variables cannot be found explicitly. There are two principal reasons for this analytical intractability. The first one is due to the non linearity of the dynamic system and the second one is due to the non linearity of the market equilibrium equations from which prices and wages obtain. This is why the exploration of these models has progressed by means of a mix of numerical methods and mathematical analysis. Very recently, two papers by Baldwin (2001) and Ottaviano (2001) have made substantial advancements in the analytical and numerical exploration of these models. Our paper complements these two. We illustrate a selection of simple and efficient methods for the approximation of unknown functional relationships. We show that a great deal of improvement in the precision of approximation can be gained by using these methods.

Precision of approximation is always desirable but it becomes of particular importance in models of the new economic geography. The reason is that this class of models exhibits catastrophic behavior and multiple equilibria. That is, a small change in the value of a parameter of the model may cause dramatic changes in the long run equilibria. Further, for any given value of parameters, there may be more than one stable equilibrium. The implication of this is that a small approximation error in the key functional relationship may lead to large errors in the prediction of outcomes. Precision becomes even more important when the models are brought into policy analysis. When analyzing policy matters, such as tax competition for instance, the addition of policy variable makes the models even less tractable. The use of simple and efficient approximation methods gives more freedom of modeling because it removes the constraint of having to arrive to a reduced form. There are also other advantages. By utilizing a reliable approximation method it is possible to obtain accurate information on the transitional dynamics. This is typically important in policy analysis where the speed of adjustment and the time path of endogenous variable often constitute a criterion for the evaluation of different policies.

The application of numerical methods to economics is in itself a vast field of research (Judd, 1998). We illustrate three simple numerical methods: Chebyshev interpolation, orthogonal collocation and Galerkin's method. They can be utilized for the exploration of highly non-linear models and, we believe, are particularly suitable for the exploration of new economic geography model.

In section 2 we review the related literature. In section 3 we illustrate Chebyshev's

interpolation method and we compare it to Lagrange interpolation. In section 4 we illustrate two simple projection methods: Orthogonal Collocation and Galerkin. In section 5 we apply these methods to the Core-Periphery model. Conclusions are drawn in section 6.

2 Review of the Literature

The literature on the new economic geography has grown large, useful reviews of this literature include Fujita and Thisse (1996, 1997), Brülhart (1997), and Ottaviano and Puga (1997) while the state of the art in the new economic geography as of the end of last century is neatly expounded in Fujita, Krugman and Venables (1999). The review that follows presents the literature from the point of view of the effort made to overcome the mathematical intractability of the core-periphery model and of its variants. It therefore leave aside other contributions that, though important in the literature, were not directly aimed to this purpose.

In reviewing the literature and in the rest of the paper we take the core-periphery model as the natural example. Naturally the approximation methods we illustrate in this paper apply to all its variants.

The dynamic mechanism of the core-periphery model rests on the two-way interaction between the international distribution of the mobile factor of production (the state variable) and the indirect utility differential between countries: the indirect utility differential at any point in time is function of the state variable which, in turn, evolves over time in response to the indirect utility differential. The indirect utility differential depends on real wages which are determined by market-clearing conditions in the product market. The mathematical intractability of the model results from two sources. The first source is in that the market equilibrium equations cannot be solved explicitly for the wages as function of the state variable. The second source is in that the dynamic system is non linear. The second source of intractability is particularly problematic when it is assumed that the mobile factor (*workers*) have forward looking expectations. Indeed, with forward looking expectations the “flash-diagram” technique does not give us enough informations about the dynamics of the system. As pointed out by Baldwin (2001) the literature has followed two approaches to deal with these problems. A first approach - known as the informal analysis approach - uses a mix of numerical and mathematical techniques. This approach has been utilized in particular in the exploration of the static expectations version of the core-periphery model. A second approach has devoted effort to modify the original structure of the core-periphery model in order arrive to a linear system of differential equations. This approach has been followed in particular when dealing with forward

looking expectations. We review these two approaches before coming to the contribution of our paper.

The informal approach is utilized in the very first paper of the new economic geography, where Krugman (1991a) deals with the non linearity of the market equilibrium equations by solving them numerically in correspondence of ten equidistant values of the state variable. These numerical solutions give the wage differential between countries in correspondence of the ten chosen values of the state variable. By connecting all these points with straight segments he draws the phase line for the myopic expectations version of the model. Given the non explicitable form of the market equilibrium equations, the analytical exploration of the model is necessarily confined to the symmetric equilibrium and to the core-periphery configurations. This mix of numerical solutions for the wage differential and mathematical analysis of local stability is pushed further by Puga (1999) in a rather richer set up and it is also adopted in Fujita, Krugman and Venables (1999). This method makes a parsimonious use of numerical analysis since it does not require to approximate the functional relationship between the indirect utilities and the state variable. Qualitative informations about the dynamics of the model can nevertheless be gleaned. However, it leaves us with virtually no knowledge of the transitional dynamics and with less than satisfactory knowledge of the dynamics when forward-looking expectations are considered. Part of the research that followed tried to overcome these problems by modifying the structure of the core-periphery model in such a way to arrive to a linear system of differential equations.

A first linear model is proposed by Krugman (1991b). The focus of his paper is on the multiplicity of agglomeration paths and on the role of history and expectations in determining the long run outcomes.¹ Dispersion forces are ruled out by assumption. The agglomeration force is introduced by postulating a linear relationship between the real wage differential and the state variable. The relationship is such that the wage rate depends positively on population, thus the region with the largest proportion of labour has also the highest real wage. The resulting dynamics is characterized by a system of two linear differential equations. The system is wholly unstable and, therefore, agglomeration emerges in the long-run if the initial equilibrium is perturbed. Further, there is a domain around the symmetric equilibrium - called the overlap region - where for any given value of the state variable there correspond two rational-expectation-consistent transition path, each leading to a different long run spatial outcome. This means that there is an indeterminacy about the long-run outcome that is not dependent on the initial

¹Two papers with similar focus are Matsuyama (1991), whose agglomeration forces rely upon technological externalities, and Galí (1995) whose model features monopolistic competition and internal increasing returns. Fukao and Bénabou (1993) show that a technical mistake in Krugman (1991b) is inconsequential on the essence of the results.

condition. A limitation of this model is that the relationship between wages and the state variable, although it builds on a number of intuitively appealing rationales, is not explicitly microfounded. The replacement of microfoundations with an ad hoc linear relationship was the modeling sacrifice that made possible to arrive to a tractable system of differential equations.

Ottaviano (1999) improves on this by introducing two simplifications to the core-periphery model that lead to a linear relationship between indirect utility differentials and the state variable without having to sacrifice the microfoundations of the model. His first simplification is to assume that the intertemporal elasticity of substitution just offsets the elasticity of substitution among varieties. The second simplification is in the intersectoral mobility of labor. He retains the assumption made in the core-periphery model that international mobility is costly but he abandons the assumption of intersectoral immobility of labour and replace it with instantaneous mobility. This second simplification leads to factor price equalization, which then leads to a linear relationship between the indirect utility differential and the state variable. These two clever simplifications enable him to arrive to a system of two linear differential equation in a model that is fully microfounded in the spirit of pecuniary externalities. His findings confirm that the qualitative results of Krugman (1991b) apply to a model with microfoundations.

Ottaviano, Tabuchi and Thisse (2001) take a more radical departure from the monopolistic competitive market structure assumed in the core-periphery model. They propose a model with linear demand functions where, like in monopolistic competition, there is no strategic interaction between firms but, unlike the monopolistic competition, firms respond to information coming from the market as a whole. The linearity of demand functions leads to a system of linear differential equations whose dynamic properties can again be analyzed by use of mathematical methods. This model, which is fully microfounded, also features the overlap region and the indeterminacy of the long-run equilibrium.

The last two papers have the merit of being fully microfounded, of featuring both agglomeration and dispersion forces, and of being analytically solvable. However, they miss some of the richness of the core-periphery model in the fact that the break-point and the sustain-point coincide. Thus, the coexistence of a stable symmetric equilibrium with stable agglomeration outcomes is ruled out. In this respect the literature has made two further advancement due to Baldwin (2001) and Ottaviano (2001). Both papers are fully microfounded, they feature agglomeration and dispersion forces, they deal with myopic and forward looking expectations and, moreover, the break and sustain point are distinct.

Baldwin (2001) first advancement concerns the analysis of the core-periphery model with myopic expectations. He uses the Liapunov's direct method to analyze the global dynamic stability of the core periphery model. The challenge in the use of this mathe-

mathematical method is to find a Liapunov function that satisfies certain properties and that is applicable to the model. Baldwin's elegant overtaking of this difficulty allows him to perform a global stability analysis of the myopic version of the core-periphery model. He shows that the results of the informal methods utilized in the earlier literature are confirmed by the formal analysis. The second thrust of Baldwin's paper is the use of numerical methods to find the saddle paths of the system when forward-looking expectations are considered. In order to find the saddle paths he exploits the fact that the stable arm of a saddle-path-stable system becomes the unstable arm of the system when time is run backward. Thus, in reverse time, starting from any point off the equilibrium, the system will converge towards the stable arm. By utilizing numerical methods in reverse time he is then able to characterize the transitional dynamics of the non linear system. Naturally, before proceeding to the numerical exploration of the model, the functional relationship between indirect utilities and the state variable is needed. Since this function cannot be found explicitly he uses Lagrange interpolation to approximate it in two steps. First, he solves numerically the market equilibrium equations for the real wages in correspondence of 25 values of the state variable (like Krugman 1991a). Then, a polynomial (of order 17th) is fitted to approximate the unknown function in correspondence of the 25 numerical solutions.

Ottaviano (2001) makes a further analytical improvement. With a simple and elegant modification to the core-periphery model he obtains an explicit expression for the indirect utility differential. His only deviation from the core-periphery model is in assuming that the marginal component of the cost function is undertaken in terms of the immobile factor instead of the mobile factor. This suffices to yield an explicit, though rather complex, expression for indirect utility differential as functions of the state variable. The resulting model retains all the richness of the core-periphery model, especially in that a stable equilibrium may coexist symmetric equilibrium with stable core-periphery outcomes. Besides its intricacy, the analytical expression for the indirect utility differential allows, at least in principle, to push the analytical exploration of the model to its utmost. However, given the non integrability of the indirect utility differential the resulting dynamic system is obviously non-solvable in both expectational regimes. This poses again the same problems of intractability. Ottaviano's further contribution is to deal with this problem in an analytical way. He notes that in the presence of multiple equilibria the linearization of the model is unsatisfactory even for local stability analysis. Therefore, he uses the perturbation method (introduced in economics by Matsuyama, 1991) for global stability analysis. This is certainly an advancement. He notes, however, that this method can be utilized only around specific values of parameters (rate of time preference equal zero in his case). The exploration of the model far from particular values of parameters requires

once again the use of numerical methods.

Our paper contributes to this literature by introducing three simple and efficient approximation methods: Chebyshev interpolation, Orthogonal Collocation and Galerkin. Like Baldwin (2001), we use numerical methods to compute the saddle paths and the transitional dynamics. The main difference with his paper is in that we approximate the indirect utility differential by use of more efficient methods than Lagrange interpolation. These methods guarantee the highest accuracy of approximation. Further, by using these methods, the utility differential can be approximated by a simple algebraic polynomial to which the perturbation method utilized by Ottaviano (2001) could be applied. We argue, however, that the advantage of the approximation methods is best appreciated when they are utilized for the complete exploration of the model. Using efficient approximation methods makes it possible to confidently analyze all variants of the core-periphery model, regardless of the non-linearity of the indirect utility differential and that of the resulting dynamic system. Naturally, an appealing approximation method should be precise and, possibly, simple. In the remainder of the paper we illustrate three methods that, we believe, have both of these features.

3 An Introduction to Approximation Theory

In this section we illustrate the advantage of Chebyshev’s interpolation over Lagrange’s. We start by laying out a few elements of Approximation Theory.

Let $X \equiv [a, b]$, $a \neq b$, be a closed interval on R , and let $C[X]$ be the set of all continuous real functions $f : X \rightarrow R$. In the sequel we discuss how to obtain a “good” approximation in the *sup norm*² of an arbitrary function $f \in C[X]$, i.e. how to find an approximating function $\hat{f} \in C[X]$ such that $\|f - \hat{f}\| < \varepsilon$ for some $\varepsilon \in R_{++}$. Following the current literature, we focus on linear approximations; in particular, we consider only finite degree *polynomials* $p : X \rightarrow R$ as approximating functions.

Definition 1 (Polynomial) *Let $\Phi \subset C[X]$. A linear combination $p(x) \equiv \sum_{j=1}^n c_j \phi_j(x)$ where $\mathbf{c} \equiv [c_1, \dots, c_n] \in R^n$ and $\phi_j(x) \in \Phi$ for $j = 1, 2, \dots, n$, is called a polynomial in the elements of Φ , or a polynomial in the ϕ_j .*

Definition 2 (Approximable function) *Let again $\Phi \subset C[X]$. A function $f \in C[X]$ is approximable by polynomials in the ϕ_j if for each $\varepsilon \in R_{++}$ there is a $p(x)$ such that $\|f - p\| < \varepsilon$.*

²The *sup norm* on $C[X]$ is defined as $\|f\| \equiv \max_{x \in X} |f(x)|$.

A subset of $C[X]$ often utilized is the sequence of powers $\{x^j\}_{j=0}^{\infty}$. The linear combinations of the first n elements in $\{x^j\}$, $p_n(x) \equiv \sum_{j=0}^n c_j x^j$, are called *algebraic polynomials* of degree n . A key result in approximation theory is offered by Weierstrass' Theorem.

Theorem 3 (Weierstrass) *Each $f \in C[X]$ is approximable by algebraic polynomials.*

Proof. See Schumaker (1981, p. 92). ■

Weierstrass' Theorem implies that any $f \in C[X]$ can be expressed as an infinite sum of powers, $f(x) = \sum_{j=0}^{\infty} c_j x^j$, i.e. that continuous functions can be approximated arbitrarily well by finite order algebraic polynomials.³

However, an essential question remains at this stage unanswered: for a given n , how can we select a "good" vector of coefficients \mathbf{c} ? There are several ways to do that, *interpolation* is the simplest way. Lagrange interpolation is probably the most commonly utilized and we review it here with the purpose of comparing it with the three methods we illustrate in this paper.

3.1 Lagrange Interpolation

Interpolation methods simply force the approximating function to cross the true function at a certain number of points. More formally:

Definition 4 (Interpolating Polynomial) *Let $\{x_j\}_{j=1}^n$ be a set of n distinct points in X , called interpolation nodes, and let $y_j = f(x_j)$ for $j = 1, 2, \dots, n$ and for some $f : X \rightarrow R$. The polynomial $p(x) = \sum_{j=1}^n c_j \phi_j(x)$, where $\phi_j(x) \in \Phi \subset C[X]$, interpolates the function f at the points x_j if $p(x_j) = y_j$ for $j = 1, 2, \dots, n$.*

In general, the interpolating polynomial is not necessarily unique. The following Theorem guarantees uniqueness if the interpolating polynomial is an algebraic polynomial.

Theorem 5 (Uniqueness of interpolating polynomial) *Given $n + 1$ distinct points in X and as many real numbers $y_j = f(x_j)$ for some $f : X \rightarrow R$, there exists a unique interpolating algebraic polynomial of degree n .*

Proof. It can be easily show that the elements of the sequence of powers $\Phi = \{x^j\}_{j=0}^n$ are linearly independent.⁴ Therefore, the system $\sum_{j=0}^n c_j x_k^j = y_k$ for $k = 0, 1, \dots, n$ admits one and only one solution (note however that not necessarily $c_j \neq 0$ for all j). ■

³The family of monomials $\{x^j\}$ can be considered a *basis* for the space of continuous real functions defined on a closed interval of R . More generally, a family of functions Φ is a basis for $C[X]$ if any $f \in C[X]$ can be written as an infinite sum of elements of Φ , $f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)$.

⁴The functions ϕ_j are linearly independent if the exactly identified linear system $\sum_{j=1}^n c_j \phi_j(x_k) = 0$ for $k = 1, \dots, n$ and for any $\{x_j\}_{j=0}^n \subset X$ admits only the trivial solution $c = 0$. In this case the set Φ is called a Chebyshev system. We prefer to omit Chebyshev's name in this definition in order not to create a terminological overlap with the more important Chebyshev theorem, Chebyshev polynomials, and Chebyshev interpolation utilized below.

The following Theorem sets an upper bound for the corresponding approximation error in the sup norm.

Theorem 6 (Upper bound of approximation error) *Let $\{x_j\}_{j=0}^n$ be a set of $n + 1$ distinct points in X , and let $p_n(x)$ be the algebraic polynomial that interpolates a function $f \in C^{(n+1)}[X]$ at these points. Then:*

$$\|f - p_n\| \leq \frac{\|f^{(n+1)}\|}{(n+1)!} \|W\| \quad (1)$$

where $W(x) \equiv \prod_{j=0}^n (x - x_j)$ and f^n is the n -order derivative of f .

Proof. See Carothers (1998, p. 72). ■

The upper bound in (1) depends on $\|W\|$. Clearly, W is an algebraic polynomial of order $n+1$ and leading coefficient 1. Any algebraic polynomial is completely characterized by its zeros.⁵ The zeros of W are, by construction, the $n + 1$ interpolation nodes, and therefore $\|W\|$ is a function of the x_j 's. Hence, a judicious choice of the interpolation nodes should allow us to minimize the approximation error. Usually, following what is known as *Lagrange interpolation*, the nodes are chosen to be $n + 1$ equidistant points in X . A better choice of the nodes is offered by *Chebyshev interpolation*.

3.2 Chebyshev Interpolation

Chebyshev Interpolation is the first of the three methods we illustrate in this paper. It consists in choosing the interpolation nodes in a way to minimize $\|W\|$. It is convenient to start by defining *Chebyshev polynomials*.

Definition 7 (Chebyshev Polynomial) *The family of functions $T_n : [-1, +1] \rightarrow [-1, +1]$ defined as:*

$$T_n(x) \equiv \cos[n \arccos(x)] \quad (2)$$

for $n \geq 0$ is a Chebyshev polynomial.⁶

We recall from the previous section that in order to minimize the approximation error $\|W\|$ we should find the *polynomial of least deviation from zero* (in the sup norm) among all algebraic polynomials of degree $n + 1$ and leading coefficient 1, and use its zeros as interpolation nodes. This is where *Chebyshev's Theorem* comes to our help.

⁵Being the set of algebraic polynomials a Chebyshev system, there is one and only one algebraic polynomials that crosses the horizontal axes at $n + 1$ distinct points.

⁶Note that T_n , being an algebraic polynomial of order n , is defined over the whole real line; however, its particularly useful properties hold on $[-1, +1]$ only.

Theorem 8 (Chebyshev) *The algebraic polynomial of degree n and of least deviation from zero with leading coefficient 1 on $[-1, +1]$ is $2^{1-n}T_n(x)$.*

Proof. See Lorentz (1986, Th. 11, p. 31). ■

Note that $T_0(x) = 1$, $T_1(x) = x$, and that:⁷

$$T_n(x) = 2xT_{n-1} - T_{n-2}(x) \quad (3)$$

for $n \geq 2$. Each $T_n(x)$ is an algebraic polynomial of degree n with leading coefficient 2^{n-1} . The domain of Chebyshev polynomials can be extended to general intervals $X \equiv [a, b] \in R$ via the change of variable $y = 2(x - a) / (b - a) - 1$, where $y \in [-1, +1]$. Hence, the algebraic polynomial of degree n and of least deviation from zero with leading coefficient 1 on $[a, b]$ is:

$$\frac{(b - a)^n}{2^{2n-1}} T_n \left(2 \frac{x - a}{b - a} - 1 \right) \quad (4)$$

Chebyshev's Theorem has an important implication:

Claim 9 *The norm of W is minimized if the interpolation nodes x_j correspond to the $n + 1$ zeros of $T_{n+1} [2(x - a) / (b - a) - 1]$.*

Given that $\cos[(n + 1)t] = 0$ for $t = [(2j - 1)\pi] / [2(n + 1)]$, where $j = 1, \dots, n + 1$, the following:

$$x_j = \cos \left[\frac{2j - 1}{2(n + 1)} \pi \right] \quad (5)$$

defines the $n + 1$ zeros of $T_{n+1}(x)$ over $[-1, +1]$ in decreasing order. All zeros are real, simple, and lie in $(-1, +1)$. Furthermore, $T_{n+1}(x)$ and $T_n(x)$ have no common zeros.

We can now establish an upper bound for the approximation error under Chebyshev interpolation.

Theorem 10 (Upper bound of approximation error) *Let $p_n(x)$ be the algebraic polynomial of degree n that interpolates a function $f \in C^k[X]$, $1 \leq k \leq n$, at the zeros of T_{n+1} . Then there exists a constant $\xi \in R_{++}$ such that:*

$$\|f - p_n\| \leq \left[\frac{2}{\pi} \ln(n + 1) + 1 \right] \frac{\xi}{n^k} \|f^{(k)}\| \quad (6)$$

Proof. See Rivlin (1990, p. 14). ■

Taking (6) to the limit, we have that $\lim_{n \rightarrow \infty} \|f - p_n\| = 0$ for any $f \in C^k[X]$: using Chebyshev interpolation, we can approximate any $f \in C^k[X]$ arbitrarily well. Furthermore, the degree n needed to reach a “good” approximation decreases with k .

⁷The last relationship follows from $\cos(nt) = 2 \cos(t) \cos[(n - 1)t] - \cos[(n - 2)t]$.

3.3 A simple Example

In this section we give an intuitive guide to Lagrange and Chebyshev interpolation methods illustrated formally in the previous section. In parenthesis we refer to the definitions, theorems, and claims of the previous section.

In approximation problems we deal with a known function that, for reasons due to power of calculus or to the complicated form of the function, we want to approximate with a simpler one. Suppose we want to approximate the function $f(x) = x^5$ in $X \equiv [0, 1]$ in the “best” possible way according to some metric. The metric we choose is the sup norm. Naturally, for the approximation problem to be solvable the function f must be approximable (definition 2). Our starting point is the Weierstrass theorem which says that any continuous function is approximable by an *algebraic* polynomial. The first step then is to choose the order of the approximating algebraic polynomial which, usually, depends on the power of calculus we have at our disposal. To make things simple, suppose we choose an algebraic polynomial of order 3. We then approximate $f(x)$ with $p_3(x) \equiv \sum_{j=0}^3 c_j x^j$. We call $p_3(x)$ the *interpolating* algebraic polynomial because we will require $p_3(x)$ to interpolate the function $f(x)$ in correspondence of some chosen points in X called the interpolation *nodes* (definition 4). For instance, let x_0, \dots, x_3 be four interpolation nodes, then we require $p_3(x_j) = f(x_j)$ for $j = 0, \dots, 3$, that is:

$$c_0(x_0)^0 + c_1(x_0)^1 + c_2(x_0)^2 + c_3(x_0)^3 = f(x_0) \quad (7)$$

$$c_0(x_1)^0 + c_1(x_1)^1 + c_2(x_1)^2 + c_3(x_1)^3 = f(x_1) \quad (8)$$

$$c_0(x_2)^0 + c_1(x_2)^1 + c_2(x_2)^2 + c_3(x_2)^3 = f(x_2) \quad (9)$$

$$c_0(x_3)^0 + c_1(x_3)^1 + c_2(x_3)^2 + c_3(x_3)^3 = f(x_3) \quad (10)$$

In sum, our approximation problem requires us to do two things: first we have to choose the approximation nodes, second we have to compute the vector of coefficients \mathbf{c} .⁸ Choosing the interpolation nodes optimally improves grandly the accuracy of approximation.

3.3.1 Lagrange interpolation.

The method commonly utilized is the Lagrange interpolation. It requires the approximating function to interpolate the true function at a number arbitrarily chosen interpolation nodes in X . Usually the interpolation nodes are chosen to be equidistant in X . In our example the interpolation nodes would be: $x_0 = -3/5$; $x_1 = -1/5$; $x_2 = 1/5$; $x_3 = 3/5$.

⁸In general, there is an issue of uniqueness of \mathbf{c} here, but we bypassed it by choosing an algebraic polynomial as approximating function (Theorem 5).

Replacing these interpolation nodes in system (7)-(10) gives us the coefficients of the interpolating algebraic polynomial. The result is shown in Figure 1. The continuous line is $f(x)$ and the dashed line is $p_3^L(x)$ where the superscript simply denotes the fact that we utilized Lagrange interpolation. The approximation is nice but it is possible to do better by choosing the interpolation nodes optimally. This is what Chebyshev interpolation does.

3.3.2 Chebyshev Interpolation.

We start by noticing that the approximation error $\|W\|$ is itself an algebraic polynomial of order $n+1$ and its magnitude depends on the the $n+1$ approximation nodes (Theorem 6). We can therefore minimize $\|W\|$ by a judicious choice of the approximation nodes. Here is where the Chebyshev theorem comes to our help. Chebyshev theorem and its implication (Claim 9) tell us that the approximation error is minimized if we choose the interpolation nodes at the zeros of a Chebyshev polynomial. Its zeros can be easily found (equation 5). Computing the four interpolation nodes at the zeros of Chebyshev polynomial in our simple example gives: $x_0 = \frac{1}{2}\sqrt{(2 + \sqrt{2})}$; $x_1 = \frac{1}{2}\sqrt{(2 - \sqrt{2})}$; $x_2 = -\frac{1}{2}\sqrt{(2 - \sqrt{2})}$; $x_3 = -\frac{1}{2}\sqrt{(2 + \sqrt{2})}$. Replacing these values in system (7)-(10) and solving gives us the vector of coefficients \mathbf{c} . Let us refer to the resulting interpolating polynomial as $p_3^C(x)$ where the superscript indicates that we have utilized Chebyshev interpolation nodes. The dotted line in Figure 1 represents $p_3^C(x)$. Even a simple visual inspection reveals that the approximating polynomial $p_3^C(x)$ is far more precise than $p_3^L(x)$. It is striking that such a simple method gives such a remarkable improvement in the precision of approximation. All it takes is to computed the interpolation nodes according to a simple formula.

4 Projection methods

In the previous section the function to be approximated was known. Yet, in new economic geography models the function to be approximated is only implicitly defined by a system of equations and it cannot be found explicitly.⁹ Typically, the problem presents itself in the following form: $g[x, f(x)] = 0$. We could proceed in a similar way to what we have done in the previous section, with the difference that the values of $f(x_j)$ would have to be found numerically. Thus, in the first step, we could solve the implicit equation numerically to get the $\hat{f}(x_j)$. These values would replace the $f(x_j)$ in system (7)-(10). The system would then give us the vector \mathbf{c} .

While this procedure would still give us a better approximation than Lagrange's

⁹An exeption is equation (13) in Ottaviano (2001) to which Chebishev interpolatoin could be applied.

method, the precision can be further improved by the use of projection methods. The advantage of projection methods is that they do two steps at once in an efficient way: they find $\hat{f}(x_j)$ and \mathbf{c} in one single procedure instead of two separate steps. In this paper we illustrate and apply to the core-periphery model two projection methods: *orthogonal collocation* and *Galerkin's method*. Before passing to the illustration and application of the methods we need a brief introduction to projection methods.

4.1 Introduction

Let again $X \equiv [a, b]$ be a closed subset of R , and let $f \in C[X]$ be *implicitly* defined by a functional equation of the form $g[x, f(x)] = 0$ where $g : R \rightarrow R$ is another element of $C[X]$. This kind of functional equations do not usually admit closed form solutions.

Let us approximate the unknown function f with a polynomial in the elements of $\Phi \subset C[X]$, where Φ is a basis for $C[X]$:

$$f(x) \approx p(x, \mathbf{c}) \equiv \sum_{j=1}^n c_j \phi_j(x) \quad (11)$$

The *residual function* is defined as $R(x, \mathbf{c}) \equiv g[x, p(x, \mathbf{c})]$. Note that, if $p(x, \mathbf{c})$ approximates $f(x)$ arbitrarily well, then $R(x, \mathbf{c}) \approx 0$ for all $x \in X$.

We need to introduce the concept of functional orthogonality. This concept is important for the choice of the elements of Φ and is a key element of projection methods.

Definition 11 Let $w : X \rightarrow R$ be an almost everywhere positive and Riemann integrable function on X .¹⁰ The function w is called *weighting function*.

Definition 12 Let $f, g \in C[X]$. Given a weighting function w , we can define an inner product on $C[X]$ as $\langle f, g \rangle \equiv \int_a^b f(x) g(x) w(x) dx$.

Definition 13 Let $\Psi = \{\psi_j\}$ be a set of functions. The elements of Ψ are mutually orthogonal with respect to the weighting function w if and only if $\langle \psi_k, \psi_j \rangle = 0$ for all $k \neq j$.

The following theorem shows Chebyshev polynomial are mutually orthogonal.

Theorem 14 The Chebyshev polynomials T_n are orthogonal on $[-1, +1]$ with respect to the weighting function $w(x) \equiv (1 - x^2)^{-\frac{1}{2}}$.

¹⁰In other words, $w(x)$ is positive, except possibly at finitely many points, and has a finite Riemann integral on X .

Proof. By definition:

$$\int_{-1}^{+1} \frac{T_k(x) T_j(x)}{\sqrt{1-x^2}} dx = \int_0^\pi \cos(k\theta) \cos(j\theta) d\theta = \begin{cases} 0, & k \neq j \\ \pi, & k = j = 0 \\ \pi/2, & k = j \neq 0 \end{cases} \quad (12)$$

■

The orthogonality of Chebyshev polynomials explains why they are preferable to the simple powers $\{x^j\}_{j=0}^\infty$ as elements of the basis Φ . We already know that any algebraic polynomial can be uniquely expressed as a finite sum of Chebyshev polynomials, that is: $\sum_{j=0}^n \alpha_j x^j = \sum_{j=0}^n c_j T_j(x)$ for adequately defined $\mathbf{c} \in R^n$. Note however that the monomials $\{x^j\}$ are *not* mutually orthogonal, and therefore, in some sense, the “information” carried by x^j overlaps partially with the “information” carried by x^{j-z} or x^{j+z} for $z \neq j$. Each Chebyshev polynomial, instead, is orthogonal to any other member of the family, and conveys therefore a different “piece of information.” From a purely numerical point of view, the coefficients in $\sum_{j=0}^n c_j T_j(x)$ are better identified than the coefficients in $\sum_{j=0}^n \alpha_j x^j$, and this improves the performance of all our solution procedures.

Orthogonality is also a key element of projection methods. The key idea of projections methods is that, if \mathbf{c} solves $R(x, \mathbf{c}) = 0$ for some $x \in X$, it solves $R(x, \mathbf{c}) \varphi(x) = 0$ too, where $\varphi(x)$ can be any suitable function $\varphi : X \rightarrow R$. The optimal vector $\hat{\mathbf{c}} \in R^n$ is then identified by imposing n *orthogonality conditions* among $R(x, \mathbf{c})$ and a set of *directions* $\{\varphi_j\}_{j=1}^n$:

$$\langle R(x, \mathbf{c}), \varphi_j \rangle = 0, \quad j = 1, 2, \dots, n \quad (13)$$

The orthogonality conditions render the identification of \mathbf{c} more efficient. The choice of the weighting function w and the set of directions $\{\varphi_j\}_{j=1}^n$ characterize the particular projection method used. We lay out two of the most popular ones: Orthogonal Collocation and Galerkin.

4.2 Orthogonal Collocation

The simplest projection method is *collocation*. For any arbitrary weighting function, collocation requires that $\langle R(x, \mathbf{c}), \delta(x - x_j) \rangle = 0$ for $j = 1, 2, \dots, n$, where δ is the Dirac delta function, and the x_j 's are n *collocation nodes* in X .

The previous expression has a simple intuitive interpretation. As interpolation requires the interpolating algebraic polynomial to cross the approximated function at a given set of points, collocation requires the approximating polynomials to exactly solve the functional

equation $g[x, f(x)] = 0$ at some n distinct points in X . Hence, collocation imposes that:

$$R(x_j, \mathbf{c}) = 0, \quad j = 1, 2, \dots, n \quad (14)$$

Note that (14) is simply a system of n nonlinear equations in n unknowns that can be numerically solved using Newton or Quasi-Newton methods. Collocation effectively transforms a functional equation into a more manageable system of nonlinear equations. The solution to (14), denoted $p(x, \hat{\mathbf{c}})$, will almost exactly represent f at the given points x_j , but will only approximate it over the remaining part of X . The accuracy of this approximation increases with n , while the perfect fit is reached asymptotically being p an algebraic polynomial.

Note that collocation does not impose any particular requirement on the interpolating polynomial $p(x, \mathbf{c})$. However, as we already know, choosing Chebyshev polynomials as basis for Φ and choosing the interpolation nodes to be the zeros of T_{n+1} minimizes the approximation error in the sup norm. Collocation performed at the zeros of T_{n+1} is called *orthogonal collocation* and this is what we use in our application to the core periphery model.

4.3 Galerkin's method

Galerkin method assumes that the basis Φ is a family of *orthogonal polynomials* with respect to a weighting function w . This method uses the first n elements of Φ as projection directions:

$$\langle R(x, \mathbf{c}), \phi_j \rangle = 0, \quad j = 1, 2, \dots, n \quad (15)$$

The main difference with respect to orthogonal collocation is in that the residual function is projected along mutually orthogonal directions, and therefore each condition in (15) is made “as different as possible” from the others.

If Chebyshev polynomials are used as basis functions, the integral in (15) can be numerically approximated using quadrature methods, and in particular Gauss-Chebyshev quadrature.¹¹

Claim 15 *Given the particular properties of T_n , applying the Gauss-Chebyshev quadrature formula leads us to $\langle R_n(x, \mathbf{c}), T_j(x) \rangle \approx \sum_{k=1}^m R_n(\hat{x}_k, \mathbf{c}) T_j(\hat{x}_k)$, where the $m > n + 1$ quadrature nodes $\{\hat{x}_j\}_{j=1}^m$ are the zeros of T_m .*

Note that the accuracy of the approximation increases with the number of quadrature

¹¹For details on numerical integration methods and Gaussian quadrature formulas, see Judd (1988, Sec. 7.2).

nodes, m . The orthogonality conditions in (15) can therefore be substituted by:

$$\sum_{k=1}^m R_n(\hat{x}_k, \mathbf{c}) T_j(\hat{x}_k) = 0, \quad j = 0, 1, \dots, n \quad (16)$$

for a large enough m . As collocation does, Galerkin's method transforms the functional equation into a system of nonlinear equations that can be easily solved numerically.

4.4 A simple example

Consider the continuous function $f(x) \equiv \sin(x) + \cos(x)$. We will now approximate f over the finite interval $X \equiv [0, 5]$ using three alternative methods and compare the results. More precisely, we will approximate f using an algebraic polynomial in the T_j 's of degree n , $\hat{f}_n(x, \mathbf{c}) \equiv \sum_{j=0}^n c_j T_j\left(2\frac{x-a}{b-a} - 1\right)$, where $a = 0$ and $b = 5$, and will pin down the vector \mathbf{c} using Galerkin's method, orthogonal collocation, and Lagrange interpolation.

1. To apply Galerkin's method, we obtain the m zeros of T_m on $[a, b]$, $\{\hat{x}_k\}_{k=1}^m$, and numerically solve the system for \mathbf{c} using Broyden's method,¹²:

$$\sum_{k=1}^m \left[\hat{f}_n(\hat{x}_k, \mathbf{c}) - f(\hat{x}_k) \right] T_j\left(2\frac{\hat{x}_k - a}{b - a} - 1\right) = 0, \quad j = 0, 1, \dots, n \quad (17)$$

2. To apply the orthogonal collocation method, we obtain the $n + 1$ zeros of T_{n+1} on $[a, b]$, $\{\tilde{x}_j\}_{j=0}^n$, and solve the system $\hat{f}_n(\tilde{x}_j, \mathbf{c}) = f(\tilde{x}_j)$ for $j = 0, 1, \dots, n$.
3. Finally, to apply the standard Lagrange interpolation method, we choose $n + 1$ equally spaced points in $[a, b]$, $\{x_j\}_{j=0}^n$, and solve the system $\hat{f}_n(x_j, \mathbf{c}) = f(x_j)$ for $j = 0, 1, \dots, n$.

Before illustrating the results a few words on the Broyden numerical method are in order. The convergence properties of Broyden's method, as of all variants of Newton's method, are extremely sensitive to the initial guess for \mathbf{c} . If the guess is "good", the method converges very fast, but it can, and probably will, fail to converge if the guess is far from the solution. This sensitivity on the initial condition increases with the dimension of the problem to solve, i.e. with the degree n : finding good initial guesses for large problems is extremely difficult. Therefore, we follow a *continuation* approach: we start with $n = 1$, i.e. with a linear approximation, and then use the corresponding result as a starting guess for the problem of order $n + 1$. Iterating on this procedure we can safely solve very

¹²Broyden's method is a variant of Newton's method that avoids the explicit calculation of the Jacobian at each iteration. For more details on non-linear solvers see Judd (1998, Chapter 5)

		$n = 1$	$n = 3$	$n = 5$	$n = 15$
		Approximation error			
	Gal.	0.41	3.89e-2	1.69e-3	4.41e-13
Avg. Abs.	Col.	0.36	3.94e-2	1.66e-3	4.06e-13
	<i>Int.</i>	<i>0.59</i>	<i>8.10e-2</i>	<i>5.57e-3</i>	<i>2.48e-11</i>
	Gal.	0.43	3.86e-2	1.55e-3	4.37e-13
Med. Abs.	Col.	0.28	3.48e-2	1.31e-3	3.12e-13
	<i>Int.</i>	<i>0.23</i>	<i>8.60e-3</i>	<i>2.55e-4</i>	<i>1.31e-14</i>
	Gal.	0.46	4.49e-2	1.99e-3	5.09e-13
Std.	Col.	0.44	5.10e-2	2.22e-3	5.32e-13
	<i>Int.</i>	<i>0.91</i>	<i>16.95e-2</i>	<i>1.47e-2</i>	<i>1.41e-10</i>
	Gal.	0.74	8.07e-2	3.79e-3	9.43e-13
Max. Abs.	Col.	1.23	13.15e-2	5.86e-3	1.26e-12
	<i>Int.</i>	<i>2.57</i>	<i>71.75e-2</i>	<i>7.94e-2</i>	<i>1.20e-9</i>

Table 1: Properties of approximation errors.

large problems: the computational effort spent on the intermediate steps is more than compensated by the robustness and accuracy of the final solution. The method outlined here can be efficiently implemented in any matrix-oriented programming language: we use MATLAB 6.1 and run our programs on a 800Mhz Pentium III PC under Windows ME.

Coming to the precision of approximation we note that an approximated solution obtained using orthogonal collocation or Lagrange interpolation almost perfectly solves (by construction) the residual function at the nodes. Hence, to assess the quality of the approximation we compute the approximation error in points that are *not* collocation or interpolation nodes. The properties of the approximation error are compared in Table 1 for $n = 1, 3, 5$, and 15 and $m = 30$. We report the average, median, standard deviation, and maximum value of the absolute approximation error, $\left|f(x) - \hat{f}_n(x, \mathbf{c})\right|$, over 100 equally spaced points in $[a, b]$.

An algebraic polynomial of order 15 fits our function almost perfectly over the approximation interval. The Galerkin method generates the best approximations for a given degree of the polynomial, but collocation performs remarkably well, given its simplicity and computational efficiency. Lagrange interpolation outperforms both other methods as far as the median absolute error is concerned, but is clearly worse under all other points of view. For $n = 15$, the Galerkin method produces a maximum absolute approximation error that is of three orders of magnitude smaller than the corresponding figure for the Lagrange method. Figure 2 plots the actual function, its approximations, and the approximation errors for $n = 3$. Figure 2 helps us to understand these differences: the Lagrange method performs very well near the center of the interval, but quite poorly at the extremes. The Galerkin and collocation methods, instead, generate slightly less

accurate approximations at the center of the interval, but are remarkably accurate at the extremes.

5 An application to the Core-Periphery model

The structure of the model is as follows. The world is composed of two countries (1 and 2). The world population is normalized to one and is composed of μ workers and $1 - \mu$ farmers. Workers and Farmers are the only factors of production and the parameter μ is constant. While workers can migrate, farmers are assumed to have country specific skills which make them internationally immobile and it is further assumed that each country is endowed with $(1 - \mu)/2$ of them.

There are two goods being produced: A and X . Commodity A is assumed to be a homogeneous good produced by use of farmers only and traded internationally at zero costs. Commodity X is a differentiated commodity produced in a monopolistic competitive market structure. The production of X requires workers only and it is assumed that X is traded at an iceberg type of trade cost; i.e., for each unit sent only a fraction $\tau \in (0, 1]$ arrives at its destination. Given the utility function specified below, both goods are produced in each country as long as $\mu < 1/2$. We assume this inequality throughout the paper so that the price of A and the wage of farmers can both be normalized to one in both countries and serve as the numeraire. The input of workers for x units of output in the X sector is $F + bx$. Profit maximization requires the price to be at a constant mark up over marginal cost. Denoting the elasticity of demand with σ it is convenient to normalize $F = 1/\sigma$ and $b = \sigma/(\sigma - 1)$ so that the profit-maximizing price for sales at home and abroad are $p_{ii} = w_i$ and $p_{ij} = w_i/\tau$ respectively, where w_i is workers' wage in country i . Free entry ensures that profits are zero in equilibrium. The zero profit condition gives the optimal scale of the firm, which is constant, and - given the normalization - is equal to 1.

Individuals enjoy consumption according to the felicity function: $c_i = X^\mu A^{1-\mu}$, where X is a CES aggregate with elasticity σ of the N varieties of X produced in the world. Given the felicity function, indirect utility in country i is w_i/P_i^μ , where P_i is the true price index for individuals in country i . The expressions for the price indexes are the following:

$$P_1 = [\lambda w_1^{1-\sigma} + (1 - \lambda) w_2^{1-\sigma} \tau^{\sigma-1}]^{\frac{1}{1-\sigma}} \quad (18)$$

$$P_2 = [\lambda w_1^{1-\sigma} \tau^{\sigma-1} + (1 - \lambda) w_2^{1-\sigma}]^{\frac{1}{1-\sigma}} \quad (19)$$

The market equilibrium equations for commodity X are the following:

$$\left(\frac{w_1}{P_1}\right)^{1-\sigma} Y_1 + \left(\frac{w_1}{\tau P_2}\right)^{1-\sigma} Y_2 = w_1 \quad (20)$$

$$\left(\frac{w_2}{\tau P_1}\right)^{1-\sigma} Y_1 + \left(\frac{w_2}{P_2}\right)^{1-\sigma} Y_2 = w_2 \quad (21)$$

where Y_1 and Y_2 represent national income and are given by the following expressions:

$$Y_1 = \frac{1-\mu}{2} + \lambda\mu w_1 \quad (22)$$

$$Y_2 = \frac{1-\mu}{2} + (1-\lambda)\mu w_2 \quad (23)$$

By Walras law the equilibrium conditions for Y can be ignored. Equations (20) and (21) determine the wage rate in each country as function of the distribution of workers between countries.

5.1 Static expectations

Workers migrate to the country where the indirect utility is the highest. The migration flow is regulated by the following difference equation:

$$\Delta\lambda_{t+1} = \frac{\lambda_t(1-\lambda_t)}{\gamma}\omega(\lambda_t) \quad (24)$$

where $\gamma \in R_{++}$ is a parameter representing the cost of migration, and:

$$\omega(\lambda_t) \equiv \frac{w_{1t}}{P_{1t}^\mu} - \frac{w_{2t}}{P_{2t}^\mu} \quad (25)$$

The initial condition $\lambda_0 \in [0, 1]$ is exogenously given.

We approximate the equilibrium wage rates w_1 and w_2 as functions of λ with finite order algebraic polynomials. More precisely, we approximate each w_i with a finite order polynomial in the T_j 's:

$$w_i(\lambda) \approx \hat{w}_i(\lambda, \mathbf{c}) = \sum_{j=0}^n c_{ij} T_j(2\lambda - 1), \quad i = 1, 2 \quad (26)$$

For the perfect symmetry of (20)-(21), it turns out that $\hat{w}_2(\lambda, \mathbf{c}) = \hat{w}(-\lambda, \mathbf{c}) \equiv \hat{w}_1(-\lambda, \mathbf{c})$. Therefore a unique set of $n + 1$ coefficients is sufficient to fully characterize both wages.

The residual function is then defined as:

$$R_n(\lambda, \mathbf{c}) \equiv \left[\frac{\hat{w}(\lambda, \mathbf{c})}{P_1(\lambda)} \right]^{1-\sigma} Y_1(\lambda) + \left[\frac{\hat{w}(\lambda, \mathbf{c})}{\tau P_2(\lambda)} \right]^{1-\sigma} Y_2(\lambda) - \hat{w}(\lambda, \mathbf{c}) \quad (27)$$

where:

$$Y_1(\lambda) = \frac{1-\mu}{2} + \lambda\mu\hat{w}(\lambda, \mathbf{c}) \quad (28)$$

$$Y_2(\lambda) = \frac{1-\mu}{2} + (1-\lambda)\mu\hat{w}(-\lambda, \mathbf{c}) \quad (29)$$

$$P_1(\lambda) = \left[\lambda\hat{w}(\lambda, \mathbf{c})^{1-\sigma} + (1-\lambda)\hat{w}(-\lambda, \mathbf{c})^{1-\sigma} \tau^{\sigma-1} \right]^{\frac{1}{1-\sigma}} \quad (30)$$

$$P_2(\lambda) = \left[\lambda\hat{w}(\lambda, \mathbf{c})^{1-\sigma} \tau^{\sigma-1} + (1-\lambda)\hat{w}(-\lambda, \mathbf{c})^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \quad (31)$$

We compare now the performance of our three alternative methods under the following benchmark parameterization:

$$\mu = 0.4, \quad \sigma = 3, \quad \tau = 0.15, \quad \gamma = 1.15$$

To apply Galerkin's method, we set $m = 50$ and numerically solve for \mathbf{c} , using again Broyden's method, the system $\sum_{k=1}^m R_n(\hat{\lambda}_k, \mathbf{c}) T_j(2\hat{\lambda}_k - 1) = 0$ for $j = 0, 1, \dots, n$, where the $\hat{\lambda}_k$'s are the m zeros of T_m on $[0, 1]$. To apply orthogonal collocation, instead, we solve the system $R_n(\tilde{\lambda}_j, \mathbf{c}) = 0$ for $j = 0, 1, \dots, n$, where the $\tilde{\lambda}_j$'s are the $n + 1$ zeros of T_{n+1} on $[0, 1]$. Finally, to apply Lagrange interpolation we solve the system $R_n(\lambda_j, \mathbf{c}) = 0$ for $j = 0, 1, \dots, n$, where the λ_j 's are $n + 1$ equally spaced points in $[0, 1]$.

We let the degree of the approximating polynomial vary from 1 to 39, and follow the previously described continuation approach. In other words, for the last of our experiments we run the solution procedure 39 times, increasing the degree of the polynomial at each iteration. The time (in seconds) needed to perform *all* computations is reported at the bottom of Table 2. As we can see, the whole procedure is extremely fast: the last experiment is generally completed in more or less two seconds.

Table 2 summarizes the empirical distribution of the wage equation residuals, in absolute terms, over 100 equally spaced points in $[0, 1]$. Note however that the wage equation residuals are only indirectly linked to the true approximation error, i.e. the difference between the approximated wage function and the true one. The Galerkin method produces a good approximation even with a polynomial of degree 9, and the accuracy reached by the last experiment is not far from the machine's precision. Orthogonal collocation performs again remarkably well: the reported statistics for the collocation residuals are generally of the same order of magnitude of the Galerkin residuals. Surprisingly enough,

		$n = 1$	$n = 9$	$n = 19$	$n = 29$	$n = 39$
Wage Equation Residuals						
Avg. Abs.	Gal.	0.11	6.30e-5	8.59e-8	3.33e-10	2.39e-13
	Col.	0.12	8.72e-5	1.38e-7	4.98e-10	4.11e-13
	<i>Int.</i>	<i>0.14</i>	<i>2.83e-3</i>	<i>1.13e-3</i>	<i>1.09e-4</i>	<i>5.58e-5</i>
Med. Abs.	Gal.	0.12	6.24e-5	7.63e-8	3.06e-10	1.96e-13
	Col.	0.13	6.87e-5	9.51e-8	3.22e-10	2.79e-13
	<i>Int.</i>	<i>0.04</i>	<i>6.52e-6</i>	<i>2.20e-8</i>	<i>2.70e-12</i>	<i>7.93e-14</i>
Std.	Gal.	0.12	7.32e-5	1.02e-7	3.99e-10	3.01e-13
	Col.	0.11	1.35e-4	2.09e-7	7.87e-10	5.87e-13
	<i>Int.</i>	<i>0.25</i>	<i>1.18e-2</i>	<i>6.80e-3</i>	<i>7.76e-4</i>	<i>4.83e-4</i>
Max. Abs.	Gal.	0.18	1.67e-4	2.33e-7	9.36e-10	5.65e-13
	Col.	0.39	6.26e-4	8.77e-7	3.46e-9	2.15e-12
	<i>Int.</i>	<i>0.96</i>	<i>8.98e-2</i>	<i>6.19e-2</i>	<i>7.48e-3</i>	<i>4.79e-3</i>
Computational Time						
Sec.	Gal.	0.00	0.11	0.38	0.77	1.42
	Col.	0.00	0.16	0.43	1.10	1.81
	<i>Int.</i>	<i>0.00</i>	<i>0.11</i>	<i>0.44</i>	<i>0.88</i>	<i>2.26</i>

Table 2: Wage equation residuals.

however, collocation takes longer to converge to a solution, even if it is in principle a less computationally intensive method. Lagrange interpolation, instead, performs poorly: the order of magnitude of the average absolute residual in the last column is eight times higher than the corresponding figure for the Galerkin method (this means almost 7,000 times bigger), while the order of magnitude of the maximum absolute residual is even ten times higher (over 18,000 times bigger). The standard deviation of Lagrange is 13,000 times bigger than Galerkin. These differences are clearly remarkable. Only the median absolute residual is in line with the results for the other two methods. This makes again clear that the Lagrange method performs well at the center of the interval and poorly at the extremes. But at the extremes, away from the symmetric equilibrium, is exactly where analytical methods meet their limitations and numerical analysis is most needed.

The wage equation residuals for the last experiment performed with the Galerkin method are plotted in Figure 3a, and show that the approximation's accuracy is roughly uniform over $[0, 1]$. Figure 3b plots the approximated wage functions.

Figure 4a plots the right-hand side of (24), and can therefore be interpreted as a phase diagram. We note that $\lambda = 1/2$ is the unique stable steady state, while $\lambda = 0$ and $\lambda = 1$ are both unstable steady states. Iterating on (24), we can solve for the transitional dynamics. Figure 4b plots the adjustment path for λ_t , together with the corresponding growth rate, when $\lambda_0 = 0.6$. Convergence is achieved after thirty periods. The transitional dynamics is the additional information we obtain by utilizing the approximation and projection

methods we have illustrated.

5.2 Rational expectations

If individuals are forward looking they use the model to predict the future value of wages. Under perfect foresight, they can compute the indirect utility in any country at any time from the future value of wages. Therefore, the representative individual will choose migration to optimally allocate its labour as follows:

$$\begin{aligned} \max_{\{m_s, \lambda_{s+1}\}_{s=t}^{\infty}} \quad & U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left[\lambda_t \frac{w_{1t}}{P_{1t}^{\mu}} + (1 - \lambda_t) \frac{w_{2t}}{P_{2t}^{\mu}} - \varphi(m_t) \right] \\ \text{s.t.} \quad & \Delta \lambda_{t+1} = m_t \end{aligned} \quad (32)$$

where:

$$\varphi(m_t) \equiv \frac{\gamma m_t^2}{2\lambda_t(1 - \lambda_t)} \quad (33)$$

is the household's cost of migration, and the initial condition $\lambda_0 \in (0, 1)$ is given.

The first order conditions for problem (32) can be written as:

$$m_t = \frac{\lambda_t(1 - \lambda_t)}{\gamma} \nu_t \quad (34)$$

$$\nu_t = \beta \left[\omega(\lambda_{t+1}) + \nu_{t+1} + \frac{(1 - 2\lambda_{t+1})\gamma m_{t+1}^2}{2\lambda_{t+1}^2(1 - \lambda_{t+1})^2} \right] \quad (35)$$

$$\lambda_{t+1} = \lambda_t + m_t \quad (36)$$

where ω is defined as in (25) and ν_t is the costate variable. For the Envelope Theorem, ν_t corresponds to the shadow value of migration in terms of the representative individual's utility. Under our assumptions, equations (34)-(36), together with the following transversality condition:

$$\lim_{t \rightarrow \infty} \beta^t \nu_t \lambda_{t+1} = 0 \quad (37)$$

are jointly necessary and sufficient for problem (32).

Conditions (34) and (35) can be combined to obtain the usual Euler equation:

$$\omega(\lambda_{t+1}) + \left[1 + \frac{(\frac{1}{2} - \lambda_{t+1})\gamma m_{t+1}}{\lambda_{t+1}(1 - \lambda_{t+1})} \right] \frac{m_{t+1}}{\lambda_{t+1}(1 - \lambda_{t+1})} = \frac{\gamma m_t}{\beta \lambda_t(1 - \lambda_t)} \quad (38)$$

5.2.1 Phase diagram analysis

To study the dynamic properties of the model from a qualitative point of view, we need first of all to characterize the $\Delta \lambda_{t+1} = 0$ and $\Delta m_{t+1} = 0$ schedules. Imposing the

conditions $\lambda_{t+1} = \lambda_t$ and $m_{t+1} = m_t$ on (34) and (35), we conclude that:

$$\Delta\lambda_{t+1} = 0 \rightarrow m = 0 \quad (39)$$

$$\Delta m_{t+1} = 0 \rightarrow \frac{\gamma m}{\beta\lambda(1-\lambda)} = \omega(\lambda') + \left[1 + \frac{(\frac{1}{2} - \lambda')\gamma m}{\lambda'(1-\lambda')} \right] \frac{m}{\lambda'(1-\lambda')} \quad (40)$$

where $\lambda' = \lambda + m$.

Equation (40) defines implicitly a continuous function $\bar{m}(\lambda) : [0, 1] \rightarrow R$ that can be approximated with a polynomial in the T_j 's:

$$\bar{m}(\lambda) \approx \bar{m}(\lambda, \mathbf{c}) = \sum_{j=0}^n c_j T_j(2\lambda - 1) \quad (41)$$

The residual function becomes then the following:

$$R_n(\lambda, \mathbf{c}) \equiv \omega(\lambda') + \left[1 + \frac{(\frac{1}{2} - \lambda')\gamma\bar{m}(\lambda, \mathbf{c})}{\lambda'(1-\lambda')} \right] \frac{\bar{m}(\lambda, \mathbf{c})}{\lambda'(1-\lambda')} - \frac{\gamma\bar{m}(\lambda, \mathbf{c})}{\beta\lambda(1-\lambda)} \quad (42)$$

where $\lambda' = \lambda + \bar{m}(\lambda, \mathbf{c})$.

To pin down the vector \mathbf{c} using our alternative methods we follow the procedure outlined in the previous Section. Note that, for a given set of λ'_j 's, the wage differential $\omega(\lambda'_j)$ is a linear function of $w_1(\lambda'_j)$ and $w_2(\lambda'_j)$. The latter, in turn, are non-linear functions of λ'_j , and can be obtained by numerically solve for $\{w'_{1j}\}_{j=0}^n$ the system:

$$\left[\frac{w'_{1j}}{P_1(\lambda'_j)} \right]^{1-\sigma} Y_1(\lambda'_j) + \left[\frac{w'_{1j}}{\tau P_2(\lambda'_j)} \right]^{1-\sigma} Y_2(\lambda'_j) = w'_{1j}, \quad j = 0, 1, \dots, n \quad (43)$$

where:

$$Y_1(\lambda'_j) = \frac{1-\mu}{2} + \lambda'_j \mu w'_{1j} \quad (44)$$

$$Y_2(\lambda'_j) = \frac{1-\mu}{2} + (1-\lambda'_j) \mu w'_{2j} \quad (45)$$

$$P_1(\lambda'_j) = \left[\lambda'_j (w'_{1j})^{1-\sigma} + (1-\lambda'_j) (w'_{2j})^{1-\sigma} \tau^{\sigma-1} \right]^{\frac{1}{1-\sigma}} \quad (46)$$

$$P_2(\lambda'_j) = \left[\lambda'_j (w'_{1j})^{1-\sigma} \tau^{\sigma-1} + (1-\lambda'_j) (w'_{2j})^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \quad (47)$$

taking into account that, for the symmetry of equations (20)-(21), $w'_{2j} = w'_{1(n-j)}$.

Table 3 summarizes the performance of our solution methods. As we can see, to approximately solve for the $\Delta m_{t+1} = 0$ locus is far more computationally expensive than

		$n = 1$	$n = 9$	$n = 19$	$n = 29$	$n = 39$
Euler Equation Residuals						
Avg. Abs.	Gal.	0.11	6.07e-5	3.81e-7	3.00e-9	2.30e-11
	Col.	0.42	1.66e-4	1.40e-6	1.16e-8	8.76e-11
	<i>Int.</i>	<i>1.17</i>	<i>7.81e-2</i>	<i>0.13</i>	<i>6.58e-2</i>	<i>0.17</i>
Med. Abs.	Gal.	0.12	6.65e-5	4.09e-7	3.24e-9	2.19e-11
	Col.	0.07	1.48e-4	8.40e-7	5.56e-9	5.64e-11
	<i>Int.</i>	<i>0.06</i>	<i>3.77e-5</i>	<i>1.47e-7</i>	<i>8.89e-10</i>	<i>8.93e-10</i>
Std.	Gal.	0.14	6.87e-5	4.45e-7	3.57e-9	2.79e-11
	Col.	1.43	2.43e-4	2.87e-6	2.54e-8	1.99e-10
	<i>Int.</i>	<i>3.67</i>	<i>0.38</i>	<i>0.92</i>	<i>0.43</i>	<i>1.20</i>
Max. Abs.	Gal.	0.59	1.11e-4	1.21e-6	1.08e-8	8.50e-11
	Col.	8.84	1.15e-3	1.69e-5	1.46e-7	1.27e-9
	<i>Int.</i>	<i>21.99</i>	<i>2.55</i>	<i>6.45</i>	<i>3.00</i>	<i>8.43</i>
Computational Time						
Sec.	Gal.	1.38	24.82	69.90	114.96	160.71
	Col.	0.06	1.76	8.90	25.59	58.81
	<i>Int.</i>	<i>0.05</i>	<i>1.76</i>	<i>8.84</i>	<i>27.02</i>	<i>91.34</i>

Table 3: Euler equation residuals: the $\Delta m_{t+1} = 0$ locus.

solve for the wage rates, and the average size of the Euler equation residuals, in absolute terms, is larger. A good approximation is however reached using Galerkin's method with a polynomial of degree 19, after slightly more than a minute of wait. Note that the statistics generated by the orthogonal collocation method are generally of an order of magnitude higher. In this case, however, collocation is computationally far more efficient than Galerkin's method. The reason is simple: in this set of experiments, Broyden's solution method proved to be unable to guarantee convergence for the large systems implied by Galerkin's method, and therefore we switched to the standard Newton's method. The time differential has been spent in numerically computing the Jacobian for the residual function, a computationally intensive task. A quick look at the statistics for the Lagrange method is enough to convince us that standard interpolation is *not* a feasible alternative to projection methods if a uniformly good approximation is our goal. Figure 5 plots the phase diagram for the rational expectation model.

The velocity vectors can be easily computed too. For given values of m and λ we can numerically solve equation (38) for m' , taking into account that $\lambda' = \lambda + m$; hence, the implied variations Δm and $\Delta \lambda$ are readily available. Figure 6 plots the velocity vectors for a subset of equally spaced points on the phase plane.

5.2.2 Transitional dynamics

The population share λ is the only state - or backward-looking - variable in the system. The value of migration ν is a costate - or forward-looking - variable, and therefore the control variable m is forward-looking too. In fact, the solution to (34)-(36) is pinned down by an initial condition for λ and a final condition for m , implicit in the transversality condition. Being the structure of problem (32) completely recursive, its solution can be represented by a *time invariant* policy correspondence that satisfies (34)-(36) together with the transversality condition. Under some assumptions,¹³ the policy correspondence becomes a unique continuous policy *function*, i.e. a one-to-one relationship between m and λ that is continuous over $[0, 1]$. This happens, for instance, when the system is saddle-path stable: in this case, for each initial condition λ_0 there is a unique path converging to the steady state, and this unique path is fully characterized by the unique policy function.

It turns out that, under our parameterization, a unique and continuous policy function $m(\lambda)$ actually exists. Hence, we are allowed to approximate it with a polynomial in the T_j 's:

$$m(\lambda) \approx \hat{m}(\lambda, \mathbf{c}) = \sum_{j=0}^n c_j T_j \left[2 \frac{\lambda - a}{b - a} - 1 \right] \quad (48)$$

where $a \in [0, 1]$, $b \in [0, 1]$, and $a < b$. Note that $[a, b]$ does not necessarily coincide with $[0, 1]$: if the dynamic system presents more than one steady state, we may be interested in approximating the stable (or unstable) saddle path near each steady state in turn.

The residual function becomes the following:

$$R_n(\lambda, \mathbf{c}) \equiv \omega(\lambda') + \left[1 + \frac{(\frac{1}{2} - \lambda') \gamma \hat{m}(\lambda', \mathbf{c})}{\lambda' (1 - \lambda')} \right] \frac{\hat{m}(\lambda', \mathbf{c})}{\lambda' (1 - \lambda')} - \frac{\gamma \hat{m}(\lambda, \mathbf{c})}{\beta \lambda (1 - \lambda)} \quad (49)$$

where $\lambda' = \lambda + \hat{m}(\lambda, c)$ and $\omega(\lambda)$ is defined as in (25).

Solving equation (49), however, satisfies the necessary conditions only. To be considered a solution to (32), the policy function has to generate a *stationary* time series for m , satisfying therefore the transversality condition. Being the solution unique, the optimal policy function is the only one that jointly satisfies the first order and transversality conditions.

We strictly follow the procedure described in the previous Section, and summarize the relevant statistics in Table 4. Evidently, Galerkin's method dominates its alternatives from all points of view except the median absolute residual and the computational cost. A good approximation is reached with a polynomial of degree 29 after slightly more than two minutes of wait. Note that collocation is still a feasible alternative that trades some

¹³See Stockey and Lucas (1988) for details.

		$n = 1$	$n = 9$	$n = 19$	$n = 29$	$n = 39$
Euler Equation Residuals						
Avg. Abs.	Gal.	0.11	3.61e-3	1.54e-4	7.32e-6	5.96e-7
	Col.	0.33	3.81e-3	2.18e-4	1.43e-5	7.90e-7
	<i>Int.</i>	<i>0.75</i>	<i>1.64e-2</i>	<i>4.44e-3</i>	<i>1.55e-3</i>	<i>1.15e-3</i>
Med. Abs.	Gal.	0.11	2.58e-3	8.93e-5	3.96e-6	3.51e-7
	Col.	0.06	2.98e-4	8.16e-6	3.10e-7	4.14e-8
	<i>Int.</i>	<i>0.05</i>	<i>1.11e-6</i>	<i>1.03e-9</i>	<i>1.06e-12</i>	<i>6.34e-13</i>
Std.	Gal.	0.14	6.14e-3	3.09e-4	1.77e-5	1.28e-6
	Col.	1.10	1.57e-2	9.84e-4	6.94e-5	4.43e-6
	<i>Int.</i>	<i>2.31</i>	<i>8.32e-2</i>	<i>2.65e-2</i>	<i>1.01e-2</i>	<i>7.73e-3</i>
Max. Abs.	Gal.	0.57	3.08e-2	1.81e-3	1.11e-4	7.69e-6
	Col.	6.82	0.10	6.87e-3	4.68e-4	3.11e-5
	<i>Int.</i>	<i>13.84</i>	<i>0.56</i>	<i>0.18</i>	<i>7.08e-2</i>	<i>5.47e-2</i>
Computational Time						
Sec.	Gal.	1.76	28.07	84.14	152.20	250.96
	Col.	0.06	1.82	8.57	24.72	58.11
	<i>Int.</i>	<i>0.11</i>	<i>1.70</i>	<i>8.40</i>	<i>25.65</i>	<i>61.41</i>

Table 4: Euler equation residuals: the policy function.

accuracy at the extremes of the interval in exchange of computational efficiency. Both projection methods clearly outperform Lagrange interpolation, which seems unable to provide acceptable approximations near the extremes of the interval.

Figure 7 plots the Euler equation residuals for our last experiment: as we can see, the approximation errors are larger at the boundaries, but remarkably uniform on the remaining part of the approximation interval. Figure 8 shows the optimal policy function $m(\lambda)$, together with the $\Delta m_{t+1} = 0$ and $\Delta \lambda_{t+1}$ loci. Finally, Figure 9 plots the adjustment paths for λ_t , m_t , and ν_t under the initial condition $\lambda_0 = 0.6$.

6 Conclusions

A feature of a family of models of economic geography appeared in the last decade is that, although they are simple in the basic structure, they cannot be solved explicitly. Hence, the need for numerical exploration of these models. In this paper we have illustrated three simple methods that give a remarkable improvement in the precision of approximation with respect to the commonly utilized Lagrange interpolation. We have applied this methods to the core-periphery model and we have characterized the dynamics of both its static and forward-looking expectations versions.

Coming to the merits of each of the methods, we have seen that Chebyshev interpolation is especially useful when the function to be approximated is known. In most

cases however, the functional relationship to be approximated is unknown. In these cases Chebyshev interpolation still improves remarkably the precision of approximation. Projection methods are yet more efficient. The advantage of projection methods is that the numerical solution of the implicit equation and the estimation of the coefficients of the approximating polynomial are done simultaneously. This, combined with the use of Chebyshev polynomial as elements of the interpolating polynomial and of optimally chosen interpolation nodes, reduces the approximation error to a minimum. Intuitively, this gain in efficiency can be explained by noticing that the use of Chebyshev polynomials makes the elements of the approximating polynomial orthogonal to each other and this improves the precision of estimation.

The precision of approximation is always a welcome feature but it is of particular importance in models that exhibit catastrophic behavior. The reason is that in such models small errors of approximation may lead to large errors in the prediction of the outcomes. For instance, there could be cases in which the Lagrange interpolation would give us the result that the core-periphery outcome is stable while a more precise approximation would reveal that such outcome is in fact unstable. Another way of seeing this is that the precision of approximation is particularly important in the neighborhood of where the catastrophe is likely to occur. In the case of the core-periphery model this means near the points where the slope of the indirect utility differential changes sign (at the local maximum and minimum) and also near the core-periphery outcomes. We have shown that the approximation methods, not only give a better approximation overall, but that the improvements in precision with respect to the Lagrange approximation are particularly significant near these points (i.e., far from the symmetric equilibrium).

An appealing feature of these methods is their simplicity. The simplest of them (Chebyshev interpolation) only requires to compute the interpolation nodes according to a simple formula. Even the more sophisticated projection methods only require a minimal programming effort and a negligible computation time. What makes the methods really attractive, however, is their efficiency. In all cases, with a minimal effort they give remarkable improvements in the precision of approximation. In some cases passing from Lagrange to orthogonal collocation or Galerkin makes the approximation error thousands of times smaller.

The methods we have illustrated become even more useful when the models are utilized for policy analysis. Accuracy is obviously very important in these cases. But there are also other advantages. First the approximation methods allow us to obtain (accurately) the information that are typically important in policy analysis, such as the transition dynamics. Second, a reliable approximation method gives more freedom of modeling because it removes the constraint of having to arrive to a reduced form. In this way the

introduction of policy parameters in the model can be done in a richer and freer way without having to worry that the model be solvable.

Simple and intuitive model structures are always desirable. The trouble is, that even simple and insightful structures may give rise to unsolvable models. The approximation methods we have illustrated, we think, can be of great help in such situations.

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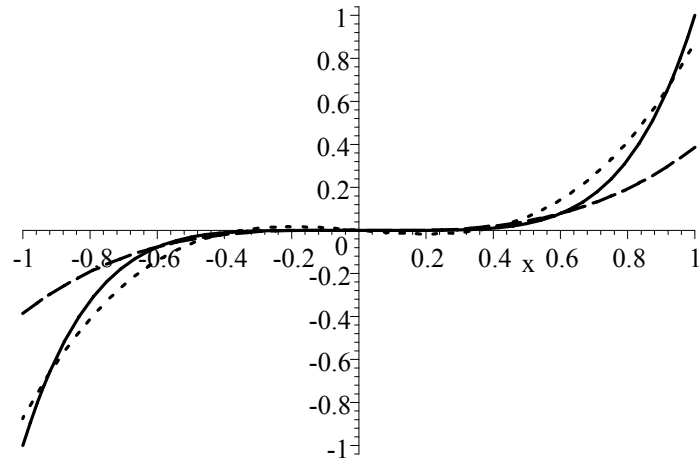


Figure 1: $f(x)$ =solid, $p_3^L(x)$ =dash, $p_3^C(x)$ =dot

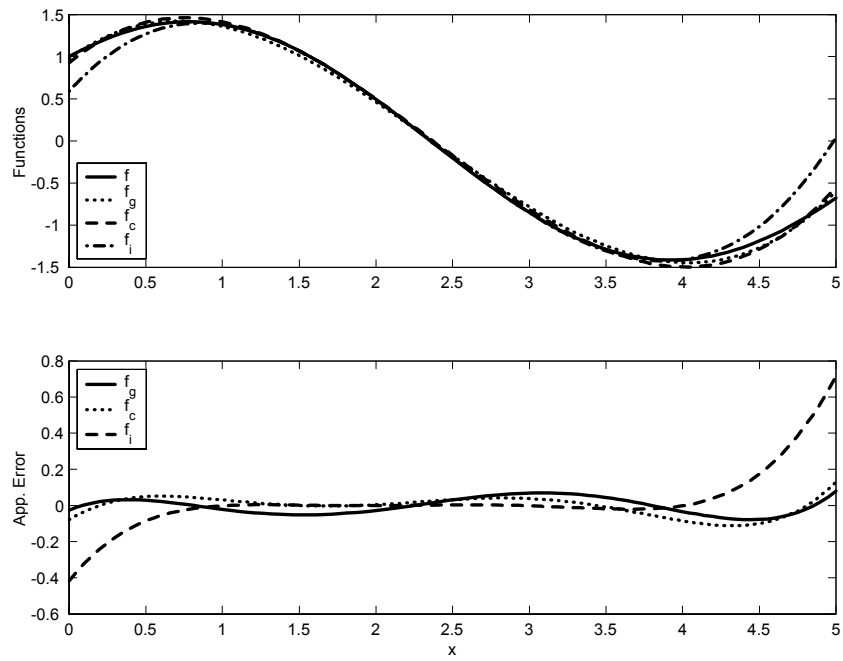


Figure 2: Approximation of a simple function.

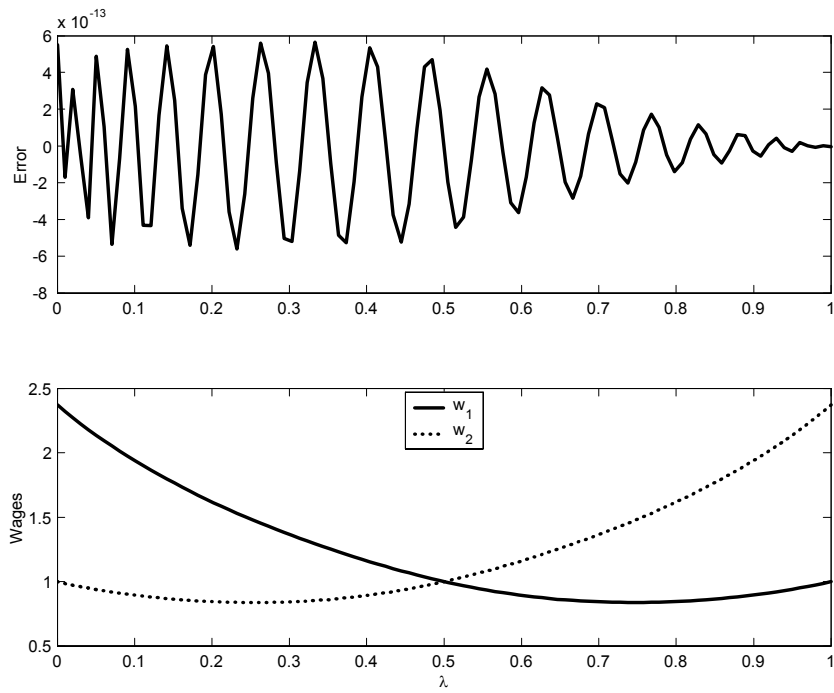


Figure 3: Wage equation residuals and approximated wages.

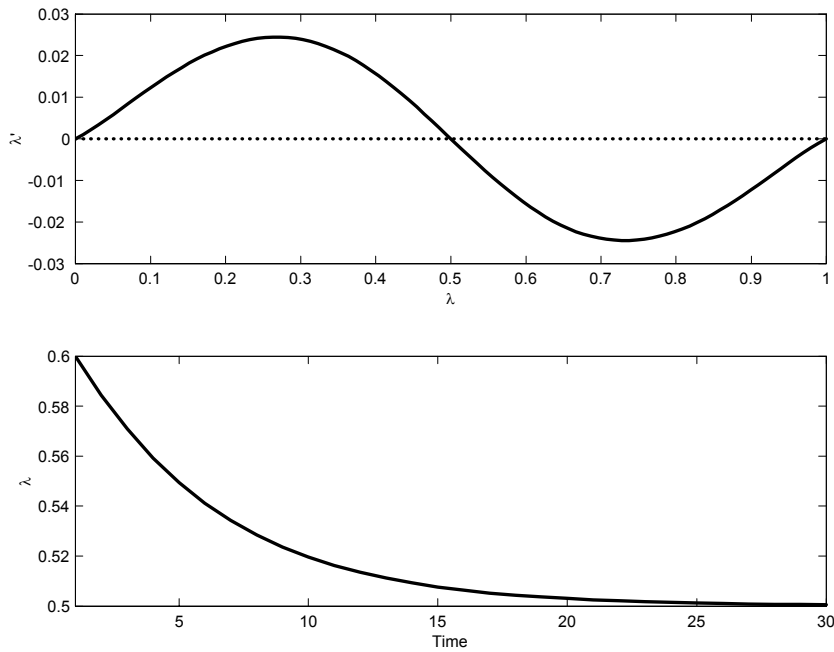


Figure 4: Phase diagram and transitional dynamics (static expectations).

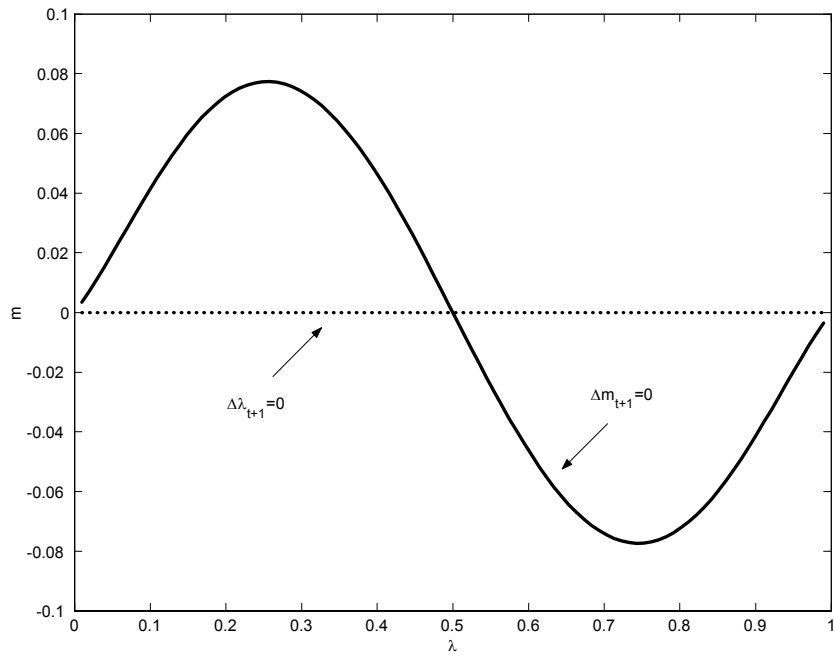


Figure 5: Phase diagram (rational expectations).

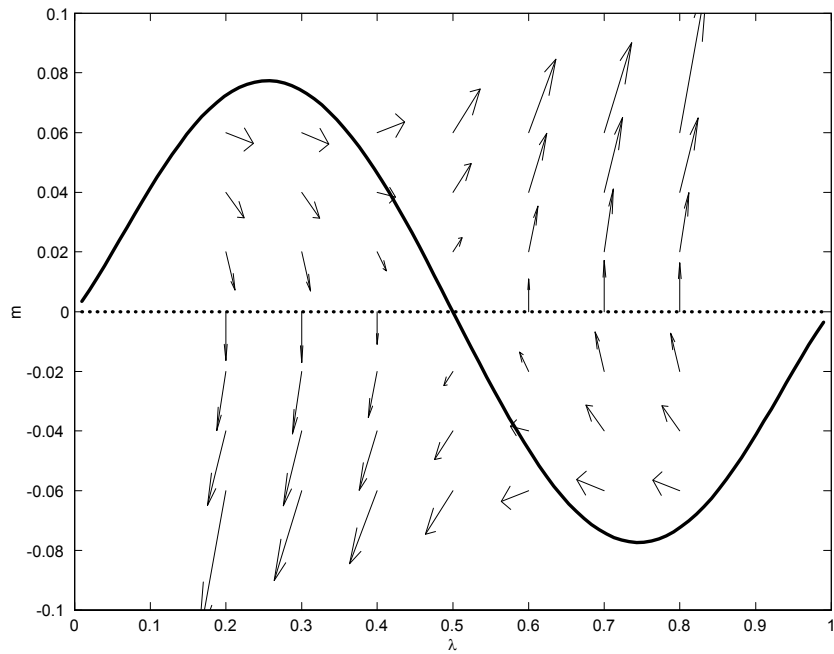


Figure 6: Velocity vectors (rational expectations).

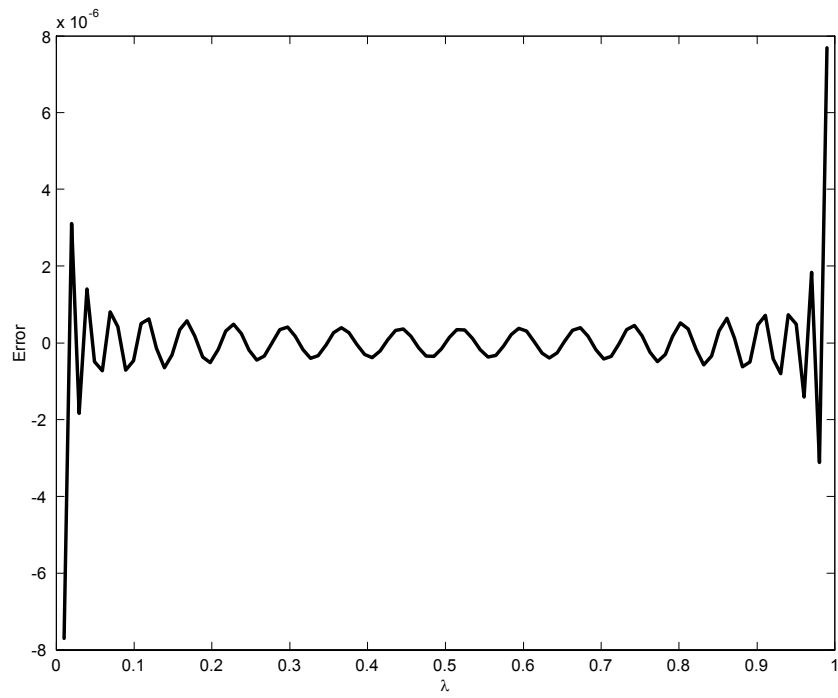


Figure 7: Euler equation residuals (rational expectations).

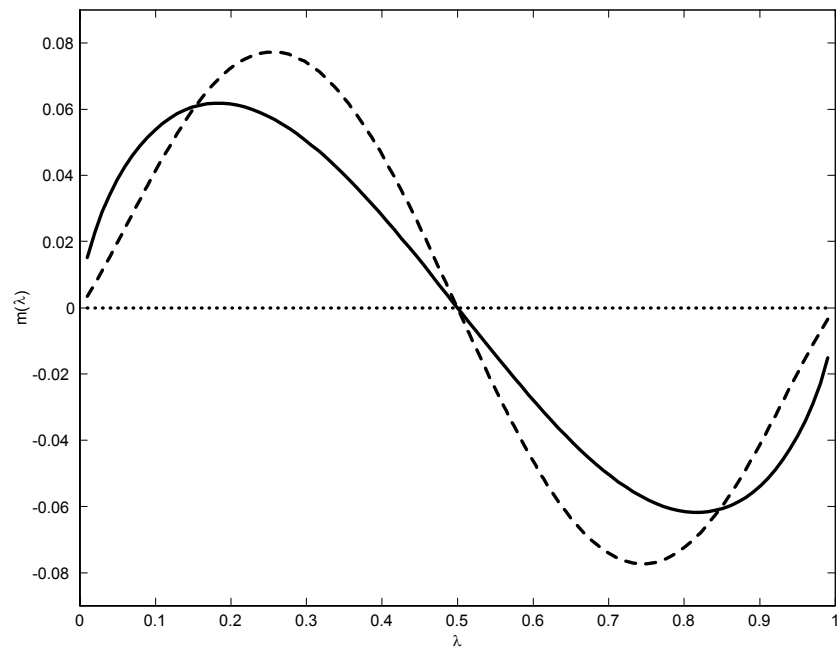


Figure 8: The policy function (rational expectations).

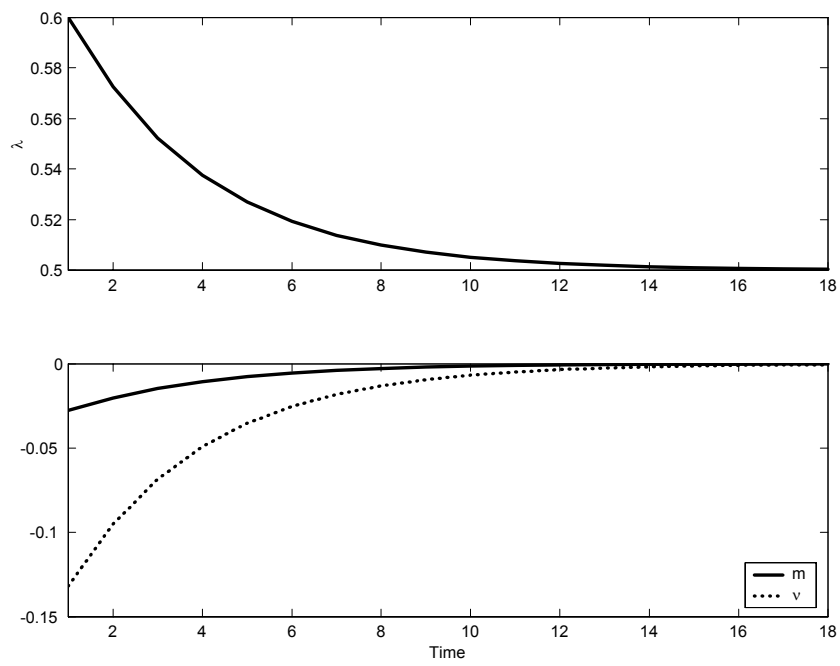


Figure 9: Transitional dynamics (rational expectations).