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Indeterminacy, Underground Activities and Tax Evasion.[*]

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Abstract

This paper introduces underground activities and tax evasion into a one sector dynamic general equilibrium model with external effects. The model presents a novel mechanism driving the self-fulfilling prophecies, which is triggered by the reallocation of resources to the underground sector to avoid the excess tax burden. This mechanism differs from the customary one, and it is complementary to it. In addition, the explicit introduction of an (even tiny) underground sector allows to reduce aggregate degree of increasing returns required for indeterminacy, and for having well behaved input demand schedules (in the sense they slope down).

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Keywords: Indeterminacy and Sunspots, Tax Evasion and Underground Activities.

1 Introduction

Self-fulfilling beliefs due to aggregate production externalities are a viable hypothesis to explain economic fluctuations (see e.g. Farmer and Guo [8]; Benhabib and Farmer [3]; Farmer [9]). This class of one-sector models, however, is affected by several undesirable features: e.g. a "too high" degree of aggregate increasing returns to scale for having indeterminacy, and an upward sloping labor demand schedule.[1]

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[1]Specifically, this class of one sector models requires returns to scale greater that 1.6, while recent estimates suggest that the United States economy returns to scale are no larger than 1.2 (see, among the others, Basu and Fernald [2], Sbordone [19], Jimenez and Marchetti [14]).

The literature proposes several solutions to overcome this difficulty: Wen [23] introduce endogenous variable capacity utilization of capital stock and/or labor hoarding, Benhabib and Farmer [4] use a multi-sector economy, Guo and Lansing [12] bring in capital maintenance, and Perli [17] explicitly introduces household production. The general spirit behind these contributions is to add at least one additional dimension to the baseline model.

This paper introduces underground activities and tax evasion into a one sector dynamic general equilibrium model with external effects, and shows that these phenomena are a possible source of local indeterminacy of the equilibrium path.[2]

We presents a one-sector dynamic general equilibrium model in which there are three agents: firms, households and a government; furthermore, there is one homogeneous consumption good and three production factors: regular labor, underground labor, and a capital stock. Government levies income taxes on regularly produced income flows, and labor taxes on regular labor services, and balances its budget (in expected terms) for each period. Firms and households, being subject to distortionary taxation, use the underground labor input to evade taxes. Government faces tax evasion originating from the underground sector, and coordinates strategy to address abusive tax evasion schemes.[3]

The model presents a novel mechanism driving the self-fulfilling prophecies, which is triggered by the reallocation of resources to the underground sector to avoid the excess tax burden. This mechanism differs from the customary one, and it is complementary to it. It turns out that prophecies of a higher expected income, triggered by a sunspot shock, are self fulfilled through a resource reallocation toward the underground labor services, which allows to evade taxes, and therefore to have additional resources (the tax wedge) for satisfying the higher desired consumption profile. In addition, the explicit introduction of distortionary taxation combined with these phenomena into a one-sector general equilibrium model allows to reduce aggregate degree of increasing returns required for indeterminacy, and for having well behaved demand schedules for production inputs (in the sense that slope down).

The model is calibrated for the United States economy, where the size of underground economy ranges

[2]There is no universal agreement on what defines the underground economy. Most recent studies use one of more of the following definitions: (a) unrecorded economy (failing to fully or properly record economic activity, such as hiring workers off-the-book); (b) unreported economy (legal activity meant to evade the tax code); (c) illegal economy (trading in illegal goods and services). Obviously, the difficulty in defining the sector extends to the estimation of its size. We are concerned with the size of the underground economy as encompassing activities which are otherwise legal but go unreported or unrecorded.

[3]Violations of the Internal Revenue Service Codes may result in civil penalties and/or criminal prosecution, which we model as a surcharge factor over customary tax rates (more details to come).

between 5 percent of GNP (Tanzi, [21]**)** and 9 percent of GDP (Schenider and Enste, [20]; Paglin [16]). Even though these figures are below the OECD countries average (17 percent, according to Schneider and Enste [20]), they still represent a significant amount of resources absconded from tax collection.[4] Notice, more importantly, that even a tiny underground sector matters for inducing local indeterminacy of the equilibrium path; and the United States underground sector's size is above this threshold.

The paper is organized as follows. Section 2 details the model; Section 3 presents the topological properties of stationary state, discusses conditions for indeterminacy and describes the theoretical mechanism. Next Section 4, presents the model's implication for the overall level of returns to scale and Section 5 checks the results' robustness through a sensitivity analysis. Finally Section 6 concludes, while proofs and derivations are included in the Appendix.

2 The Model

2.1 Firms

Assume that there exist a continuum of firms, uniformly distributed over the unit interval. Production technology for the homogenous good uses three inputs: physical capital, regular labor services, and underground labor services. The production function of firm $j \in [0, 1]$ reads:

$$y_t^j = A_t (k_t^j)^{\alpha} (n_{M,t}^j)^{1-\alpha-\rho} (n_{U,t}^j)^{\rho}, \;\; \alpha, \rho \in \,]0, 1[\,, \tag{1}$$

where k_t^j denotes capital stock, $n_{M,t}^j$ is regular labor and $n_{U,t}^j$ represents underground labor, and the quantity A_t is an aggregate production externality (defined below).[5] Production function follows a "*moonlighting production scheme*", where underground labor services use the same capital stock that is used by regular labor.[6] We could imagine, for example, that the same firm produces in the regular economy by day, and in the

[4]Shadow, informal or underground activities are a fact in many countries, and there are significant indications that this phenomenon is large and increasing. Schneiner and Enste [20] show that the estimated average size of the underground sector (as a percentage of total GDP) over 1996-97 in developing countries is 39 percent, in transition countries 23 percent, and in OECD countries about 17 percent.

[5]The model implicitly assumes that firms always use some underground labor services. In this regards this model applies to economy where there exist at least one firm hiring at least one worker on the underground labor market. We think, however, that this is still a general formulation, because it would be difficult to find economies without any form of tax evasion. In addition, official GDP estimates incorporate an estimate of the contribution produced by underground sector. This is a useful information for parameterizing the model (more details to come).

[6]Bajada [1] defines moonlighting as failure to report income from a second job; or profit-businesses that are paid in cash and do not report this additional income i.e. hair dressers may report fewer clients than they really service; expenditures over–reporting to

underground economy by night.

The aggregate production externality A_t is defined below:

$$A_t = \underbrace{\left\{(K_t^{\alpha} N_{M,t}^{1-\alpha-\rho}\right\}^{\eta}}_{\text{Marshallian Ext.}} \underbrace{\left\{N_{U,t}^{\rho}\right\}^{\zeta}}_{\text{Underground Labor Ext.}} \qquad \zeta, \eta \geq 0, \zeta \gtreqless \eta, \tag{2}$$

where capital letters denote aggregate quantities (in a perfect foresight symmetric equilibrium; details below).[7] We distinguish between the "regular" externality $\left\{(K_t^{\alpha} N_{M,t}^{1-\alpha-\rho}\right\}^{\eta}$ that is related to the well known Marshallian effect, and the underground labor input specific external effect $\left\{N_{U,t}^{\rho}\right\}^{\zeta}$.[8] Once we allow for labor heterogeneity at the firm and individuals' level, it is a natural to do the same at the aggregate one.[9] Section 4.2 discusses in more details how to pin down values for η and ζ.

As firms are homogeneous, overall level of output for a given (and equal for all firms) level of inputs utilization is given by:

$$Y_t = A_t \int_j \left\{(k_t^j)^{\alpha} (n_{M,t}^j)^{1-\alpha-\rho} (n_{U,t}^j)^{\rho}\right\} dj = K_t^{\alpha(1+\eta)} N_{M,t}^{(1-\alpha-\rho)(1+\eta)} N_{U,t}^{\rho(1+\zeta)}.$$

Increasing returns to scale are a pure aggregate phenomenon (as equation (1) suggests), and returns to scale are constant at the firms' level, as each firm takes K_t, $N_{M,t}$ and $N_{U,t}$ as given for all $t = 1, 2, ..., T, ...$.

Firms try to evade taxes on labor services, by allocating labor demand to underground labor market. Firms, however, may be detected evading, with probability $p \in (0, 1)$, and forced to pay the statutory tax rates on labor (τ_N), increased by a surcharge factor, $s > 1$, applied to the standard tax rate. [10]

decrease the amount of taxable income; the failure to report interest earnings and barter; and the exchange of goods and services for each other. Cowell [7] offers additional details.

[7] The aggregate value of a variable z_j is defined as: $Z = \int_0^1 (z_j) dj$.

[8] In the standard one-sector model with aggregate increasing returns, the externality is specified as

$$A_t = \left\{K_t^{\alpha} N_t^{1-\alpha}\right\}^{\eta}$$

while our model explicitly distinguishes between externality associated to the economy regular side, with those generated in the underground. It represents a input-specific external effect. It differs from a sector-specific externality because our model has just one homogenous good, which is produced, however, with multiple labor inputs.

[9] The characteristics of irregular labor are different form those of the regular one, giving place to a different aggregate impact of the former. Notice, moreover, that this formulation adds generality to the analysis: when $\eta = \zeta$ and there are neither tax evasion nor distortionary taxation, the model reduces to Farmer and Guo's one.

[10] The model implicitly assumes that firms always try to evade taxes by reallocating some labor demand to the underground labor. It is a consequence of the production technology: in order to have nonzero production, $N_{U,t}^j$ must be positive in equilibrium. In other words, this model would not be suited to study and economy without tax evasion, unless we impose $\rho = 0$. But in this case, the model would reduce to a standard one sector economy without labor market segmentation, which is one of the distinctive characteristics of our contribution.

When a firm is **not detected** evading (with probability $1-p$), its profit are denoted with $\pi^j_{ND,t}$.[11] If **detected** evading (with probability p), we denote firm's profits as $\pi^j_{D,t}$; both are defined below, after normalizing the the output price to unity:

$$
\begin{array}{llll}
\pi^j_t & \rightarrow & \underset{(\sim p)}{\textbf{D}\text{etected}} & \pi^j_{D,t} = y^j_t - (1+\tau_N)w_{M,t}n^j_{M,t} - (1+s\tau_N)w_{U,t}\pi^j_{U,t} - r_t k^j_t \\
 & \searrow & & \\
 & & \underset{\sim(1-p)}{\textbf{N}\text{ot }\textbf{D}\text{etected}} & \pi^j_{ND,t} = y^j_t - (1+\tau_N)w_{M,t}n^j_{M,t} - w_{U,t}n^j_{U,t} - r_t k^j_t,
\end{array}
$$

where $w_{M,t}$ and $w_{U,t}$ denote the regular and the underground sector wages, r_t is capital remuneration rate. Finally, expected profit are computed by taking linear projection, i.e. $E\pi^j_t = (1-p)\pi^j_{ND,t} + p\pi^j_{D,t}$:

$$
E\pi^j_t = y^j_t - (1+\tau_N)w_{M,t}n^j_{M,t} - (1+ps\tau_N)w_{U,t}n^j_{U,t} - r_t k^j_t, \tag{3}
$$

where E denotes an expectation operator.

Here the parameter $s > 1$ represents the surcharge on the statutory tax rate that a firm, detected employing workers in the underground labor market, must pay.

As markets are competitive, firm's behavior is described by the first order conditions for the (expected) profit maximization, with respect to k^j_t, $n^j_{M,t}$ and $n^j_{U,t}$:

$$
\left\{
\begin{array}{c}
\frac{\partial y^j_t(A_t)}{\partial k^j_t} = r_t \\
\frac{\partial y^j_t(A_t)}{\partial n^j_{M,t}} = (1+\tau_N)w_{M,t} \\
\frac{\partial y^j_t(A_t)}{\partial n^j_{U,t}} = (1+sp\tau_N)w_{U,t}
\end{array}
\right\}. \tag{4}
$$

Concavity of the production function (recall that firms take A_t as a constant) ensures the existence of a unique solution.

[11] We assume that the probability being detected is exogenous; this assumption is meant to reflect the fact the actual probability to be controlled by the Internal Revenue Service is exogenous, because it follows a random extraction process. From a theoretical perspective, however, it would be interesting endogenize this probability. A natural way would be to assume that is depends on the evasion rate and/or on the amount of taxes evaded, for example. We leave, however, these developments to future investigation.

2.2 Households

Suppose that there exist a continuum of households, uniformly distributed over the unit interval. The $h-$th household's preference are represented by the following momentary utility function:

$$\mathcal{V}_t^h = \log(c_t^h) - B_0 \frac{(n_{M,t}^h + n_{U,t}^h)^{1+\xi}}{1+\xi} - B_1 \frac{(n_{U,t}^h)^{1+\psi}}{1+\psi}, \quad B_0, B_1 \geq 0,$$

where c_t^h denotes household's consumption flow, $n_{M,t}^h$ and $n_{U,t}^h$ denote regular and underground labor supplies; the quantity $B_0 \frac{(n_{M,t}^h + n_{U,t}^h)^{1+\xi}}{1+\xi}$, represents the overall disutility of working, while the last term, $B_1 \frac{(n_{U,t}^h)^{1+\psi}}{1+\psi}$, reflects the idiosyncratic cost of working in the underground labor market. Specifically, this cost may be associated with the lack of any social and health insurance in the underground sector. Finally, the parameters ξ and ψ represent the inverse labor supply elasticities of aggregate and underground labor supplies, respectively.

The representative household evades income taxes by reallocating labor services from regular to underground labor markets. Underground-produced income flows $w_{U,t} n_{U,t}^h$ are, therefore, not subject to distortionary income tax rate τ_Y, as the budget constraint below suggests:[12]

$$c_t^h + k_{t+1}^h = (1 - \tau_Y)\left(w_{M,t} n_{M,t}^h + r_t k_t^h\right) + w_U n_{U,t}^h + (1-\delta) k_t^h, \tag{5}$$

where k_{t+1}^h is next period capital stock, and δ denotes a quarterly capital stock depreciation rate.

Imposing a constant subjective discount rate $0 < \beta < 1$, and defining μ_t^h as the costate variable, the representative households maximizes the Lagrangian $\mathcal{L}_0^h$

$$\begin{aligned} \max_{\left\{c_t^h, n_{M,t}^h, n_{U,t}^h, k_{t+1}^h\right\}_{t=0}^{\infty}} \mathcal{L}_0^h \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^t \mathcal{V}_t^h + \\ & + E_0 \sum_{t=0}^{\infty} \mu_t^h \left\{ (1-\tau_Y) \left(w_{M,t} n_{M,t}^h + r_t k_t^h \right) + w_U n_{U,t}^h - k_{t+1}^h + (1-\delta) k_t^h \right\}, \end{aligned}$$

subject to the budget constraint (5). Optimal households' choice is characterized by the following first order conditions:

[12] Tax rate τ_Y is not a legitimate income tax, because part of produced income is not taxed. Specifically, it is an income tax applies to declared income flows. In addition, a value added tax would be part of an optimal tax policy in the presence of underground activities (although this tax also can be evaded). We abstract, however, from this tax and we leave it for further investigations.

$$\left\{\begin{array}{c}\beta^t (c_t^h)^{-1} = \mu_t^h \\ \beta^t B_0 (n_{M,t}^h + n_{U,t}^h)^{\xi} = \mu_t^h (1-\tau_Y) w_{M,t} \\ \beta^t B_0 (n_{M,t}^h + n_{U,t}^h)^{\xi} + \beta^t B_1 (n_{U,t}^h)^{\psi} = \mu_t^h w_{U,t} \\ E_t \left\{ \mu_{t+1}^h \left[(1-\delta) + (1-\tau_Y)\, r_{t+1} \right] \right\} = \mu_t^h \\ \lim_{T\to\infty} E_0 \mu_T^h k_T^h = 0. \end{array}\right\} \tag{6}$$

2.3 Government

The government budget reads

$$\tau_Y (w_{M,t} N_{M,t} + r_t K_t) + sp\tau_N w_{U,t} N_{U,t} + \tau_N w_{M,t} N_{M,t} = G_t, \tag{7}$$

where the left hand side denotes expected government revenues that are allocated to aggregate government expenditure G_t, which is is assumed to be wasteful. Capital letters denote aggregate equilibrium quantities (defined below). Government balances its budget in expected terms since tax revenues collected from the underground side corporate sector depend on the probability of being detected p.

2.4 Perfect Foresight Equilibrium

We focus on a perfect foresight equilibrium in which households maximize utility and firms make zero profits. In equilibrium the aggregate consistency requires that $y_t = Y_t$, $k_t = K_t$, $n_{M,t} = N_{M,t}$, $n_{U,t} = N_{U,t}$, $c_t = C_t$, where lower capital letters denote individual equilibrium quantities, and capital letters denote aggregate equilibrium quantities[13]. As a result, the first order conditions characterizing the equilibrium are given by:

[13] Note that the aggregate resource constraint holds: $C_t + I_t + G_t = Y_t$. Substitute first the expression of G_t from (7) into the resource constraint $C_t + I_t + G_t = Y_t$; then, by manipulating the resulting expression, the aggregate version of households' budget (5) obtains.

$$B_0(N_{M,t}+N_{U,t})^{\xi}=(C_t)^{-1}(1-\tau_Y)\frac{(1-\alpha-\rho)}{(1+\tau_N)}\frac{Y_t}{N_{M,t}} \tag{8}$$

$$B_0(N_{M,t}+N_{U,t})^{\xi}+B_1(N_{U,t})^{\psi}=(C_t)^{-1}\frac{\rho}{1+sp\tau_N}\frac{Y_t}{N_{U,t}} \tag{9}$$

$$(C_{t+1})^{-1}\left((1-\delta)+(1-\tau_Y)\,\alpha\frac{Y_{t+1}}{K_{t+1}}\right)=(C_t)^{-1} \tag{10}$$

$$K_{t+1}=\left((1-\tau_Y)\left(\frac{(1-\alpha-\rho)}{(1+\tau_N)}+\alpha\right)+\rho\right)K_t^{\phi_1}N_{M,t}^{\phi_2}N_{U,t}^{\phi_3}+(1-\delta)\,K_t-C_t \tag{11}$$

$$\lim_{t\to\infty}(C_t)^{-1}K_t=0 \tag{12}$$

where $Y_t=K_t^{\phi_1}N_{M,t}^{\phi_2}N_{U,t}^{\phi_3}$, $\phi_1=\alpha\,(1+\eta)$, $\phi_2=(1-\alpha-\rho)\,(1+\eta)$ and $\phi_3=(1+\zeta)\,\rho$.

2.5 Stationary State

Proposition 1 shows that the model has a unique stationary state for capital stock, and unique values for equilibrium regular and underground labor services. The stationary state quantities are derived under perfectly elastic labor supply schedules ($\xi=\psi=0$).[14]

Proposition 1 *For $\xi=\psi=0$ there exist a unique stationary capital stock $K^\star>0$, and a unique stationary equilibrium for regular labor supply $N_M^\star>0$, and unique stationary ratio $\left(\frac{N_U}{N_M}\right)^\star$ such that:*

$$\begin{aligned}
K^\star &\simeq \left(\frac{(1-\tau_Y)\,\alpha}{\beta^{-1}-1+\delta}\right)^{\frac{1}{1-\alpha(1+\eta)}}\left(\frac{1+\tau_N}{1+sp\tau_N}\frac{B_0\rho\,(1-\alpha-\rho)^{-1}}{(B_0+B_1)\,(1-\tau_Y)}\right)^{\frac{\rho(1+\zeta)}{1-\alpha(1+\eta)}}\left(N_M^\star\right)^{\frac{(1-\alpha)(1+\eta)}{1-\alpha(1+\eta)}};\\
N_M^\star &\simeq \frac{(1-\alpha-\rho)}{B_0\left((1-\tau_Y)\left(\bar{\Psi}_M+\frac{\beta^{-1}-1+\delta}{1-\tau_Y}\right)+\bar{\Psi}_U-\delta\right)}\left(\frac{\beta^{-1}-1+\delta}{\alpha\,(1+\tau_N)}\right);\\
\left(\frac{N_U}{N_M}\right)^\star &= \frac{B_0\rho\,(1-\alpha-\rho)^{-1}}{(B_0+B_1)\,(1-\tau_Y)}\frac{1+\tau_N}{1+ps\tau_N},
\end{aligned}$$

where $\frac{(1-\alpha-\rho)}{(1+\tau_N)}\frac{\beta^{-1}-1+\delta}{(1-\tau_Y)\alpha}=\overline{\Psi}_M$ and $\frac{\rho}{(1+sp\tau_N)}\frac{\beta^{-1}-1+\delta}{(1-\tau_Y)\alpha}=\overline{\Psi}_U$.

Proof. see Appendix. ■

[14] This is a customary assumption, commonly accepted in this literature; see among the many Farmer and Guo [8]. It also allows to find a closed form for the stationary state. Indeterminacy arises, however, for ξ, $\psi\neq 0$, as well. It can be shown, in addition, that the larger the aggregate increasing returns to scale, the larger is the ξ,ψ parameter region where indeterminacy arises.

The stationary equilibrium value for capital stock is derived under an approximation, which is necessary in order to obtain a closed form for $K^\bigstar$; it can be shown, however, that results would not be significantly different if we derived its value numerically.

3 Topological Properties and an Original Theoretical Mechanism

3.1 Topological Properties

To study topological properties of the stationary state derived in Proposition 1, we arrange the system of linearized equations so to obtain a planar dynamical system in the state vector $\widehat{S}_t = \left(\widehat{K}_t; \widehat{C}_t\right)$ (hat-variables denote percentage deviations from their steady state values):

$$E_t \widehat{S}_{t+1} = \mathcal{W} \widehat{S}_t + \Omega \mathcal{E}_{t+1},$$

where $\mathcal{E}_{t+1}$ is a 2×1 vector of one step ahead forecasting errors satisfying $E_t \mathcal{E}_{t+1} = 0$ and Ω is a coefficient matrix. Its first element $\widehat{K}_{t+1} - E_t \widehat{K}_{t+1}$ equals zero, since K_{t+1} is known at period t; denote the second element with $\tilde{\varepsilon}_c = \widehat{C}_{t+1} - E_t \widehat{C}_{t+1}$.

We now introduce the following standard definition.

Definition 1 *The stationary state $K^\bigstar$ is called locally indeterminate if there exists $\epsilon > 0$ such that from any K_0 belonging to $\left(K^\bigstar + \epsilon, K^\bigstar - \epsilon\right)$ there are infinitely many equilibrium paths converging to the steady state.*

Now, when the model has a unique equilibrium (i.e., one of the eigenvalues of $\mathcal{W}$ lies outside the unit circle), the optimal decision rule for consumption does not depend on the forecasting error, $\tilde{\varepsilon}_c$.[15] If both eigenvalues of $\mathcal{W}$ lie inside the unit circle, however, the model is indeterminate in the sense that any value of $\widehat{C}_t$ is consistent with equilibrium given $\widehat{K}_t$. Hence, the forecasting error $\tilde{\varepsilon}_c$ can play a role in determining the equilibrium level of consumption. Under indeterminacy the decision rule for consumption at time t take the special form

$$\widehat{C}_t = w_{21} \widehat{K}_{t-1} + w_{22} \widehat{C}_{t-1} + \omega_2 \tilde{\varepsilon}_{c,t},$$

[15] Specifically, in that case $\widehat{C}_t$ can be solved forward under the expectation operator E_t to eliminate any forecasting errors associated with future consumption. Then the optimal decision rules at time t depend only on the current capital stock $\widehat{K}_t$.

where w_{21} and w_{22} are the second row elements of the matrices $\mathcal{W}$ and Ω. The condition $E_t\tilde{\varepsilon}_{c,t+1} = 0$ ensures that rational agents do not make systematic mistakes in forecasting future based on current information. Since $\tilde{\varepsilon}_{c,t}$ can reflect a purely extraneous shock, it can be interpreted as shock to autonomous consumption.

3.2 Conditions for Indeterminacy

Theorem1 derive necessary and Sufficient conditions for local indeterminacy of the equilibrium path (that is for both eigenvalues of the $\mathcal{W}$ matrix lying inside the unit circle) under the assumption of perfectly elastic labor supply schedules ($\xi = \psi = 0$).[16]

Theorem 1 *For $\xi = \psi = 0$ the model equilibrium is locally indeterminate when the following necessary (NC) and sufficient (SC) conditions hold:*

$$\begin{aligned} \boldsymbol{NC} &: \quad \beta < (1+\eta)\left\{\left(1-(1-\alpha-\rho)\frac{\tau_N}{1+\tau_N}\right)+\rho\left(\frac{1}{(1-\tau_Y)(1+sp\tau_N)}-1\right)\right\} \\ \boldsymbol{SC} &: \quad \frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1} < \rho(1+\zeta)+(1-\alpha-\rho)(1+\eta) < \max\left\{\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right\}, \end{aligned}$$

where; $\underline{\mathcal{R}}$, and $\overline{\mathcal{R}}$ are two quantities (defined in the appendix) such that $1 < \underline{\mathcal{R}} < \overline{\mathcal{R}}$.

Proof. see Appendix. ■

To present a neat economic interpretation it is convenient to rewrite the necessary condition in terms of the steady state share of disposable income, and the sufficient conditions in terms of elasticities and cross-elasticities of the demand schedules for capital, regular and underground labor with respect to the three production inputs.

3.2.1 The Necessary Condition.

The necessary condition suggests that distortionary taxation and tax evasion, combined together, allow to reduce the degree of increasing returns to scale (the parameter η) required for having indeterminacy.[17] It is here necessary to distinguish between τ_N and τ_Y.

[16]See footnote 14.

[17]When there is no tax evasion (and therefore no underground labor: when $\tau_N = \tau_Y = \rho = 0$) the necessary conditions reduces to $\beta < (\eta+1)$. The condition for indeterminacy in the standard Farmer and Guo model is $\frac{1}{1-\alpha} < 1+\eta$, which is, however, a necessary and sufficient condition. We would get it by combining together our necessary and sufficient conditions for no-evasion scenario.

Taking τ_Y as given, that the higher the probability of being detected evading p (and/or the penalty surcharge factor s), the more difficult becomes to allocate resources to the underground sector, and the smaller the quantity $\left(\frac{1}{(1-\tau_Y)(1+sp\tau_N)} - 1\right)$ gets. Consider the extreme case where tax evaders are punished with an infinitely large penalty (that is $s \to \infty$). The NC reads: $\beta < (1+\eta)\left\{\left(1-(1-\alpha-\rho)\frac{\tau_N}{1+\tau_N}\right)-\rho\right\}$, suggesting that parameter region for indeterminacy shrinks when tax evasion becomes extremely costly, and it fails if labor taxes are too high ($\tau_N > \frac{\beta-(1+\eta)(1-\rho)}{\alpha(1+\eta)-\beta}$, for example).[18] Indeed, when tax evasion is extremely costly/risky and when taxes are higher than a certain threshold, they "*tax away*" the externality, in the spirit of Guo and Lansing [11].

The picture is different if we take τ_N as given. An increase in income tax rate τ_Y monotonically increases the quantity $\left(\frac{1}{(1-\tau_Y)(1+sp\tau_N)} - 1\right)$, easing, by this hand, the necessary condition. That happens because there is no probability of being detected evading income taxation (on the households' side). Therefore, the higher income tax rate, the higher would be underground labor supply; in this sense, resources would be reallocated toward an input that ensures a tax-free externality. In this case we cannot claim that higher income tax rates tax away the externality.

In summary, this analysis suggests that underground economy and tax evasion can be considered as additional economic phenomena inducing local indeterminacy of the equilibrium path.

3.2.2 The Sufficient Condition

The sufficient conditions can be re-written in terms of the cross-elasticity of regular (underground) labor demand with respect underground (regular) labor services ($\varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} = \rho(1+\zeta)$ and $\varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D} = (1-\alpha-\rho)(1+\eta)$, respectively).

$$\frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1} < \varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} + \varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D} < \max\left\{\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right\}$$

This condition suggests that the labor demand schedules should have a sufficiently large response to changes in equilibrium employment (that is the lower bound inequality $\frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1} < \varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} + \varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D}$), but, at the same time, that this response should not be too large (that is the upper bound inequality $\varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} + \varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D} < \max\left\{\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right\}$). This condition represents a building block of the theoretical mechanism described

[18] The NC condition fails when $\beta > (1+\eta)\left\{\left(1-(1-\alpha-\rho)\frac{\tau_N}{1+\tau_N}\right)-\rho\right\}$. This inequality can be recast in terms of τ_N obtaining $\tau_N > \frac{\beta-(1-\rho)(1+\eta)}{\alpha(1+\eta)-\beta}$. Notice, moreover, that the quantity $\frac{\beta-(1-\rho)(1+\eta)}{\alpha(1+\eta)-\beta}$ is quite small for reasonable parameters' value.

below (see Section 3.3).

To better appreciate the economic intuition behind this sufficient condition, then, it can be rewritten in terms of elasticity of labor demand schedules to changes in capital stock ($\varepsilon_{\widehat{K}}^{\widehat{L}_M^D}$ and $\varepsilon_{\widehat{K}}^{\widehat{L}_U^D}$); it yields:

$$\varepsilon_{\widehat{K}}^{\widehat{L}_M^D} \text{ and } \varepsilon_{\widehat{K}}^{\widehat{L}_U^D} > \frac{s_I}{s_I + s_C} \left\{ 1 + \frac{(1-\delta)(1-\beta)}{\delta} \left(\varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} + \varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D} \right) \right\},$$

suggesting that regular and underground labor demands should reacts more to changes in capital stocks rather than changes in labor services, *ceteris paribus*. [19] In other words, the shifts of labor demands driven by changes in capital stock should be "larger" than those driven by changes in labor services. Again, this result is important for understanding model's mechanism presented below.

3.3 The Theoretical Mechanism

The theoretical mechanism driving the economy under self-fulfilling beliefs differs from the customary one, and this section describes it by focusing on the labor markets.

Suppose a sunspot shock hits the economy.[20] Agents, then, formulate a conjecture on future income and consumption, according to which they believe to be more wealthy. Hence, they would like to consume more and - at the beginning - to work less. The household first order conditions suggest that both labor supply schedules would shift up (Figure 1).

Now comes the key difference compared with customary one sector model where labor demand slopes upward. In our model both labor demand schedules are sloping down for our parameterization, and therefore the economy would begin plummeting into a recession. [21] But, in a perfect foresight equilibrium agents are aware that they could reduce their tax burden, increase their disposable income, and therefore could afford the higher conjectured consumption, by allocating resources to the underground input. Notice that this is possible only when the necessary condition is satisfied.[22]

[19] Technically speaking $\varepsilon_{\widehat{K}}^{\widehat{L}_M^D}$ and $\varepsilon_{\widehat{K}}^{\widehat{L}_U^D}$ should be larger than $\varepsilon_{\widehat{N}_U}^{\widehat{L}_M^D} + \varepsilon_{\widehat{N}_M}^{\widehat{L}_U^D}$, which is also reduced by quantities $\frac{s_I}{s_I+s_C}$ and $\frac{(1-\delta)(1-\beta)}{\delta}$.

[20] In a discrete time model like this, however, everything happens simultaneously, and the following description has just an explanatory purpose.

[21] When labor demand schedule is upward sloping, as in the benchmark one sector model with indeterminacy, an upward shift of labor supply schedules increases equilibrium labor services; this supports an increase in income flow, and generates an expansion. This would be consistent with the households' conjectures, and would lead the economy into a self-fulfilled expansion.

[22] The necessary condition suggests that "*distortionary taxation should not drain too much resources away from the private sector, in order to allow the latter to form self-fulfilling beliefs*".

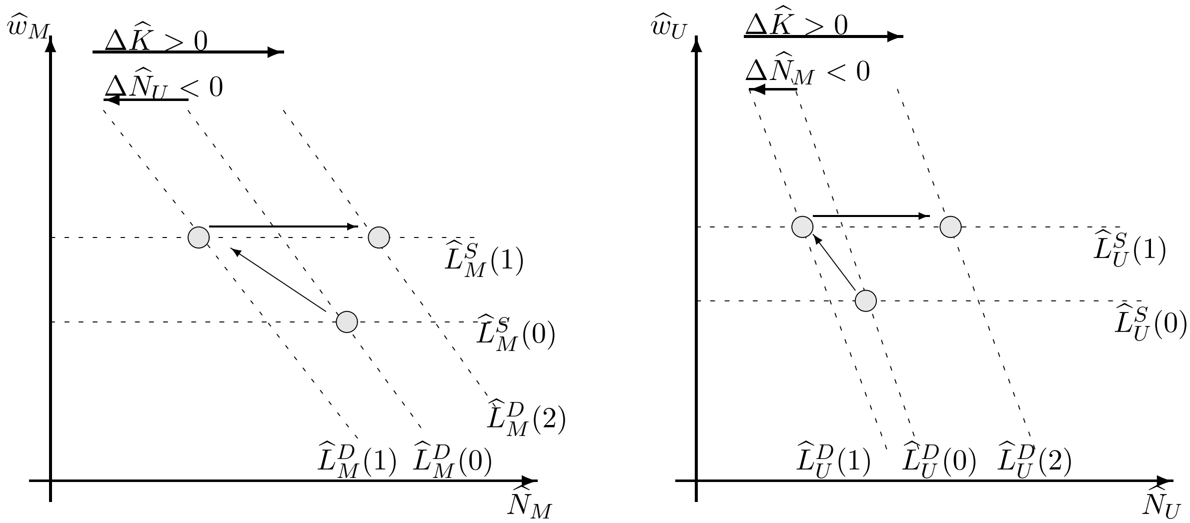

Figure 1: **Theoretical Mechanism**. Regular $\left(L^S_M\right)$ and underground $\left(L^S_U\right)$ labor supply schedules shift upward (to $L^S_{M,S}(1)$) after a positive sunspot shock; economy would enter into a recession ($\Delta N_M < 0$ and $\Delta N_U < 0$), as labor demands $L^D_M(0)$ and $L^D_U(0)$ are negatively sloped. The cross-interaction between labor markets would further strengthen the inward shifts of labor demands. But, in a perfect foresight equilibrium, the labor input reallocation toward underground labor input would increase households' disposable income, who can consume and invest more. This triggers the capital accumulation ($\Delta K > 0$) that shifts out both labor demand schedules to $L^D_M(2)$ and $L^D_U(2)$.

The resource reallocation toward underground labor would then trigger an expansionary mechanism, because it increases returns to capital, regular wage, and therefore equilibrium capital stock and equilibrium regular labor. This in turn, would raise underground labor wage rate, inducing a further expansion in the economy.[23] The sufficient condition ensures that the expansionary shifts of labor demand schedules due to capital stock dominates.[24] This mechanism, eventually, would push the economy into an expansion, which fulfills the initial prophecy of higher consumption.

4 Numerical Results

The introduction of underground economy ensures that labor demand schedules are well behaved (in the sense that slope down in the wage-employment plane). To have a complete understanding of the model, we then calibrate it for the United States economy and compare the impulse response functions to a positive sunspot shock with those generated with the benchmark one sector model. Finally a sensitivity analysis exercise shows the results' robustness, and underlines that even a tiny underground sector (in terms of GDP percentage points) matters for inducing local indeterminacy of the equilibrium.

4.1 Well behaved labor demand schedules

Both labor demand schedules are well behaved (in the sense that they slope down), compared to standard one-sector economy models where the necessary and sufficient condition for indeterminacy requires that labor demand is upward sloping. Just observe that in our model $\frac{\partial \widehat{L}_M^D}{\partial \widehat{N}_M} = (1+\eta)(1-\alpha-\rho)-1 < 0$ and $\frac{\partial \widehat{L}_U^D}{\partial \widehat{N}_U} = (1+\zeta)\rho - 1 < 0$ are both negative, as long as the following condition holds:

Condition 1 *Linearized labor demand schedules slope in the planes* $\left(\widehat{N}_M, \widehat{w}_M\right)$ *and* $\left(\widehat{N}_U, \widehat{w}_U\right)$ *down if* $\eta^* < \frac{\alpha+\rho}{1-\alpha-\rho}$ *and* $\zeta^* < \frac{1-\rho}{\rho}$.

Condition 1 is satisfied for our parameterizations n.1 and n.2 (Table II): for $\alpha = 0.23$ and $\rho = 0.088$, the regular externality parameter should be less than $\eta^* = 0.46$ and the underground specific externality

[23] To better appreciate this claim, observe that the regular and underground labor demand schedules can be written as follows from firm's first order conditions (4): $w_M = L_M^D(\overset{-}{N}_M; \overset{+}{N}_U, \overset{+}{K})$ and $w_U = L_U^D(\overset{-}{N}_U; \overset{+}{N}_M, \overset{+}{K})$ respectively. The labels denote the sign of corresponding partial derivatives.

[24] Recall that the second sufficient condition suggests that "*the shifts of labor demands driven by changes in capital stock should be larger than those driven by changes in labor services.*" Hence the increases in capital stock pushes the economy toward the expansion.

parameter should be less than $\zeta^* = 11.50$.[25] Notice that in a model with no underground labor input (that is for $\rho = 0$), this condition would be much more restrictive and not compatible with the stationary state being indeterminate. Indeed, for $\rho = 0$ regular labor demand schedule slopes down if $\eta^* < 0.2987$, and for this figure the model stationary state is saddle path stable. Hence the introduction of underground labor input eases the conditions for having a sloping down labor demand schedule. A similar argument applies for underground labor demand schedule.

4.2 Parameterization and Impulse Response Functions

Parameterization. The model is parameterized for the U.S. economy; calibration is based on seasonally adjusted series from 1970 to 2001, expressed in constant 1995 prices. Actual data for the United States economy are drawn from Farmer [9]. The system of equations we use to compute the dynamic equilibria of the model depends on a set of 14 parameters. **Five** pertain to household preferences, $(B_0, B_1, \xi, \psi, \beta)$, **four** to the institutional context (the probability of a firm being detected evading p, and the surcharge factor s, the two tax rates τ_Y and τ_N), and the remaining **five** parameters to technology (the factor shares α, ρ, the capital depreciation rate δ, the "regular" externality parameter η and the one associated to the underground labor ζ). A starred parameter denotes the precise calibrated value. **Table I** below includes calibrated values of all parameters.

Table I

The calibration of parameters that are more closely related to tax evasion and the underground economy deserve more attention.[26] For the *probability of being detected* p, we rely on Joulfaian and Ride [15], which estimate that the probability of auditing in the US ranges between 4.6% and 5.7%. We choose the higher value, $p^* = 0.057$, but results do not significantly change if we consider the lower value 4.7%.

To stop tax evasion schemes, the Internal Revenue Service has recently undertaken a national coordinated strategy to address abusive tax evasion schemes.[27] Violations of the Internal Revenue Code may result in civil penalties, which includes a fraud penalty up to 75% of the underpayment of tax attributable to the fraud in

[25] This result is robust to a large set of parameters, as long as the regular externality η is sufficiently small. The quantity $(1-\alpha-\rho)$ is in fact positive and small. If η gets too big, the slope of regular labor demand schedule $(1+\eta)(1-\alpha-\rho)-1$ becomes positive.

[26] Busato and Chiarini [5] calibrate a model for the Italian Economy, and Conesa, Diaz-Moreno and Galdon-Sanchez [6] calibrate a model for the Spanish economy.

[27] For more details about the Internal Revenue Service policy regarding abusive schemes, refer to Internal Revenue Service Public Announcement Notice 97-24, which warns taxpayers to avoid abusive tax evasion schemes that advertise bogus tax benefits.

addition to the taxes owed. Therefore we set the *surcharge factor* $s^* = 1.75$.[28]

The calibration of the externality parameters η and ζ little more attention, as well. They are calibrated by using the regular labor demand schedule and the aggregate production function. More precisely, rewrite these two equations in terms of the empirically-known macroeconomic ratios $\frac{K}{N_M}$, $\frac{N_U}{N_M}$, take a logarithmic transformation, and then solve for η and ζ. We obtain $\eta^* = 0.62$ and $\zeta^* = 0.15$.[29]

Table II compares the baseline and two alternative model's parameterization with the benchmark model (Farmer and Guo [8]) and with actual data.

$$\eta^* = \frac{-\ln\frac{(1-\alpha-\rho)}{1+\tau_N} - \alpha\ln\left(\frac{K}{N_M}\right) - \rho\ln\left(\frac{N_U}{N_M}\right) + \ln w_M}{\left(\alpha\ln\left(\frac{K}{N_M}\right) + \ln N_M + \rho\ln\left(\frac{N_U}{N_M}\right)\right)} \text{ and } \zeta^* = \frac{\ln Y - \alpha\left(1+\eta^*\right)\ln\left(\frac{K}{N_M}\right) - \rho\ln\left(\frac{N_U}{N_M}\right) - \left(1+\eta^*\right)\ln N_M}{\rho\ln\left(\frac{N_U}{N_M}\right)}$$

Table II

This parameterization, however, presents a quite large degree of aggregate increasing returns to scale. It is reasonable to consider it as an upper bound for the calibrated values and as a useful indication for the relative sizes of the two parameters. The baseline parameterization and the alternative parameterization in Table II show that the degree of increasing returns can be lowered even more. Section 5 offers additional details[30].

Impulse Response Functions. It is then interesting to compare the model's impulse response function (solid line) to an i.i.d. positive sunspot of shock with those of the benchmark one (dashed line, in **Figure 3** below).

A sunspot shock increases consumption, equilibrium total employment and production output. Investment falls on the impact (to accommodate the higher consumption profile), but then recovers. All dynamic responses follow a non-monotone pattern, and revert back to the stationary state.

[28] Violations may also result in criminal prosecution; in this case there are penalties up to five years in prison for each offense.

[29] More precisely, rewrite these the regular labor demand schedule and the aggregate production function in terms of $\frac{K}{N_M}$, $\frac{N_U}{N_M}$, take a logarithmic transformation, and then solve for η and ζ. The final expressions read:

$$\eta^* = \frac{-\ln\frac{(1-\alpha-\rho)}{1+\tau_N} - \alpha\ln\left(\frac{K}{N_M}\right) - \rho\ln\left(\frac{N_U}{N_M}\right) + \ln w_M}{\left(\alpha\ln\left(\frac{K}{N_M}\right) + \ln N_M + \rho\ln\left(\frac{N_U}{N_M}\right)\right)} \text{ and } \zeta^* = \frac{\ln Y - \alpha\left(1+\eta^*\right)\ln\left(\frac{K}{N_M}\right) - \rho\ln\left(\frac{N_U}{N_M}\right) - \left(1+\eta^*\right)\ln N_M}{\rho\ln\left(\frac{N_U}{N_M}\right)}$$

Data Source: Y: NIPA Table 1.4.1 (revised Feb 2004); N_M : NIPA Table 6.4B (revised Feb 2004); GDP Deflator: NIPA Table 1.5.1 (revised Feb 2004); Capital labor ratio $\frac{K}{N_M}$: Ghosal [10].

[30] The computations show here and in section 5 have been done not taking into account the approximation in the derivation of the steady state values discussed in section 2.5. Anyway the error is sufficiently small.

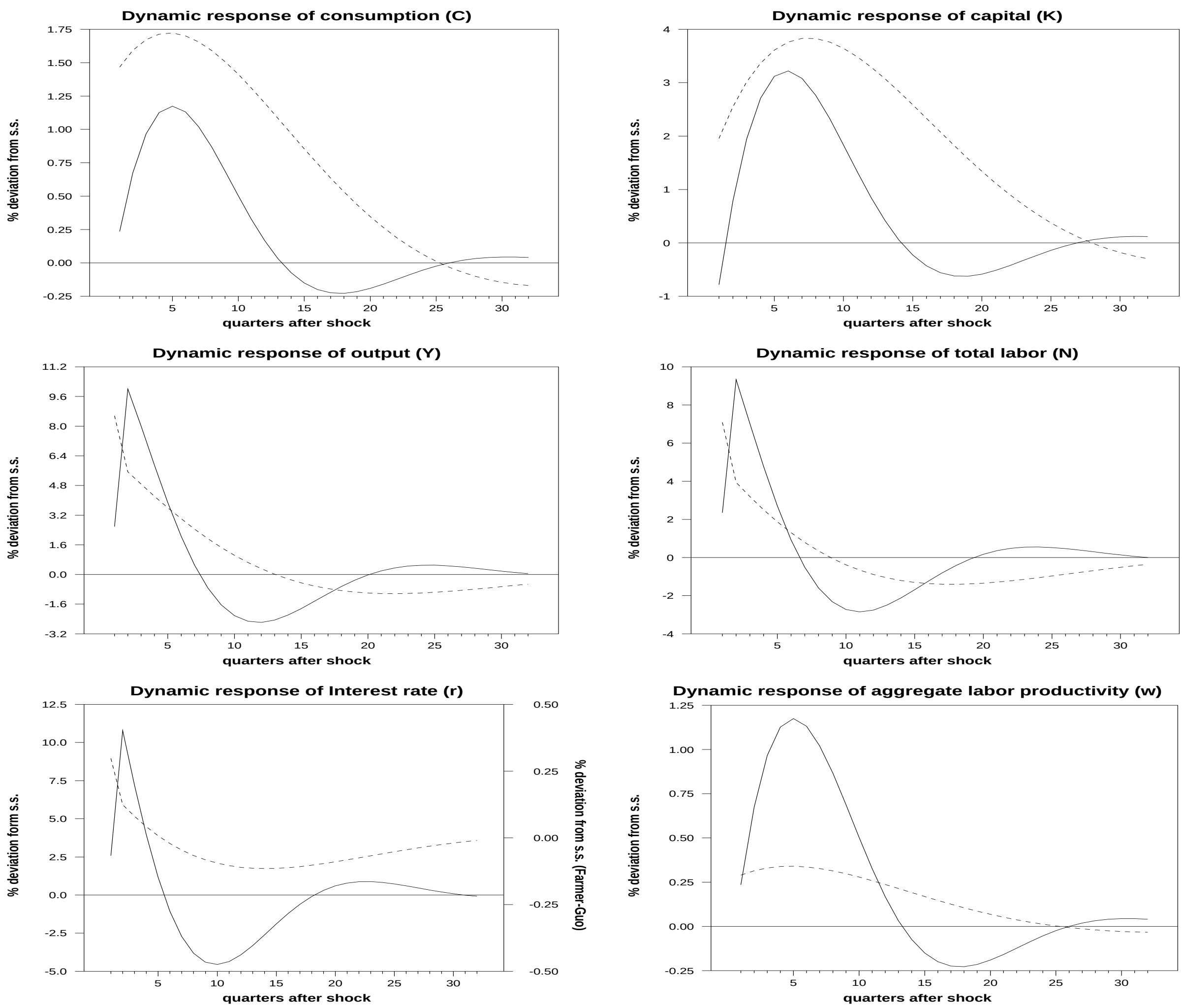

Figure 2: **Impulse Response Functions**. The figure shows the first 32 quarter response of main endogenous variables to a positive sunspot shock, for the baseline parameterization in Table II ($\eta = 0.44$; $\zeta = 0.28$). The response patterns would not be qualitatively different for the other parameterizations under which the stationary state is indeterminate. Notice that the response functions in the bottom-left panel are plotted in dual scale.The curves are the quarterly percentage deviations from a baseline scenario where there is no uncertainty.

The qualitative response patterns of aggregate quantities are comparable with those of the benchmark one sector model, as the Figure suggests.[31] Notice, however, that the mechanism underlining a class of model with labor heterogeneity (i.e. regular and underground labor) is completely different from the customary one, as Section 3.3 suggests. In addition, a distinctive characteristic of our model is the much stronger responses of capital interest rate (r) and aggregate labor productivity (w). This is the consequence of the propagation mechanism operating in our model.

In other words, the model predicts that a positive sunspot shock under indeterminacy should make investment more appealing (in order to self-fulfill the expansionary expectations). A natural way to verify this issue is to rewrite the Euler equation, isolating the covariance term between marginal utility of consumption and investment returns $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$, i.e.

$$E_t\left(C_t^{-1}\right) = \beta E_t\left(C_{t+1}^{-1} R_{t+1}\right) \Rightarrow E_t\left(C_t^{-1}\right) = \beta E_t\left(C_{t+1}^{-1}\right) E_t\left(R_{t+1}\right) + \beta Cov\left(C_{t+1}^{-1}, R_{t+1}\right).$$

Now, investment becomes more appealing when the returns to saving is high in times when marginal utility of consumption is low. Hence, the lower the covariance $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$, the more appealing investment is. We should expect, therefore, that the stronger the self-fulfilling prophecies, the higher the correlation between consumption and returns. A numerical exercise confirms this claim. When self fulfilling prophecies are low (degree of returns to scale equal to 1.37) then $Cov\left(C_{t+1}^{-1}, R_{t+1}\right) = 0.63$; consistently with our claim, when self fulfilling prophecies get stronger (degree of returns to scale equal to 0.41) then $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$ monotonically decrease to 0.19.

5 Sensitivity analysis

For the baseline parametrization (Section 4.2, Table II), the model's attractor is a sink; the matrix $\mathcal{W}$ of the linearized system $E_t\widehat{S}_{t+1} = \mathcal{W}\widehat{S}_t + \Omega\mathcal{E}_{t+1}$, (previously defined) has two complex conjugated eigenvalues: the two roots equals $0.8297 + 0.2185i$ and $0.8897 - 0.2185i$. This sensitivity analysis restricts its attention to the externality parameters η and ζ. **Table III** shows that the value of ζ can be significantly lowered and that of η increased with a little subsequent increase in the degree of returns to scale - which however remains

[31] Such a comparison would be reasonable, because the official GDP estimates for the United States incorporate an estimate of underground sector contribution

below the benchmark one-sector model. In this sense the Table suggests that there exist a trade-off between ζ and η.

Table III

It is then important to underline that the equilibrium path is still locally indeterminate even when the underground sector does not contribute to the externality effect at all, i.e. when ζ is equal to zero - as can be seen from the two last rows of the table. The model still presents indeterminacy for an even smaller (as a percentage of the GDP) underground economy sector ($\left(\frac{N_U}{N}\right)^*$ lowered from 0.088 to 0.04), as the last two rows of the Table show.

6 Conclusions

This paper proposes a one-sector dynamic general equilibrium model augmented with tax evasion and underground activities; it includes an additional kind of labor services along with the standard capital and regular labor, and a proper taxation structure. The model displays increasing returns to scale due to externalities in both regular and underground inputs, capable to induce local indeterminacy of the equilibrium path.

The theoretical mechanism driving such a model departs from the customary one. It turns out that prophecies of a higher expected income are self fulfilled through a resource reallocation toward the underground labor services, which allows to evade taxes, and therefore to have additional resources (the tax wedge) for satisfying the higher desired consumption profile. The explicit introduction of these phenomena into a one-sector general equilibrium model allows to to have well behaved demand schedules for production inputs (in the sense that slope down).

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Tables

Table I: **Parameterization**.

Institutional Setting	$p^* = 0.057$	$s^* = 1.75$	
Disutility of Working	$B_0^* = 2.50$	$B_1^* = 2.00$	
Preferences	$\xi^* = 0.00$	$\psi^* = 0.00$	$\beta^* = 0.984$
Technology	$\alpha^* = 0.23$	$\rho^* = 0.088$	$\delta^* = 0.025$
Effective Tax Rates	$\tau_N^* = 0.153$	$\tau_Y^* = 0.1186$	

Notes: The parameters ξ^*, β^*, ψ^*, α^*, δ^* are from Farmer and Guo [8]; B_0^* and B_1^* are set to match the labor share of regular and underground labor services $\left(\frac{N_U}{N}\right)^* = 0.088$ and $\left(\frac{N_M}{N}\right)^* = 0.912$, following Schneider and Enste [20], where N^* is calibrated to 0.33; p^* and s^* are from Joulfaian and Ride [15]; the effective income tax rate τ_Y^* is computed from the "Effective Tax Rates, 1979-1997", Table H-1a, prepared by the Congressional Budget Office; social security tax rate is taken from www.socialsecurity.com; we choose the value applying for the 1990 and later, which equals $\tau_N^* = 0.153$ (http://www.ssa.gov/OACT/ProgData/taxRates.html).

Table II: **Externality parameters.**

	η^*	ζ^*	Aggregate degree of increasing returns to scale
Baseline Parameterization	0.44	0.28	1.43
Alternative Parameterization	0.39	0.28	1.38
Benchmark (Farmer and Guo [8])	0.60	$-$	1.60
Actual Data (Farmer and Guo [8])			≤ 1.20

Notes: η^* : calibrated regular externality parameter; ζ^* : calibrated underground externality parameter; overall degree of returns to scale reads: $(1+\eta)(1-\rho)+(1+\zeta)\rho$.

Table III: **Sensitivity Analysis**.

$\left(\frac{N_U}{N}\right)^*$	η^*	ζ	eigenvalues	Attractor Topological Properties	Aggregate degree of increasing returns to scale
0.088	0.29	1.5	$0.9629 \pm 0.2428i$	sink	1.396
0.088	0.29	1.3	$0.9292 \pm 0.2912i$	sink	1.378
0.088	0.29	1.0	$0.7115 \pm 0.4421i$	sink	1.352
0.088	0.29	0.9	$-0.01; 0.19$	stable node	1.343
0.088	0.29	0.7	$0.7; 2.0$	saddle	1.326
0.088	0.45	0.28	$0.8437 \pm 0.2097i$	sink	1.435
0.088	0.4	0.28	$0.666 \pm 0.2263i$	sink	1.389
0.088	0.39	0.28	$0.5509 + 0.1487i$	sink	1.38
0.088	0.38	0.28	$-0.06; 0.66$	stable node	1.371
0.088	0.3	0.28	$0.8; 1.5$	saddle	1.298
0.04	0.40	1	$0.8911 \pm 0.2278i$	sink	1.424
0.04	0.35	1.2	$0.8238 \pm 0.3020i$	sink	1.384
0.04	0.30	1.5	$0.3481 \pm 0.3781i$	sink	1.348
0.04	0.29	1.5	$0.56; 15.1$	saddle	1.338
0.088	0.44	0.0	$0.6304 \pm 0.1254i$	sink	1.401
0.04	0.40	0.0	$0.5833 \pm 0.1801i$	sink	1.384

Notes: $\left(\frac{N_U}{N}\right)^*$: underground labor labor share; η : regular externality parameter; ζ : underground-specific externality parameter. Labor demand schedules are well behaved (sloping down) for all parameterizations included in the table.

Appendix

Proof of Proposition 1.

Evaluating at the stationary state the firms' and households first order conditions yields:

$$
\begin{aligned}
B_0 &= C^{-1}\left(1-\tau_Y\right)w_M && (1)\\
C^{-1}w_U &= B_0+B_1 && (2)\\
r &= \frac{\beta^{-1}-1+\delta}{1-\tau_Y} && (3)\\
C &= \left(1-\tau_Y\right)\left(w_M N_M + rK\right) + w_U N_U - \delta K && (4)\\
\tilde{w}_M &= \left(1+\tau_N\right)w_M = \left(1-\alpha-\rho\right)K^{\alpha(1+\eta)}N_M^{(1-\alpha-\rho)(1+\eta)-1}N_U^{\rho(1+\zeta)}\\
\tilde{w}_U &= \left(1+\tilde{\tau}_N\right)w_U = \rho K^{\alpha(1+\eta)}N_M^{(1-\alpha-\rho)(1+\eta)}N_U^{\rho(1+\zeta)-1}
\end{aligned}
$$

where $\tilde{\tau}_N = sp\tau_N$. Rewrite the Euler Equation (3) as

$$\frac{\beta^{-1}-1+\delta}{1-\tau_Y} = \alpha K^{\alpha(1+\eta)-1}N_M^{(1-\alpha-\rho)(1+\eta)}N_U^{\rho(1+\zeta)} = \alpha K^{\alpha(1+\eta)-1}N_M^{(1-\alpha)(1+\eta)}\left(\frac{N_U}{N_M}\right)^{\rho(1+\zeta)} \quad (5)$$

where the quantity $\left(\frac{N_U}{N_M}\right)^{\rho}\frac{N_U^{\rho\zeta}}{N_M^{\rho\eta}}$ is approximated as $\left(\frac{N_U}{N_M}\right)^{\rho(1+\zeta)}$.

Claim 1 *Ratio* $\frac{N_U}{N_M}$ *is stationary.*
Proof. *Combining (1) with (2), yields*

$$\frac{B_0}{\left(B_0+B_1\right)\left(1-\tau_Y\right)}\left(\frac{\rho}{1-\alpha-\rho}\right)\left(\frac{1+\tau_N}{1+\tilde{\tau}_N}\right) = \left(\frac{N_U}{N_M}\right)^{\star} \quad (6)$$

Notice that the stationary labor ratio $\left(\frac{N_U}{N_M}\right)^{\star}$ *is independent of any input-specific externality.* ■

Claim 2 *The quantity* $K^{\alpha(1+\eta)-1}N_M^{(1-\alpha)(1+\eta)}$ *is stationary*
Proof. *Combining (5) with (6), we obtain:*

$$\frac{\beta^{-1}-1+\delta}{\left(1-\tau_Y\right)\alpha}\left[\frac{B_0}{\left(B_0+B_1\right)\left(1-\tau_Y\right)}\left(\frac{\rho}{1-\alpha-\rho}\right)\left(\frac{1+\tau_N}{1+\tilde{\tau}_N}\right)\right]^{-\rho(1+\zeta)} = K^{\alpha(1+\eta)-1}N_M^{(1-\alpha)(1+\eta)} \quad (7)$$

which establish our claim, since the left hand side is constant. ■

Claim 3 *There exists a unique stationary equilibrium for aggregate labor supply* $N_M^{\star} > 0$.
Proof. *Now consider the feasibility constraint (4):*

$$C = K\left(\left(1-\tau_Y\right)\left(\frac{w_M N_M}{K}+r\right)+\frac{w_U N_U}{K}-\delta\right) \quad (8)$$

and observe that the quantities $\frac{w_M N_M}{K}$ and $\frac{w_U N_U}{K}$ are stationary, too:

$$\frac{w_M N_M}{K} = \frac{1-\alpha-\rho}{1+\tau_N}\frac{\beta^{-1}-1+\delta}{(1-\tau_Y)\,\alpha} = \bar{\Psi}_M \tag{9}$$

$$\frac{w_U N_U}{K} = \frac{\rho}{1+\tilde{\tau}_N}\frac{\beta^{-1}-1+\delta}{(1-\tau_Y)\,\alpha} = \bar{\Psi}_U \tag{10}$$

Substituting then (9), (10) into feasibility constraint (8), yields:

$$C = K\left((1-\tau_Y)\left(\bar{\Psi}_M + \frac{\beta^{-1}-1+\delta}{1-\tau_Y}\right) + \bar{\Psi}_U - \delta\right) \tag{11}$$

To derive the stationary equilibrium for total labor supply, substitute (11) into (1) to obtain:

$$\frac{B_0\left((1-\tau_Y)\left(\bar{\Psi}_M + \frac{\beta^{-1}-1+\delta}{1-\tau_Y}\right) + \bar{\Psi}_U - \delta\right)}{(1-\tau_Y)\,(1-\alpha-\rho)} = K^{\alpha(1+\eta)-1} N_M^{(1-\alpha)(1+\eta)} \left(\frac{N_U}{N_M}\right)^{\rho(1+\zeta)} N_M^{-1}$$

Now, since $K^{\alpha(1+\eta)-1} N_M^{(1-\alpha)(1+\eta)}$ and $\frac{N_U}{N_M}$ are constant, the above equation gives the stationary value of $N_M^{\bigstar}$ in proposition 1. ■

The final step is to compute the stationary equilibrium value for $K^{\bigstar}$. Combining (7) with the value of $N_M^{\bigstar}$ it turns out that $K^{\bigstar}$ is stationary, and it reads

$$K^{\bigstar} \simeq \left(\frac{(1-\tau_Y)\,\alpha}{\beta^{-1}-1+\delta}\right)^{\frac{1}{1-\alpha(1+\eta)}} \left(\frac{1+\tau_N}{1+\tilde{\tau}_N}\frac{B_0}{(B_0+B_1)\,(1-\tau_Y)}\frac{\rho}{1-\alpha-\rho}\right)^{\frac{\rho(1+\zeta)}{1-\alpha(1+\eta)}} \left(N_M^{\bigstar}\right)^{\frac{(1-\alpha)(1+\eta)}{1-\alpha(1+\eta)}}.$$

■

Proof of Theorem 1. Preliminaries. Linearize equilibrium conditions, and rearrange them into the following 2×2 dynamic system in the variables $\widehat{K}_t$ and $\widehat{C}_t$:

$$\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s_I}J_3 & -\frac{1}{s_I}J_4 \\ \frac{J_1}{J_2 s_I}J_3 & \frac{1}{J_2} - \frac{J_1}{J_2 s_I}J_4 \end{bmatrix} \begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t \end{bmatrix}, \tag{12}$$

where the $J\prime s$ are defined below. Define the set of our model parameters by $\mathbf{P}$, and a function $\varphi(\mathbf{P})$ such that: $\varphi(\mathbf{P}) : \mathbf{P} \longmapsto \mathbb{R}$.

$$\begin{aligned} J_1 &= \frac{[1-\beta(1-\delta)]}{\gamma}\{\alpha(1+\eta)\,[1+\varphi(\mathbf{P})]-1\}; \\ J_2 &= \{1+[1-\beta(1-\delta)]\,[\varphi(\mathbf{P})]\}; \\ J_3 &= \delta\alpha(1+\eta)M\,(1+\varphi(\mathbf{P})) + (1-\delta)s_I; \\ J_4 &= \delta\gamma M\,(\varphi(\mathbf{P})) + \delta s_C, \end{aligned}$$

where $\varphi(\mathbf{P}) = \frac{\varphi_1(\mathbf{P})}{\varphi_2(\mathbf{P})}$ ($\varphi_1(\mathbf{P}) : \mathbf{P} \longmapsto \mathbb{R}$ and $\varphi_2(\mathbf{P}) : \mathbf{P} \longmapsto \mathbb{R}$ are defined below) is a continuous function mapping the $\mathbf{P}$-parameter space into the reals such that : $\varphi(\mathbf{P}) : \mathbf{P} \longmapsto \mathbb{R}$; $s_C = \frac{C^{\bigstar}}{Y^{\bigstar}}$ and $s_I = \frac{\alpha\beta\delta(1-\tau_Y)}{1-\beta(1-\delta)} = \frac{I^{\bigstar}}{Y^{\bigstar}}$ denote the steady state investment and consumption ratios; the quantity M is equal to the (steady state) share of income net of taxes, and it is defined as $M = 1 - s_G$, where $s_G = s_C + s_I$, and $s_G = \tau_Y + (1-\tau_Y)(1-\alpha-\rho)\frac{\tau_N}{(1+\tau_N)} + \{sp\tau_N(1-\tau_Y) - \tau_Y\}\,\rho\frac{1}{(1+sp\tau_N)}$. The function $\varphi_1(\mathbf{P})$ and $\varphi_2(\mathbf{P})$ are precisely defined below:

$$\begin{aligned}
\varphi_1(\mathbf{P}) &= (1+\eta)(1-\alpha-\rho)\{1+[\psi-(1-q)\xi]W\}+(1+\zeta)\,\rho(1+\xi q W);\\
\varphi_2(\mathbf{P}) &= \xi+1-(1+\eta)(1-\alpha-\rho)-(1+\zeta)\rho+\\
&\quad +\xi W\{\psi q-(1-q)[1-(1+\eta)(1-\alpha-\rho)]-q(1+\zeta)\rho\}+\\
&\quad +W\psi[1-(1+\eta)(1-\alpha-\rho)],
\end{aligned}$$

where W is equal to $1-\frac{w_M^{\star}(1-\tau_Y)}{w_U^{\star}}$, and $q=\frac{N_M^{\star}}{N^{\star}}$.

Gandolfo [13] (ch. 5) states necessary and sufficient conditions for a discrete dynamical system (like system (12)) to display local indeterminacy of the equilibrium path. In terms of our notation, they read:

$$\frac{(J_3-s_I)(1-J_2)+J_1J_4}{s_IJ_2} > 0 \tag{13}$$

$$\frac{(J_3+s_I)(1+J_2)-J_1J_4}{s_IJ_2} > 0 \tag{14}$$

$$\frac{s_IJ_2-J_3}{s_IJ_2} > 0. \tag{15}$$

Strategy. To derive indeterminacy conditions in terms of the parameters of our interest (those belonging to set **P**) we use a constructive argument, which is made of the following four steps.

- **Step 1**. Rewrite (13)-(15) in terms of $\varphi(\mathbf{P}):\mathbf{P}\longmapsto\mathbb{R}$, and therefore in terms of parameters $\mathbf{P}$;
- **Step 2**. Define two subsets of the reals $S_1\subset\mathbb{R}$ and $S_2\subset\mathbb{R}$ ($S_1\cap S_2=\varnothing$) in which the model display (S_1) and does not display (S_2) indeterminacy, respectively;
- **Step 3**. Show that the subset S_1 has a non-empty interior, and therefore that there exist parameters' values for which the stationary state is indeterminate;
- **Step 4**. Invert the function $\varphi(\mathbf{P})$ on the subset S_1, and derive, by this hand, conditions on the parameters $\mathbf{P}$ for the stationary state being indeterminate.

Step 1. Rewrite equations (13)-(15) in terms of $\varphi(\mathbf{P}):\mathbf{P}\longmapsto\mathbb{R}$. Algebraic manipulations yield:

$(\mathbf{I}.)$ $\frac{\delta(M-s_I)[\beta(1-\delta)-1]\{[1-\alpha(1+\eta)](1+\varphi(\mathbf{P}))\}}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))}>0.$

$(\mathbf{II}.)$ $\frac{\{\delta(M-s_I)[1-\beta(1-\delta)][1-\alpha(1+\eta)]+2[\delta\alpha(1+\eta)M+s_I(1-\beta(1-\delta))]\}(\varphi(\mathbf{P}))}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))}+\frac{\delta(M-s_I)[1-\beta(1-\delta)][1-\alpha(1+\eta)]+2[\delta\alpha(1+\eta)M+s_I(2-\delta)]}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))}>0.$

$(\mathbf{III}.)$ $\frac{[s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)](\varphi(\mathbf{P}))+\delta[s_I-M\alpha(1+\eta)]}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))}>0.$

Conditions **(I.)-(III.)** are necessary and sufficient for the equilibrium capital accumulation being local indeterminate.

Step 2. Conditions **(I.)-(III.)** define a system of inequalities, which, in turns, defines two subsets of the reals $S_1\subset\mathbb{R}$ and $S_2\subset\mathbb{R}$ ($S_1\cap S_2=\varnothing$) defined as follows:

- $S_1\subset\mathbb{R}$: model displays indeterminacy (all inequalities **(I.)-(III.)** are satisfied);
- $S_2\subset\mathbb{R}$ model does not displays indeterminacy (at least one inequality among **(I.)-(III.)** is not satisfied).

In other words, if $\varphi(\mathbf{P}) \in S_1$ the equilibrium is indeterminate, and if $\varphi(\mathbf{P}) \in S_2$ the equilibrium is not indeterminate.

Step 3. Notice that the conditions **(I.)-(III.)** share the same denumerator, and they are all functions of $\varphi(\mathbf{P})$. Hence they are functions $\mathcal{C}_i : \mathbb{R} \longmapsto graph(\varphi(\mathbf{P})) \subseteq \mathbb{R}$. The zeros of these functions are the values delimiting the intervals over which the conditions are (are not) simultaneously satisfied. They are:

$$\begin{aligned}
R_{13}^0 &= -1 \\
R_{14}^0 &= -\frac{\delta(M-s_I)\left[1-\beta(1-\delta)\right](1-\alpha(1+\eta))+2\left[\delta\alpha(1+\eta)M+s_I(2-\delta)\right]}{\delta(M-s_I)\left[1-\beta(1-\delta)\right](1-\alpha(1+\eta))+2\left[\delta\alpha(1+\eta)M+s_I(1-\beta(1-\delta))\right]} \\
R_{15}^0 &= -\frac{\delta s_I-\delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} \\
R_D^0 &= -\frac{1}{1-\beta(1-\delta)},
\end{aligned}$$

where R_D^0 denotes the zero of the common denumerator. It is convenient to rewrite the conditions **(I.)-(III.)** in terms of the values delimiting the intervals S_1 and S_2. Algebraic manipulations yield to the following necessary and sufficient condition for indeterminacy:

$$\max\left(R_D^0, R_{14}^0\right) < R_{15}^0 < R_{13}^0 \qquad (\bigstar)$$

The following theorem, together with Step 4, show that there exist a set of parameters for which condition ($\bigstar$) is satisfied.

Theorem 2 (Non Empty Parameter Space for Indeterminacy) *The model equilibrium is locally indeterminate* ***iff*** *the following inequalities hold:*

$$\max\left(R_D^0, R_{14}^0\right) < R_{15}^0 < R_{13}^0$$

Proof. We prove the theorem by proving the following preliminary claims.

Claim 1 $R_D^0 < R_{13}^0$ *and* $R_{14}^0 < R_{13}^0$.
Proof. $R_D^0 < R_{13}^0$ *directly follows from* $1-\beta(1-\delta) < 1$. *Furthermore,* R_{14}^0 *is negative for all parameters' values and its denumerator is always smaller than its numerator, as* $2-\delta > 1-\beta(1-\delta)$ *(in absolute values). So it must be* $R_{14}^0 < R_{13}^0$. ■

Claim 2 $R_{15}^0 < R_{13}^0$.
Proof. $R_{15}^0 < R_{13}^0$ *implies that* $\frac{\delta s_I-\delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} > 1$, *and we first show that this can happen* ***iff*** $s_I(1-\beta(1-\delta)) < \delta M\alpha(1+\eta)$*; in fact, if this is true, then it will be* $\delta s_I < \delta M\alpha(1+\eta)$, *as it is* $1-\beta(1-\delta) > \delta$. *In this case the numerator of* R_{15}^0 *is always negative and greater (in absolute value) than the denominator (which is also negative); then* $\frac{\delta s_I-\delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} > 1$. *We then show that the inequality* $s_I(1-\beta(1-\delta)) > \delta M\alpha(1+\eta)$ *is incompatible with conditions (13)- (15), so that the opposite must hold. Suppose (by contradiction) that it is* $s_I(1-\beta(1-\delta)) > \delta M\alpha(1+\eta)$*; then it would be either* $\delta s_I > \delta M\alpha(1+\eta)$ *or* $\delta s_I < \delta M\alpha(1+\eta)$. *In the first case the line Numerator (**III.**) has a positive slope and a negative intersection* R_{15}, *which is greater than* R_{13}*; the situations is shown in figure 3.A). It is easy to check that for all values of* $\frac{\varphi_1}{\varphi_2}$ *inside the interval* $(R_D; R_{15})$ *indeterminacy never occurs, as* R_D *is positive and* R_{15} *is negative; but also for* $\frac{\varphi_1}{\varphi_2}$ *outside the interval indeterminacy never occurs, as condition (13) (for* $\frac{\varphi_1}{\varphi_2} > R_{15}$*) and condition (15) (for* $\frac{\varphi_1}{\varphi_2} < R_D$*) are not satisfied. Now consider the case in which* $s_I(1-\beta(1-\delta)) > \delta M\alpha(1+\eta)$ *but* $\delta s_I < \delta M\alpha(1+\eta)$*; then Numerator(**III.**) has still a positive slope*

*but R_{15} is positive. By shifting the line Numerator(**III.**) in Figure 3.A), the same above argument applies and indeterminacy disappears. So, it must be $s_I(1-\beta(1-\delta)) < \delta M\alpha(1+\eta)$, and then $R^0_{15} < R^0_{13}$.* ■

Claim 3 $R^0_D < R^0_{15}$

Proof. *The inequality $R^0_D < R^0_{15}$ can be recast as i.e. that $\frac{\delta s_I - \delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} < \frac{1}{1-\beta(1-\delta)}$. Note that when the term $\delta M\alpha(1+\eta)$ (which is always $\geqq 0$) is zero, the first fraction reduces to $\frac{1}{1-\beta(1-\delta)}\delta < \frac{1}{1-\beta(1-\delta)}$. We show that when the term $\delta M\alpha(1+\eta)$ increases, passing from zero to positive numbers, the fraction $\frac{\delta s_I - \delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)}$ monotonically decreases, so that it must always be $R^0_D < R^0_{15}$. Consider $\delta M\alpha(1+\eta)$ as a function of η: when η is equal to[32] -1, the fraction collapses to $\frac{1}{1-\beta(1-\delta)}\delta$; when η increases, the term $\delta M_1\alpha(1+\eta)$ monotonically increases. Now $\frac{d(-R_{15})}{d\eta} = \frac{-\delta M\alpha}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} \left[1 - \frac{\delta s_I - \delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)}\right]$. We have seen before that $-\frac{\delta s_I - \delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta))-\delta M\alpha(1+\eta)} < -1$; but then it is $\frac{d(-R_{15})}{d\eta} < 0$ for all the parameters' values. Thus $R^0_D < R^0_{15}$.* ■

Claim 4 $R^0_{14} < R^0_{15}$.

Proof. *We demonstrate this inequality by contradiction. Assume that $R^0_{15} < R^0_{14}$; given the inequalities demonstrated above, two cases are possible: either $R^0_{14} < R^0_D$, or $R^0_D < R^0_{14}$; the first one is clearly impossible, as it would imply that $R^0_{15} < R^0_{14} < R^0_D$ and we have seen that it is $R^0_D < R^0_{15}$. Let's consider the second one: $R^0_D < R^0_{15} < R^0_{14}$; in this case the situation would be the one depicted in Figure 3.B) (recall that the slope of Numerator(**II.**) is always positive). In the interval $(R_{14}; R_{13})$ indeterminacy is impossible, as Numerator(**III.**) < 0 and Denumerator > 0; this is also true in the interval $(R_D; R_{14})$, as Numerator(**II.**) < 0, Denumerator > 0, and in the regions outside the two intervals. Then the only ordering compatible with indeterminacy is $R^0_{14} < R^0_{15}$.* ■

Claim 5 $R^0_{14} < R^0_D$ *and* $R^0_D < R^0_{14}$ *are possible and compatible with an interval for S_1 being non-empty..*

Proof. *The order between R^0_{14} and R^0_D does not affect the existence of a non empty parameter space for indeterminacy of equilibrium, as Figure 3.C) illustrates.* ■

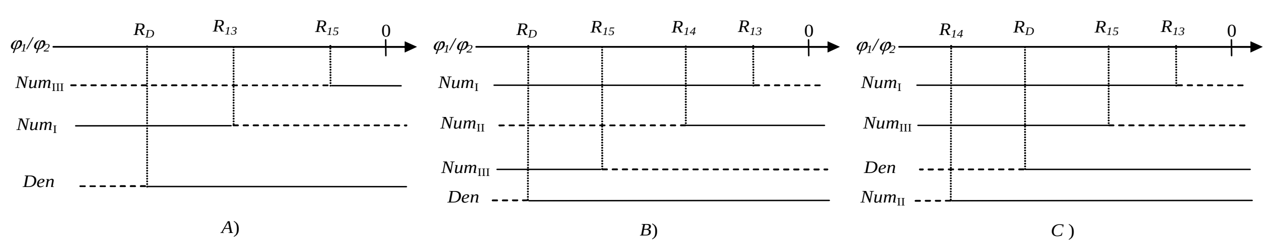

Figure 3: **Auxiliary intervals**: dotted lines represent negative values of the corresponding function of $\frac{\varphi_1}{\varphi_2}$, (i.e. the the three numerators of Conditions (**I.**) - (**III.**) and the common denumerator) while solid lines represent positive values.

The interval $\left(\max\left\{R^0_D, R^0_{14}\right\}; R_{15}\right)$ is thus a viable region for indeterminacy, as for all the values of $\varphi(\mathbf{P})$ falling in this region, the necessary and sufficient conditions for indeterminacy (**I.**)-(**III.**)) are satisfied.

[32]Obviously $\eta < 0$ is not an interesting case in our model (it could be interpreted as a *negative* externality at system level), but for the sake of the argument it can be accepted just to see what is the effect on the fraction of the term $\delta\alpha(1+\eta)M$ when the latter is arbitrary small.

In summary, we have demonstrated that: $R_D^0 < R_{13}^0$, $R_{14}^0 < R_{13}^0$, $R_{15}^0 < R_{13}^0$, $R_D^0 < R_{15}^0$, $R_{14}^0 < R_{15}^0$, $R_{14}^0 < R_D^0$ or $R_D^0 < R_{14}^0$. By merging all these inequalities together, the orderings compatible with an indeterminacy region turn out to be $R_D^0 < R_{14}^0 < R_{15}^0 < R_{13}^0$ and/or $R_{14}^0 < R_D^0 < R_{15}^0 < R_{13}^0$. This completes the proof of Theorem 2. ■

Step 4. So we have two possible orderings defining the indeterminacy region; one is given by the interval $(\overline{R_{14}^0; R_{15}^0})$, or

$$-\underline{\mathcal{R}} < \frac{\varphi_1(\mathbf{P})}{\varphi_2(\mathbf{P})} < -\overline{\mathcal{R}} \tag{16}$$

where:

$$\begin{aligned}
\underline{\mathcal{R}} &= \frac{\delta(M-s_I)\left[1-\beta(1-\delta)\right](1-\alpha(1+\eta)) + 2\left[\delta\alpha(1+\eta)M + s_I(2-\delta)\right]}{\delta(M-s_I)\left[1-\beta(1-\delta)\right](1-\alpha(1+\eta)) + 2\left[\delta\alpha(1+\eta)M + s_I(1-\beta(1-\delta))\right]} \\
\overline{\mathcal{R}} &= \frac{\delta s_I - \delta M\alpha(1+\eta)}{s_I(1-\beta(1-\delta)) - \delta M\alpha(1+\eta)}
\end{aligned}$$

The other one is given by $\left(R_D^0, R_{15}^0\right)$, or:

$$-\frac{1}{1-\beta(1-\delta)} < \frac{\varphi_1(\mathbf{P})}{\varphi_2(\mathbf{P})} < -\overline{\mathcal{R}} \tag{17}$$

Both the previous conditions suggest that in order to have indeterminacy, the ratio $\frac{\varphi_1}{\varphi_2}$ must be negative, greater (in modulus) than one and finally included between two specific values. From the above theorem, we can see that the necessary condition for indeterminacy is given by:

$$s_I(1-\beta(1-\delta)) < \delta M\alpha(1+\eta),$$

which, after some algebraic manipulation, turns out to be necessary condition (NC). As for the sufficient condition (SC), we can first write the term $\frac{\varphi_1}{\varphi_2}$, assuming, for simplicity, that ξ and ψ are equal to zero:

$$\begin{aligned}
\varphi_1 &= (1+\zeta)\,\rho + (1+\eta)(1-\alpha-\rho) \\
\varphi_2 &= 1-(1+\zeta)\rho-(1+\eta)(1-\alpha-\rho) = 1-\varphi_1
\end{aligned}$$

Note that $\varphi(\mathbf{P}) = \frac{\varphi_1(\mathbf{P})}{\varphi_2(\mathbf{P})} = \frac{(1+\zeta)\rho+(1+\eta)(1-\alpha-\rho)}{1-(1+\zeta)\rho-(1+\eta)(1-\alpha-\rho)} < -1$ $(= R_{13}^0)$.[33] Putting together (16) and (17) and explicitating with respect to φ_1, the sufficient condition (**SC**) in the main text can be obtained. This completes the proof of Theorem 1. ■

[33]Note also that the term φ_1 is equal to sum of the cross elasticities of the two labor demands: $\varepsilon_{\widehat{K}}^{\widehat{L}_M^D}$ and $\varepsilon_{\widehat{K}}^{\widehat{L}_U^D}$, as shown in the main text