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Miscounts, Duverger's Law and Duverger's Hypothesis

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Abstract

In real-life elections, vote-counting is often imperfect. We analyze the consequences of such imperfections in plurality and runoff rule voting games. We call a strategy profile a *robust* equilibrium if it is an equilibrium if the probability of a miscount is positive but small.

All robust equilibria of plurality voting games satisfy *Duverger's Law*: In any robust equilibrium, exactly two candidates receive a positive number of votes. Moreover, robustness (only) rules out a victory of the Condorcet loser.

All robust equilibria under runoff rule satisfy *Duverger's Hypothesis*: First round votes are (almost always) dispersed over more than two alternatives. Robustness has strong implications for equilibrium outcomes under runoff rule: For large parts of the parameter space, the robust equilibrium outcome is unique.

Keywords: strategic voting, plurality rule, runoff rule, Duverger's Law and Hypothesis
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1 Introduction

In real-life elections, vote-counting is often imperfect. The best indication of this fact is that, when an election outcome is close, the candidate who is behind in the first count often asks for a recount. In a substantial fraction of recounts, the initial result is actually overturned, and even if the winner does not change, vote totals generally change after a recount. Moreover, it is plausible that even recounts are often not perfect in the sense that it is not guaranteed that every vote is counted in the way that the corresponding voter intended to cast it.

In this paper we investigate the implications of a vote counting technology that is imperfect, but very close to perfect, on the equilibrium of voting games with three candidates. Specifically, we consider a setup in which each ballot is counted as an "undervote" (i.e., a vote for no candidate) with probability $\varepsilon > 0$, while it is counted correctly with probability $1 - \varepsilon$. We call ε the miscount probability. A strategy profile s is a *robust* equilibrium if there is a $\bar{\varepsilon} > 0$ such that s is an equilibrium whenever $\varepsilon < \bar{\varepsilon}$. We focus on two of the most commonly studied types of non-binary voting games, namely plurality rule games and runoff rule games.

Robustness is very similar in spirit to Selten's Trembling Hand Perfection. The aspects in which the two concepts differ are the following: First, the 'trembles' (miscounts) that we consider do not correspond to a player casting his vote for a wrong candidate. Instead they simply nullify a player's ballot. Second, our miscounts are not committed by the players themselves but are a characteristic of the vote counting technology. Thus, unlike in the case of Selten's trembles, the distribution of our miscounts does not depend on the players' identities. The chance that a ballot is lost is the same for each player.

Modeling perturbations in voting games in this way has several important advantages. The first advantage is that it has a very natural interpretation. Experience from close elections show that vote totals often change after a recount (and even then it is not clear that the last result is necessarily the "correct" one in an absolute sense). Second, the fact that the miscount probability is the same for all ballots is very convenient from a technical point of view, and this is the main advantage relative to classical trembling hand perfection. Third, and related to the second point, our way of modeling perturbations in voting games as arising from an imperfect vote counting technology produces sharper predictions in terms of reducing the set of possible outcomes than classical trembling hand perfection.

Our main results can be summarized as follows: Under plurality rule, an equilibrium is robust if and only if it satisfies Duverger's Law, that is, if all votes are concentrated on exactly two candidates. Moreover, voters vote "sincerely" for one of the two relevant candidates; that is, each voter votes for the candidate he prefers from among the two candidates who receive a positive number of votes. While robustness has strong implications for the structure of equilibrium voting behavior under plurality rule, it does not narrow down the set of possible equilibrium outcomes very much. Since votes can be distributed on any pair of candidates, the only candidate who can be excluded as possible winner of the election is the Condorcet loser (whenever one exists).

In runoff rule voting games, robustness generates (under very weak assumptions on the preference distribution) equilibria consistent with "Duverger's Hypothesis", which states that first round votes in a runoff system are typically dispersed over more than just two candidates. In terms of reducing the set of robust equilibrium outcomes, robustness is more powerful under runoff rule than under plurality rule. In fact, the robust equilbrium outcome is often unique, both in the case where a Condorcet winner exists and where it does not exist. If in a setting with a transitive majority ordering the outcome is unique, then this outcome must be the Condorcet winner. If the outcome is unique in a setting where no Condorcet winner exists then it must be the candidate with the smallest set of voters who rank that candidate lowest.

The remainder of this paper is organized as follows. In the following subsection we provide a short overview of the related literature. In Section 2 we describe the environment (voters, preferences, set of candidates). We also introduce some notation and terminology that we use throughout the paper. In Section 3, we first provide a description of our plurality rule voting games and a definition of the concept of robustness for plurality voting games. After doing so we provide an exact characterization of the robust equilibrium set of plurality voting games. Runoff voting games are analyzed in Section 4. Since runoff games are not static, we first discuss how the robustness concept is to be adapted to such games before providing a (partial) characterization of the set of robust equilibria. Section 5 discusses possible extensions of our work and provides some concluding remarks.

1.1 Related literature

Our paper contributes to the literature on multicandidate voting games. Unfortunately, the analysis of voting games faces the problem that they usually have multiple equilibria. Moreover, different Nash equilibria typically lead to different outcomes (i.e., election winners). The only case in which this problem is easily overcome is in voting games with just two alternatives, or more generally, voting games with a binary agenda in which at every stage only two alternatives are voted on. It is well known that such games are dominance solvable (see Moulin (1979)). Moreover, the unique undominated equilbrium outcome of such games is the *sophisticated* outcome as defined by Farquharson (1969). Binary agendas apply to the analysis of voting in legislative settings, but in elections with three or more candidates, weak dominance arguments lose their power and, typically, any candidate can be supported as outcome of an equilibrium in undominated strategies.

Plurality and runoff voting games with complete information. There are only few papers that consider complete information plurality or runoff voting games as we do here. Dhillon and Lockwood (2004) show that the class of plurality voting games that is dominance solvable is rather small. Sinopoli (2000) applies the classical equilibrium refinement concepts of trembling hand perfection and stability to such games and arrives at the conclusion that neither of the two refinements has much bite in terms of reducing the set of equilibrium outcomes.

Feddersen (1993), Niemi and Frank (1981) and Messner and Polborn (2007) use (variations of) the concept of coalition-proof equilibrium as a way to model coordination among voters. While coalition-proofness rules out many "unreasonable" equilibria in plurality voting games,[1]

[1] Messner and Polborn (2007) consider also runoff voting games.

both Feddersen (1993) and Messner and Polborn (2007) show that there is the drawback that, for some constellations of voter preferences, coalition-proof equilibria may not exist.

More commonly, the difficulties that a strategic analysis of voting creates are circumvented by assuming sincere voting. Sincere voting means that players vote for the candidate whom they would like most to be the winner, even if this candidate has no chance to win. Very often therefore some voters' interests would be better served if they voted for another candidate.

Plurality and runoff voting games with incomplete information. Palfrey (1989) is the first paper to propose a model that delivers equilibria consistent with Duverger's Law. He considers a plurality voting game where voters' preference types are private information and shows that, in every symmetric strategy profile, the probability that a vote for one of the two candidates with the largest expected vote shares will influence the outcome becomes infinitely large relative to the probability that a vote for a candidate with a smaller expected vote share will make a difference as the electorate grows larger and larger. Consequently, in large elections all votes have to be concentrated on two candidates only.

Myerson (2002) models voting games as Poisson games that were first introduced in Myerson (1998) and Myerson (2000). Poisson voting games are voting games where players are uncertain both about the number of voters that participate in the election and the preferences of the players who might participate. He shows that large best rewarding scoring rule voting games (like plurality voting games) always admit equilibria in which only two candidates seriously compete for victory. In contrast, large voting games with "worst punishment" scoring rules (like negative voting) do not admit such equilibria.

Our paper shares with both Palfrey (1989) and Myerson (2002) the general idea to overcome the difficulties generated by the fact that voting games are highly non-generic games through the introduction of a stochastic element. While we do not assume directly that players are uncertain about the number of other players and/or their payoffs, such forms of uncertainty in our model arise as a consequence of the assumption of an imperfect vote counting technology. After all, being uncertain about whose votes will be counted is the same as being uncertain about who will take part in the election.

Despite these similarities there are important differences between our approach and the ones of Palfrey (1989) and Myerson (2002). First of all, in the games that we consider our approach offers a rather simple way to calculate the pivot probabilities and their relative orders of magnitude which does not require the assumption of a large electorate. The simplicity of our model is particularly useful in the context of the more complex runoff voting games that have not been considered by either Palfrey (1989) or Myerson (2002). Finally, we also think that our assumption of an imperfect vote counting technology provides a very natural justification for the distributional assumptions (iid assumptions) with which we work.

A somewhat different approach to the analysis of large plurality voting games is taken in a recent paper by Myatt (2007). He considers a situation where voters can choose among two alternatives, A and B say, that they both prefer over the status quo (C say). The status quo will be replaced by A if A collects a large enough consensus in the elections. If instead neither of the two alternatives reaches a sufficiently large share of the votes the status quo prevails.

Thus if the electorate is divided (some rank A highest, others B) then voters face a coordination problem. Myatt (2007) studies this coordination problem applying global games techniques. In particular, he analyzes how the incentive to vote strategically (i.e. to vote for the less preferred one among A and B) depends on the parameters of the game.

Apart from the global games tools Myatt (2007) differs from our paper in a number of other dimensions. Most importantly, Myatt (2007) does not consider runoff voting games but only plurality voting games. Moreover, in his model with three alternatives he allows only for three preference types: $A \succ B \succ C$, $B \succ A \succ C$ and $C \succ A \sim B$. The only restriction that we impose on preferences is that we require them to be strict.

Similar restrictions on preferences are also imposed in Martinelli (2002), who considers a model where a majority of the voters have a common interest but disagree on which candidate among A and B to support due to private information; the remaining voters instead all agree that candidate C is best for them. Given these assumptions, under plurality rule there arises a tension between efficient aggregation of information and the need of coordination: If the majority voters express their private information through their votes and thus split between A and B then they risk that the minority candidate C will carry the victory. This tension does not arise under runoff rule which thus yields a higher expected payoff to the majority voters.

Myerson and Weber (1993) propose a semi-strategic approach to the analysis of plurality voting games that assumes that all voters choose optimally given a vector of 'pivot probabilities' that describes for every pair of candidates the probability that those two candidates will tie for the first place.[2] The pivot probabilities are assumed to satisfy an 'ordering condition': If the number of votes candidate i gets is less than the number of votes for candidate j, then, for any other candidate h, the pivot probability between candidates i and h is only at most ϵ times the pivot probability between candidates j and h (where ϵ is very small). Myerson and Weber (1993) show that their model allows for equilibria that are not consistent with Duverger's Law.

Also in our model voters choose on the basis of pivot probabilities that satisfy the above ordering condition. But unlike in Myerson and Weber (1993) the pivot probabilities depend on the voting behavior of the other voters and thus vary across players. In Section 3.3 we discuss in detail why this feature is crucial for our results.

Duverger's Law and Duverger's Hypothesis in models with strategic candidacy. Our paper is also related to a strand of the literature that studies under what conditions Duverger's Law and Duverger's Hypothesis obtain in the context of models with candidate entry and/or endogeneous platform choice (see for instance, Castanheira (2003) and Callander (2005)).

[2]This method has subsequently also been applied to runoff voting games; see Cox (1997).

2 The Setup

A group of N voters chooses between the elements of the set $\mathcal{C} = \{C_1, C_2, C_3\}$, which we, for concreteness, refer to as (citizen) candidates.[3] All voters are expected utility maximizers. We write $u_n(C_i)$ for the utility that voter n assigns to candidate C_i. The assumption of expected utility maximization is needed in order to have well defined preferences over lotteries defined on $\mathcal{C}$. However, our main arguments throughout the paper do not rely on the cardinal properties of the individuals' utility functions. Almost all of our results depend only on the ordinal aspects of the agents' preferences.

We also assume that no individual is indifferent between any two candidates.

Assumption 1. *$u_n(C_i) \neq u_n(C_j)$ for all $n \in \{1, \ldots, N\}$ and all distinct pairs of alternatives $C_i, C_j \in \mathcal{C}$.*

For any pair (i, j), $i \neq j$, N_{ij} denotes the set of players who like C_i best and rank C_j second. We refer to the set N_{ij} as the ij-preference group and to elements of N_{ij} as $ij-$types. The number of voters belonging to group N_{ij} is denoted by n_{ij} (i.e. $n_{ij} = \#N_{ij}$). The following definition introduces concepts and notations.

Definition 1 (Core supporters/opponents, sympathizers and winning margin)**.**

i) *The set $N_i = \cup_{j \neq i} N_{ij}$ is the set of* **core supporters** *of candidate C_i; the number of core supporters of C_i is denoted by n_i (i.e. $n_i = \sum_{j \neq i} n_{ij}$).*

ii) *The set $L_i = N_{jk} \cup N_{kj}$, $k, j \neq i$, is the set of* **core opponents** *of candidate C_i; the number of core opponents of C_i is denoted by l_i (i.e. $l_i = n_{jk} + n_{kj}$).*

iii) *$\Sigma_{ij} = \{n : u_n(C_i) > u_n(C_j)\}$ is the set of* **sympathizers** *of candidate C_i against candidate C_j; we write σ_{ij} for the number of elements of Σ_{ij}.*

iv) *$\Delta_{ij} = \max\{\sigma_{ij} - \sigma_{ji}, 0\}$ is the* **Condorcet margin** *of candidate C_i against candidate C_j. If $\Delta_{ij} > 0$ we say that candidate C_i* **Condorcet dominates** *candidate C_j. In this case we also write $C_i \succ C_j$.*

In order to exclude some notationally cumbersome and substantively rather uninteresting cases, we assume that no candidate's core support constitutes an absolute majority of the electorate, i.e., we require that $n_i < (N + 1)/2$ for all i.

We analyze plurality rule and runoff rule voting games when the vote counting technology is imperfect and each ballot is counted correctly with probability $1 - \varepsilon$, while it is subject to a *miscount* with probability $\varepsilon > 0$. We thus refer to ε as the *miscount probability*, which we assume to be known by all players. If a vote is miscounted, it is mistakenly considered an undervote. That is, miscounted votes are tabulated as a votes for "none of the candidates", not as votes for some other candidate.[4]

[3]Several of our results generalize in a straightforward way to settings with more than three candidates. We mention such extension possibilities in the discussion of the respective results.

[4]This assumption helps to simplify the analysis, as we do not need to specify relative probabilities with which a miscounted vote is counted for each of the other candidates. We conjecture that our main results would be unaffected if, instead, miscounted votes were assigned to all candidates in equal proportions.

3 Plurality Rule

3.1 Description of the Game

Under plurality rule, each voter casts a single ballot for one of the available candidates. Thus, each player's set of (pure) strategies equals the set of candidates. A generic (pure) strategy of player n is denoted by s_n. For strategy profiles we write s. We refer to s also as vote profile. The set of players who vote for candidate C_i is denoted by $V_i(s)$, and we write $v_i(s) = \#V_i(s)$ for the number of votes intended to be in favor of C_i.

The winner of the election is the candidate with the highest number of *effectively counted* votes, where ties are broken by uniform randomization. The number of effectively counted votes for candidate C_i in vote profile s is a binomially distributed random variable with mean $(1-\varepsilon)v_i(s)$.

3.2 Robustness

The set of equilibrium outcomes of a perturbed plurality voting game depends on the miscount probability which, in practice, is positive but small. We therefore focus our attention in the following on vote profiles that constitute equilibria of voting games with small, but strictly positive, miscount probabilities. We refer to such strategy profiles as *robust equilibria.*

Definition 2 (Robust equilibria)**.** *A profile $s \in \mathcal{C}^N$ is* **robust** *(or a* **robust equilibrium***) if there exists a $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, s is a Nash equilibrium of the ε-perturbated version of the plurality voting game.*

A few remarks on this concept are in order. Robustness can be seen as a refinement concept similar in spirit to trembling hand perfection (henceforth, THP). THP constrains players to play fully mixed strategies (so that each player makes mistakes with positive probability and votes for another candidate than the one he intended to). In contrast, allowing for miscounts in our model has essentially the same effects as extending players' strategy sets by the action "abstention", and to require all voters to play that strategy with probability ε.

From a technical point of view, the particularly simple structure of the miscount technology makes our robustness concept easy to handle. However, as explained in more detail below, the more important advantage that robustness offers over THP is that it has stronger implications in narrowing down the set of possible equilibrium outcome in the voting games that we consider.

Our way of modeling perturbations is also consistent with other interpretations. In particular, it naturally fits with the following costly voting model. Voters either cast a ballot in favor of one of the candidates or they abstain. The cost of casting a ballot is stochastic and depends on the voters' private state. With probability $1-\varepsilon$ a voter is in the zero cost state while with probability ε the high cost state realizes (e.g. illness on the election day). If the voting cost in the latter state is high enough to offset any possible benefits from voting, then abstaining is the strictly dominant strategy for all voters who happen to be in this state. In contrast, for all individuals for whom voting is costless, abstaining is a strictly dominated strategy.

3.3 Results

In any unperturbed plurality voting game, each player has at least one weakly dominated strategy, namely to vote for his least preferred candidate. It is well known that THP rules out that such strategies are played in equilibrium. The same holds true for our robustness concept. Allowing for miscounts implies that each candidate has a strictly positive chance to win the elections. Moreover, each candidate's winning probability is the larger the more votes he receives. Thus, voting for one's least preferred candidate is strictly dominated by voting for one's most preferred candidate. Consequently, there are no robust equilibria in which any player votes for his lowest-ranked candidate. Lemma 1 summarizes these observations.

Lemma 1. *In any perturbed plurality voting game, voting for C_i is a strictly dominated strategy for all core opponents of C_i. Thus, a voting profile in which some voter votes for his least preferred candidate cannot be a robust equilibrium.*

Proof. Omitted. □

In unperturbed voting games, (even iterated) elimination of weakly dominated strategies only rules out that candidates are supported by their core opponents. Dhillon and Lockwood (2002) show that a further round of elimination is possible in a three candidate race if and only if at least 2/3 of the voters agree on who is the worst candidate. In that case, the candidate ranked lowest by more than 2/3 of the voters receives less than 1/3 of the vote and thus cannot win. Therefore, everyone knows that a vote for this candidate is wasted, which effectively reduces the three candidate race to one with two viable candidates, and this game is dominance solvable.

Which additional implications can be derived in our framework? Remember that since in an ε-perturbed game each vote is counted with probability $1-\varepsilon$ there is a positive probability that the candidates who are in the second and third place in the *voting intentions* actually end up winning the election. However, provided that ε is small, the probability that the third placed candidate wins is actually *much smaller* than the probability that the second placed candidate wins. Formally, 'much smaller' means that the winning probability of the third placed candidate is a term of higher order (in ε) than the probability that the second placed candidate wins.

In a sense, a vote for a candidate who is not among the top two candidates (according to vote intentions) is wasted. For example, if a citizen prefers the candidate with the most votes over the runner-up, he should vote for the leader, because his vote is much more likely to be pivotal in a decision between the two candidates who receive the most vote intentions than in a decision involving any other candidate. We thus have the following result.

Lemma 2. *Let $s \in \mathcal{C}^N$ be such that $v_i(s) \geq v_j(s) > v_k(s) > 0$. Then s is not a robust equilibrium.*

Proof. See Appendix. □

Note that Lemma 2 relies on the assumption that each voter's ballot is counted with the same probability. If, instead, the probability of a miscount were to vary across voters, then the winning chances of the second and third ranked candidates would depend on the composition of their respective electorates, and this would complicate matters.

The next question is whether there are any robust equilibria with two candidates tying for second place ($v_i > v_j = v_k$). Intuitively, this is also very unlikely. For such a constellation to be robust, it would have to be true that *every* voter for candidate C_j prefers C_j over C_i (otherwise, in any perturbed game switching the vote to candidate C_i is optimal), and C_i over C_k (otherwise, switching the vote to candidate C_k is optimal). An analogous argument applies to all those who vote for C_k. Thus, a first necessary condition for $v_i > v_j = v_k$ to arise in equilibrium is that $n_{ji} = n_{ki}$. In addition, if $V_j = N_{ji}$ and $V_k = N_{ki}$, then voters of type jk and kj (i.e., candidate C_i's core opponents) would have to vote for C_i. Since Lemma 1 excludes this behavior, it follows that a second necessary condition for $v_i > v_j = v_k$ to arise in equilibrium is that $L_i = \emptyset$.

Finally, consider the possibility of a three-way tie between all candidates. In such a case, each candidate would have to get all votes from his core supporters (otherwise, a deviation to a voter's most preferred candidate would break the tie in favor of that candidate). Thus, a necessary condition for a robust equilibrium with a three-way tie is that all candidates have equally strong core supports. In addition, it must be true that every voter prefers a uniform lottery over all three candidates to having the candidate whom he ranks second for sure.

In summary, the conditions on the preference distribution under which robust equilibria with vote "dispersion" can arise under plurality rule are rather extreme.

Proposition 1. *Suppose that $n_{ij} > 0$ for all (distinct) pairs (i, j). Moreover, assume that either*

i) $\min_i n_i < \max_i n_i$ *or that*

ii) there exists one individual who prefers having his second ranked candidate for sure over a uniform lottery over all three candidates.

A voting profile s^* is robust if and only if the following conditions are satisfied.

a) Exactly two candidates receive a positive number of votes.

b) If $v_k(s^*) = 0$, then $V_i(s^*) = \Sigma_{ij}$, for $i, j \neq k$. That is, each player chooses "sincerely" between those two candidates who receive a positive number of votes.

Proof. See Appendix. □

Proposition 1 has two interesting implications. First, the Condorcet loser can never be a robust equilibrium outcome. Remember that iterated elimination of weakly dominated strategies generally does not support this conclusion. Second, it shows that plurality rule has a tendency to generate equilibria in which all votes are concentrated on two candidates. This is, of course, Duverger's law, a well known stylized fact of political systems operating under plurality rule. Note again that only applying iterated elimination of weakly dominated strategies is not sufficient to generate this result.

Second, Proposition 1 extends in a straightforward way to the case of more than three candidates (with minor adjustments to the conditions used to exclude equilibria with more than two active candidates).[5] The reason is that the main driving force — namely that a voter is much more likely to be pivotal between the two top-scorers than between any other pair of candidates — applies irrespective of the total number of candidates.

The intuition for Proposition 1 is similar to the one in Palfrey (1989), where a given number of actual voters are drawn from some known preference distribution of the population. For the limit case of very many voters, he shows that there can only be equilibria in which just two candidates receive a positive measure of votes. While the details of Palfrey's and our model are different, in both models the probability of a tie between the top vote-getter and the runner-up is relatively much larger than the probability of a tie between any other pair of candidates.

Myerson and Weber (1993) also analyze the effects of trembling in plurality elections. They define the concept of a 'voting equilibrium' as a strategy profile in which every voter acts optimally, given some vector q of "pivot probabilities" that contain, for every pair of candidates, the probability that these two candidates tie for the first place. The vector q is specified exogenously and, assumed to satisfy an 'ordering condition': If candidate C_i receives fewer votes than candidate C_j, then, for any other candidate C_k, the pivot probability between candidates C_i and C_k is only at most ε times the pivot probability between candidates C_j and C_k (where ε is very small). The error process in our model generates this property endogenously.

Myerson and Weber give the following example of a voting equilibrium in which three candidates (A, B and C) receive a positive number of votes. 30 percent of the electorate are of type 1 ($A \succ B \succ C$), another 30 percent are of type 2 ($B \succ A \succ C$), and 40 percent are of type 3 ($C \succ A \sim B$). Consider a pivot vector of $q_{AB} = 0$, $q_{AC} = 1/2$ and $q_{BC} = 1/2$, i.e., conditional on a tie occurring, the probability that it is between candidates A and C, and between B and C is 1/2 each, and the probability that the tie is between A and B is zero.[6] Given these probabilities, it is an equilibrium for each preference group to vote for their preferred candidate, so that candidate C wins under plurality rule, although he is the Condorcet loser.[7]

Myerson and Weber interpret this result as refuting Palfrey's (1989) result (derived in a slightly different setting) that states that small uncertainty in the voting process is sufficient to generate Duverger's law. It also seems to contradict our result. However, their example relies on the assumption that the vector of pivot probabilities is *the same for all voters*, no matter how these voters are supposed to vote in equilibrium. This assumption is crucial for any equilibrium involving 3 or more relevant candidates and not satisfied in our model that models an error process to generate the pivot probabilities endogenously.

To see this, note that any voter should act optimally given *his own* pivot probability vector,

[5] The exact conditions that need to be satisfied in the case with more than three candidates are spelled out in the remark after the proof of Proposition 1 in the Appendix.

[6] There are multiple pivot vectors which work, depending on the cardinal utility that types get from their first and second candidate.

[7] This result is robust to small changes in the proportion of type 1/type 2 voters. This works as follows: If there are more type 1 than type 2 voters, then all type 2 and some type 1 voters vote for B, while the rest of the type 1 voters vote for A, so that the number of votes for A and B is equal. This is supported by a q vector that has $q_{AC} < q_{BC}$ (which is possible, since Myerson and Weber's ordering condition imposes no restrictions on the pivot probabilities when A and B receive the same number of votes). These pivot probabilities make type 1 voters indifferent between voting for A and voting for B, and hence willing to randomize.

i.e. what is the probability that this voter's vote is pivotal, given *the other voters' actions?* In fact, for any finite voting population and given Myerson and Weber's ordering assumption, the vector of pivot probabilities in the given voting profile is $(q_{AB} = 0, q_{AC} = 0, q_{BC} = 1)$ for a type 1 voter, because there are more votes from other voters for candidate B than for candidate A. Hence, a type 1 voter would benefit from deviating to voting for candidate B. Similarly, the pivot probability vector for a type 2 voter, given that there are more votes of other voters for candidate A than for B, is $(q_{AB} = 0, q_{AC} = 1, q_{BC} = 0)$, and so type 2 voters would also deviate from this strategy profile.

4 Runoff Rule

4.1 Description of the Game

Under runoff rule, there are potentially two rounds of elections. In the first round, a candidate is elected if he achieves an absolute majority of all votes. If there is no such candidate, then the two candidates who received the most votes in the first round face off against each other in a runoff, and the overall winner is the candidate who gets more votes in the runoff round. Just as in the case of the plurality voting games we assume that any possible ties – both in the first and second round – are broken by uniform randomization.

In this two stage game, a strategy of player n is a pair $s_n = (s_n^1, s_n^2)$, where $s_n^1 \in \mathcal{C}$ represents the vote which the player casts in the first round election while $s_n^2 \in \{C_1, C_2\} \times \{C_1, C_3\} \times \{C_2, C_3\}$ describes his plans for the runoff round, where voting can be conditioned on the pair of candidates that competes in the runoff. In order to highlight this we write $s_n^2(ij)$ for the component of s_n^2 which corresponds to a runoff between candidates C_i and C_j.[8]

Strategy profiles are denoted by s. We write $V_i^1(s)$ and $v_i^1(s)$, respectively for the set and number of voters who cast their first round ballot for candidate C_i in strategy profile s.

Perturbations of the runoff voting game and the robustness concept that we apply to such games are defined just like in the case of plurality voting games. In particular, a perturbed runoff voting game is a runoff game where the vote count in both stages is subject to errors.[9] The probability with which a ballot is lost due to a miscount is identical (and independent) across ballots. Notice that this also means that miscounts are independently distributed across voting stages. A vote profile s is robust (or a robust equilibrium) if there is an $\bar{\varepsilon} > 0$ such that s is a Nash equilibrium in all ε-perturbed games with $\varepsilon \in (0, \bar{\varepsilon})$. As argued above, our robustness concept is closely related to THP. In the context of runoff voting games, robustness can be interpreted more specifically as 'counterpart' of *agent normal form THP*.

Under runoff rule, there are potentially two rounds of elections, so that keeping track of the

[8]In principle, we could allow for strategies that condition on first round actions in a more complex way. However, as Lemma 3 below implies, the restriction is without loss of generality since voting behavior in the second round must be sincere in any robust equilibrium.

[9]Given the possibility of miscounts in the first round we need to be careful about whether a candidate needs an absolute majority of all votes that are *effectively counted*, or an absolute majority of all votes *casted*. Mainly for technical and expositional convenience, we have decided to adopt the second assumption. However, our results are relatively robust to changing this assumption.

number of miscounts that are necessary for victories of the various candidates is somewhat more complex than in static plurality voting games. In the interest of a compact exposition, it is useful to introduce some further notation and terminology. If s is a vote profile in which candidate C_j does not obtain an absolute majority of all first round votes, then we write $T_{ji}(s)$ for the minimal number of miscounts that are necessary in order to trigger a victory of candidate C_j through a second round election against candidate C_i. Moreover, $T_j(s) = \min_{i \neq j} T_{ji}(s)$ denotes the minimal number of miscounts that are necessary in order for candidate j to win. We refer to $T_{ji}(s)$ as the *ji-miscount margin at s*; $T_j(s)$ instead is the *miscount margin of candidate C_j at s*. A *t-configuration of miscounts* (or simply, $t-$configuration) is a specific distribution of t miscounts over the voters' ballots.

4.2 Results

Our first observation is that the runoff round is a binary election. In unperturbed binary voting games voting for the less preferred candidate is a weakly dominated strategy. In the perturbed game with the possibility of miscounts, this dominance relation becomes strict.

Lemma 3. *In any robust equilibrium the voting behavior in the second round elections is sincere.*

In particular, Lemma 3 implies that a Condorcet loser cannot win the election in the second round.

In contrast to the second round, (iterated) dominance arguments do not restrict first-round behavior very much under runoff rule. Unlike under plurality rule, even voting for one's least preferred candidate may be an (iteratively) undominated strategy, as Example 1 below shows. The reason for this is that, under runoff rule, there are two possible motives to vote for a given candidate in the first stage. First, like under plurality rule, a player might want to vote for the candidate because he likes him (at least better than the most likely alternative). Second, and without parallel under plurality rule, a voter might support a candidate in the first round because that candidate is "easy to beat" by his preferred candidate in the runoff. These two motives are often in conflict with each other, and this is the fundamental reason for why the analysis of robust equilibrium behavior is much more involved under runoff rule than under plurality rule.

Example 1. *Consider the voter preferences given in Table 1. The number in the top row indicates the size of each preference group. Boxes indicate the first round voting behavior of individuals of the different preference types (for instance, individuals of the first preference group vote for candidate C_1).*

34	1	1	32	32	1
$\boxed{C_1}$	C_1	$\boxed{C_2}$	$\boxed{C_2}$	$\boxed{C_3}$	$\boxed{C_3}$
C_2	C_3	C_1	C_3	C_1	C_2
C_3	$\boxed{C_2}$	C_3	C_1	C_2	C_1

Table 1: The individual in N_{13} optimally votes for his least preferred candidate

In the voting profile in Table 1, all players, except for the only 13-type voter, vote for their most preferred candidate. Given this behavior of the other voters, the unique optimal strategy for the player in N_{13} is to vote for his least preferred candidate, C_2: A vote for C_2 secures a second round election between C_1 and C_2, which is won by C_1 (remember that all players vote sincerely in the second round). Instead, voting for C_1 or C_3 would trigger, with probability 1/2, a runoff between C_1 and C_3 which would be won by C_3. □

Thus, dominance arguments alone impose very few restrictions on first round voting behavior in runoff games. However, as the following results in this section show, our robustness concept has substantially more bite, and allows us to derive two types of results. The first one concerns the structure of the vote distribution and shows that in a robust equilibrium, *all* candidates must receive a positive number of votes (Proposition 2). The second type of results relate to the identity of the election winner. These are more complex (and therefore not useful to summarize here), but they also often narrow the set of possible equilibrium election winners relative to plurality rule, both in the case that a Condorcet winner exists and in the case that none exists.

We start by next turn to robust equilibria in which the election is decided in the first round (if there are no miscounts). Lemma 4 shows that a first round winner cannot receive any votes from voters who prefer the losing candidate with the smallest miscount margin to the winner.

Lemma 4. *Let s^* be a robust profile of the runoff voting game such that $v_i^1(s^*) \geq (N+1)/2$.*

i) $T_j(s^*) < T_k(s^*)$ implies $V_1^i(s^*) = \Sigma_{ij}$.

ii) Suppose that $T_j(s^*) = T_k(s^*)$ and $\Delta_{jk} > 0$. Then, $V_i^1(s^*) \cap \Sigma_{ji} = \emptyset$. Moreover, C_i is either a Condorcet winner or $\Sigma_{ji} = N_{ji} = (N-1)/2$.

Proof. See Appendix. □

Just as under plurality rule, the winner (C_i) gets exactly all the votes of those players who rank him above the runner up (C_j). In both cases, the underlying intuition for the result is as follows: The chance that C_i's first round victory is upset by miscounts in favor of C_j decreases with the number of votes for C_i. Thus all voters who rank C_i above C_j have an incentive to vote for C_i, and voters who prefer C_j over C_i should not vote for C_i.

Lemma 4 implies that a first round winner (C_i) can be elected only by players who prefer him over the candidate who is most likely to benefit from miscounts (C_j). Thus, it must be the case that C_i Condorcet dominates C_j. This in turn implies that a Condorcet loser (in case there is one) can never win outright in the first round. Since we also know (by Lemma 3) that a Condorcet loser cannot win in the runoff round either, we obtain the following result.

Corollary 1. *Runoff voting games do not admit robust equilibria in which the Condorcet loser wins the election.*

Even in situations where the election is decided in the first round, one important difference between plurality rule and runoff rule lies in the behavior of voters who do not vote for the winner. Under plurality rule, the ballots of those players must be concentrated on the runner up. One of our main results, Proposition 2 below, shows that this is no longer true under runoff rule: In fact, in the first round under runoff rule, all candidates must receive some votes.

However, before we can develop this result, we need to analyze whether there can be equilibria in which the election is won by C_i in the first round if C_i is not a Condorcet winner. Note that there are two cases when this situation can arise: Either, there is a Condorcet cycle, or C_i is the middle candidate in a transitive Condorcet ranking (i.e., the one who is neither the Condorcet winner nor the Condorcet loser). By Lemma 4 we know that, if candidate C_i wins outright then he Condorcet-dominates the runner-up (i.e., the candidate with the smallest miscount margin). Moreover, by Lemma 4, C_i gets exactly the votes of all players who rank him above the runner-up. However, Lemma 4 is silent about the distribution of the remaining votes among C_j and C_k.

Lemma 5 fills this gap. It shows that if C_j is the runner-up in a profile in which C_i wins outright, then among the players in $\Sigma_{ji} = N_{ji} \cup N_{jk} \cup N_{jk}$, exactly those in N_{ji} vote for C_j.

Lemma 5. *Let s^* be a robust profile such that $v_i^1(s^*) \geq (N+1)/2$ and $T_j(s^*) < T_k(s^*)$. If $\Delta_{ki} > 0$ then $V_j^1(s^*) = N_{ji}$ and $V_k^1(s^*) = L_i = N_{jk} \cup N_{kj}$. Moreover, $n_{ji} > \sigma_{ij}/2 \geq (N+1)/4 > l_i + 1$.*

Proof. See Appendix. □

The intuition for Lemma 5 is as follows. Because C_i is neither a Condorcet winner nor a Condorcet loser, and C_i Condorcet-dominates C_j, C_i must be dominated by C_k. Since C_i gets the largest number of votes it follows that any runoff election that involves C_i is more likely than a runoff among C_j and C_k. Since C_k wins a runoff against C_i it must be the case that the ki-miscount margin is strictly smaller than the jk- and kj-margin. Therefore C_j can be the runner-up only if $T_{ji}(s^*) < T_{ki}(s^*)$. But in order to satisfy this condition, C_j must receive sufficiently more votes than C_k in the first round, in order to compensate for C_k's advantage against C_i. Specifically, it must be the case that $v_j^1(s^*) - v_k^1(s^*) > \Delta_{ij}$.

Now consider the effects of shifting a vote from C_j to C_k. Doing so reduces the gap between v_j^1 and v_k^1 by two, and the $ki-$margin also decreases by two. Miscount configurations that contain exactly $T_{ki}(s^*) - 2$ miscounts and yield C_k as outcome in the post-deviation situation, can contain only miscounts of first round votes. Thus, in the pre-deviation situation, where such configurations must have led to a ij-runoff, they cannot have produced C_j as winner. For any other miscount configuration that comprises no more than $T_{ki}(s^*) - 2$ miscounts there can be no change in the outcome. To see this, note that both before and after the deviation all such configurations either end with an outright win of C_i or lead to a runoff between C_i and C_j. The deviation neither affects the votes that determine whether or not there is a second round election (i.e. C_i's first round votes) nor the votes that determine who wins the second round. But then the deviation cannot change the outcome distribution for the miscount configurations under consideration either.

Thus, the deviation's only relevant effect is a replacement of candidate C_i by candidate C_k. Hence, whether or not a voter who does not vote for C_i in s^*, votes for C_j only depends on how he ranks C_i relative to C_k. By Lemma 4 we already know that C_j and C_k together obtain exactly the votes of all players in Σ_{ji} and so it follows that $V_j^1(s^*) = \Sigma_{ji} \cap \Sigma_{ik} = N_{ji}$,

$V_k^1(s^*) = \Sigma_{ji} \cap \Sigma_{ki} = L_i = N_{jk} \cup N_{kj}$. Finally, combining this observation with the requirement that $v_j^1(s^*) - v_k^1(s^*) > \Delta_{ji}$ yields the conditions $2n_{ji} > \sigma_{ij} \geq (N+1)/2$ and $l_i + 1 < (N+1)/4$ in Lemma 5.

Lemma 5 shows that the conditions under which there may be robust equilibria in which a candidate who is not a Condorcet winner wins the elections in the first round are very restrictive. First, such a candidate must have few core opponents (less than a quarter of the population). Second, the candidate whom he Condorcet-dominates must have many core supporters who in their majority rank the election winner second. It is easy to see that the second condition can never hold when players' preferences are single peaked with respect to a linear ordering of the candidates and the Condorcet winner is in the central position. In that case, all core supporters of the Condorcet loser rank the Condorcet winner above the third candidate. Also, if there is a Condorcet cycle, the first of these conditions can be met at most by the candidate with the smallest core support. To see this, note that if there were two candidates with fewer core opponents than 1/4 of the population, then the third candidate must have more than half of the population as core opponents; but then, he must be a Condorcet loser.

Lemma 5 also shows that equilibria in which a candidate who is neither the Condorcet winner nor the Condorcet loser wins the election in the first round, must typically (and, in contrast to plurality rule) be characterized by "vote dispersion": All three candidates receive a positive number of votes. The following Proposition 2 shows that vote dispersion is a feature of all robust equilibria, provided that all candidates do have some core opponents.

Proposition 2 (Duverger's Hypothesis)**.** *Assume that $n_{ij} > 0$ for all (distinct) pairs (i, j). If either of the following two conditions holds then in every robust equilibrium each candidate receives a strictly positive number of first round votes.*

i) The Condorcet ranking is not transitive.

ii) Candidate i is the Condorcet winner, and his winning margins $\sigma_{ij} - \sigma_{ji}$ and $\sigma_{ik} - \sigma_{ki}$ are both strictly larger than one.

Proof. See Appendix. □

It is easy to see that, if there is a Condorcet cycle (i.e., if the first condition in Proposition 2 holds), then all three candidates must receive some first round votes. If, by contradiction, first-round votes were concentrated on just two candidates, then one of them must win outright, and since this winner is not a Condorcet winner, Lemma 5 applies and shows that votes cannot be concentrated on two candidates, the desired contradiction.

Lemma 5 also rules out the possibility that the Condorcet winner (in case there is one) does not receive any votes (that is, votes cannot be concentrated on the other two candidates). We therefore now focus on providing an intuition for why in situations where a Condorcet winner exists there can be no equilibria in which either the Condorcet loser or the candidate that takes the middle position in the Condorcet ranking do not receive any votes. Specifically, we consider voter preferences in which Candidate C_1 is the Condorcet winner, and Candidate C_2 has the closer winning margin in a runoff round against Candidate C_1 than Candidate C_3 (i.e.

$0 < \Delta_{12} < \Delta_{13}$). Note that this case is compatible with either C_2 being majority-preferred to C_3, or vice versa.

Now take a profile of first-round votes in which only C_1 and C_3 receive votes. By Lemma 5 such a profile is not robust if C_3 receives more votes than C_1. So assume that $v_1^1 \geq (N+1)/2$. Consider a voter who prefers C_2 over C_3 over C_1, and who switches his vote from C_3 to C_2.[10] Just like in the case above, this does not affect the probability of C_1 winning outright, as the number of votes for and against C_1 has not changed. There are two possible changes in the runoff: Either, C_2 replaces C_3, or C_2 replaces C_1 in the runoff. Since C_1 receives more votes than C_3, the first of these cases is much more likely than the second one. Moreover, the chance that C_2 wins the runoff against C_1 is much larger than the chance that C_3 wins the runoff against C_1, because the first event requires fewer miscounts in the second round than the second one. Therefore, the voter whom we have considered is better off switching his vote to C_2 instead of C_3.

This leaves us with one final two-candidate first-round vote profile to consider in which only C_1 and C_2 receive votes. We now claim that such a vote profile cannot be robust since voters in N_{32} would have an incentive to deviate from voting for C_2 to C_3.[11]

A deviation from C_2 to C_3 reduces the vote gap between C_3 and the other two candidates. This makes it more likely that C_3 survives the first stage of the election. In particular, the minimal number of miscounts that are necessary in order to trigger a runoff race between C_2 and C_3 decreases from v_1^1 to $t := v_1^1 - 1$. Now observe that any $t-$event that leads to the runoff pair (C_2, C_3) after the deviation only involves first round miscounts. Thus, any such event must have produced C_1 as outcome before the deviation. Since voters in N_{32} prefer both C_2 and C_3 over C_1 this implies that the deviation generates benefits of at least order t for individuals in N_{32}. Therefore these individuals deviate if there is no cost associated with the deviation that is of order t or lower.

What are the potential costs of the deviation? For voters in N_{32}, an undesirable outcome change is a switch from C_2 to C_1, but we argue now that such a switch requires more than t miscounts and is thus much less likely than the benefit described above. The only way how C_2 can win in the pre-deviation situation with at most $t = v_1^1 - 1$ miscounts is via a runoff between C_1 and C_2. Moreover, the only possibility that the deviation can lead to a switch of the outcome from C_2 to C_1 is by triggering a change of the runoff pair from (C_1, C_2) to (C_1, C_3). This means that events for which the deviation leads to a switch from C_1 to C_2 must involve at least the following miscounts:

a) $v_1^1 - \frac{N-1}{2}$ miscounts of first round C_1-votes (otherwise, C_1 wins outright);

b) $v_2^1 - 2$ miscounts of first round C_2-votes (otherwise, the deviation does not trigger a change in the runoff pair from (C_1, C_2) to (C_1, C_3)).

[10] Remember that a necessary condition for the robustness of the presumed vote profile is (see Lemma 4) that C_3 either obtains all votes of the sympathizers of C_3 against C_1 or it gets the votes of all sympathizers of C_2 against C_1. Since $N_{23} \subset \Sigma_{21} \cap \Sigma_{31}$ it thus follows that the vote profile can be robust only if $N_{23} \subset V_3^1$.

[11] Again, we show in the proof of Proposition 2 that these voters cannot vote for C_1 in equilibrium, and assume that they are voting for C_2.

c) Δ_{12} second round votes of individuals in Σ_{12}, for otherwise, C_2 could not win the runoff against C_1 in the pre-deviation situation.

By Lemma 4, a necessary condition for robustness of our voting profile is that either $\Sigma_{21} \subset V_2^1 \cup V_3^1 = V_2^1$ or $\Sigma_{31} \subset V_2^1 \cup V_3^1 = V_2^1$, depending on whether C_2 or C_3 is more likely to benefit from miscounts.

Assume first that $\Sigma_{21} \subset V_2^1$. In this case, summing up the miscounts in the above list yields

$$\begin{aligned} & v_1^1 - \frac{N-1}{2} + v_2^1 - 2 + \Delta_{12} \\ \geq \quad & v_1^1 - \frac{N-1}{2} + \sigma_{21} - 2 + N - 2\sigma_{21} \\ = \quad & v_1^1 - 1 + \frac{N-1}{2} - \sigma_{21} > v_1^1 - 1 = t. \end{aligned}$$

Thus, for ε sufficiently small, the probability that the deviation generates a benefit is much larger than the probability that the deviation generates a cost.

Finally, assume that $\Sigma_{31} \subset V_2^1$. This is the case only if, under the pre-deviation vote profile, C_3 is at least as likely to benefit from miscounts as C_2. But then, since C_3 becomes strictly more likely as outcome due to the deviation (i.e. the minimal number of miscounts that are required to obtain C_3 strictly decreases), it must be replacing C_1 as outcome. Moreover, there can be no cost of the same order as this benefit since in the pre-deviation situation only C_1 wins in events of that order.

Proposition 2 is a central result, because it shows that our voting model admits *only* equilibria that are consistent with *Duverger's Hypothesis* (Duverger (1963)).[12] Duverger argues that, unlike plurality rule, runoff rule favors the emergence of a multi-party system. Arguably, a necessary condition for the co-existence of more than two parties is that all the competing parties receive a positive vote share. This is exactly what Proposition 2 shows: If preferences are sufficiently diverse (i.e., all potential preference groups are represented in the population) then in every robust equilibrium of a runoff voting game all candidates must get at least some votes. If there are more than three candidates, this result generalizes as follows: There must be at least three candidates who receive a positive vote share. To the best of our knowledge our paper is the first one that proposes an equilibrium concept that generates *both* (i) under plurality rule, only equilibria that satisfy Duverger's Law, and (ii) under runoff rule, only equilibria that satisfy Duverger's Hypothesis.

Our next step in the characterization of the set of robust equilibria under runoff rule is to consider voting profiles that lead to a second round election. Lemma 6 considers situations where a Condorcet winner exists and shows that a vote profile in which the Condorcet winner does not reach the runoff round cannot be a robust equilibrium.

Lemma 6. *Assume that candidate C_i is the Condorcet winner. If a voting profile s satisfies the condition $(N+1)/2 > v_j^1(s), v_k^1(s) \geq v_i^1(s)$, then s is not a robust equilibrium.*

Proof. See Appendix. □

[12] Strictly speaking, Proposition 2 just shows that there are no equilibria in which votes are concentrated on two candidates, but not that there exist equilibria in which votes are distributed over all three candidates. However, we show in Propositions 3 and 4 below that existence holds as well.

To understand this result, assume (without loss of generality) that the Condorcet ordering is given by $C_i \succ C_j \succ C_k$. In a voting profile such as the one described in Lemma 6, the election is won by C_j after a runoff against C_k (unless miscounts trigger either a different runoff or a different outcome of the runoff round). By assumption, a majority of the electorate ranks C_i above C_j but less than a third of the electorate votes for C_i (otherwise C_i would get into the runoff). Thus, some of those who vote for C_j or C_k prefer C_i over C_j. By deviating to C_i, they can move the outcome distribution in that direction.

We next consider voter preferences for which no Condorcet winner exists.

Lemma 7. *Assume that the Condorcet ordering is not transitive.*

i) If $(N-1)/2 \geq v_i^1(s) - 1 > \max(v_j^1(s), v_k^1(s))$ then s is not a robust equilibrium.

ii) Suppose that $C_i \succ C_j$. If s is a robust equilibrium such that $(N-1)/2 \geq \max\{v_j^1(s), v_k^1(s)\} > \min\{v_i^1(s), v_j^1(s), v_k^1(s)\} = v_i^1(s)$, then $L_j \subset V_i^1(s)$.

Proof. See Appendix. □

Part i) shows that, if there is a Condorcet cycle, a voting profile that leads to a runoff can be robust only if the first round vote profile is tight. That is, the difference between the first-ranked and the second-ranked candidate is at most one vote.

To get some intuition for this result, suppose that the Condorcet cycle is given by $C_i \succ C_j$, $C_j \succ C_k$ and $C_k \succ C_i$. Consider first a voting profile where $v_i^1 - 1 > v_j^1 \geq v_k^1$. In such a profile, all voters of candidate C_i have an incentive to deviate. Those who rank C_i above C_k profit from a deviation towards C_j since it decreases the risk that in the second round election C_i faces C_k instead of C_j. Players who instead prefer C_k to C_i want to increase the probability of an ik-runoff and thus want to switch their vote towards C_k. An analogous argument applies when $v_i^1 - 1 > v_k^1 > v_j^1$.

Part ii) of Lemma 7 shows that, unless there is a three way tie among all three candidates, the one who receives the fewest votes must receive all votes of the core opponents of the dominant candidate among the other two candidates. The intuition is again straightforward. Given that there is a Condorcet cycle, the candidate who receives the fewest votes in the first round (C_i) Condorcet-dominates the winning candidate (C_j). All players who rank C_j lowest have an incentive to reduce C_j's winning chances, no matter which of the other two candidates' winning probabilities increases. They can achieve this goal by deviating to C_i, which reduces the chances of a runoff between C_j and C_k in favor of a runoff that involves C_i and is won by either C_i or C_k.

Why does the same argument not apply in the case of a three-way tie? If all three candidates get the same number of votes, then the relevant effect of any possible deviation is to eliminate the candidate who loses the vote from the race. This is not the case in voting profiles where the first- and last-ranked candidates do not get the same number of votes. In those situations there also exist deviations whose main effect is to enhance the chances of the candidate who receives the additional vote to enter the second round.

It is interesting to think about the fundamental reason for the difference between Lemmas 6 and 7. Lemma 6 shows that there is no robust voting profile that leads to a runoff without the

CW (candidate C_i). This is due to the fact that those voters who prefer C_i over the winner, but do not vote for C_i benefit from switching their vote to C_i, as they increase C_i's chance of entering (and winning) the runoff round.

Of course, when no CW exists, there are also some voters who prefer C_i over the dominant candidate among C_j and C_k. Why is it then that, when no CW exists, Lemma 7 does not rule out the possibility of robust voting profiles that will lead to a runoff? The reason is that, if candidate C_i is the CW then he beats both of the other two candidates in pairwise comparison, so that an increase in the number of votes for C_i always translates into an increase in the chances of C_i, independently of which of the other two candidates proceeds to the runoff. In contrast, in case of a Condorcet cycle, the effects of a deviation towards C_i are more ambiguous, as C_i dominates only C_j, but is dominated by C_k. Thus, whether a deviation towards C_i increases C_i's winning probability depends on which of the two runoff combinations that include C_i increases more in probability. Not all voters who rank C_i above C_j will necessarily deviate towards C_i if the relevant effect of the switch is an increase of the probability of a runoff between C_i and C_k. In this case, the deviation amounts to a replacement of C_j by C_k and therefore decreases the utility of voters in $N_{ij} \subset \Sigma_{ij}$.

In combination with Lemma 5, Lemma 7 shows an important result regarding the set of equilibrium outcomes for environments where no Condorcet winner exists. If the number of voters is not a multiple of three, then all candidates ranked lowest by more than a third of the electorate can be ruled out as robust equilibrium outcomes. Generically, there is at least one such candidate, and in many situations, it may even be possible to reduce the set of candidates who can win the election to a single candidate.

Our analysis so far has concentrated on characterizing which candidates are possible equilibrium outcomes. We now turn to the question of existence of a robust equilibrium under runoff rule. Unlike in the case of plurality rule, it is not as straightforward to see that robust equilibria always exist. Since we require equilibria to be in pure strategies, we take a constructive approach. Our first result covers cases where no Condorcet winner exists.

Proposition 3. *Suppose the Condorcet cycle is given by $C_i \succ C_j$, $C_j \succ C_k$ and $C_k \succ C_i$. Let candidate C_j have the smallest set of core opponents, i.e. $l_j < l_i, l_k$. Moreover, assume that the sets N_{ij} and N_{ki} both contain at least one individual who prefers a uniform lottery over all three candidates to having his second ranked candidate for sure. There either exists a robust equilibrium profile s such that $v_j^1(s), v_k^1(s) > v_i^1(s)$ and $|v_j^1(s) - v_k^1(s)| \leq 1$ or there is a robust profile s' such that $v_i^1(s') = v_j^1(s') > v_k^1(s')$.*

Proof. See Appendix. □

When the Condorcet ranking is transitive then there are always equilibria in which the Condorcet winner gets an absolute majority in the first round. The exact voting behavior in these equilibria often depends on the details of the preference distribution (i.e. on the absolute and relative sizes of the preference groups). This is the reason why we have to go through several different cases to show that existence holds for the entire space of voter preference distributions.

Proposition 4. *Suppose that the Condorcet ordering is given by $C_i \succ C_j \succ C_k$.*

i) If $(N+1)/3 > \sigma_{ji}, \sigma_{ki}$ *and* $\sigma_{ji} > 4\sigma_{ki} - N$ *then there exists a robust equilibrium* s *satisfying* $V_i^1(s) = \Sigma_{ij}$ *and* $1 \geq v_j^1(s) - v_k^1(s) \geq 0$.

ii) If $(N+1)/3 > \sigma_{ji}, \sigma_{ki}$ *and* $\sigma_{ji} < 4\sigma_{ji} - N$ *then there exists an equilibrium* s *satisfying* $V_i^1(s) = \Sigma_{ik}$, $V_k^1(s) = N_{ki}$ *and* $V_j^1 = L_i$.

iii) If $\sigma_{ji} > (N+1)/3, \sigma_{ki}$ *then there exists a robust equilibrium* s *satisfying* $V_i(s) = \Sigma_{ij}$, $1 \geq v_j(s) - [2\sigma_{ji} - (N+1)/2] \geq 0$ *and* $v_j^1(s) > v_k^1(s)$.

iv) If $\sigma_{ki} > (N+1)/3, \sigma_{li}$ *then there exists a robust equilibrium* s *satisfying* $V_i(s) = \Sigma_{ik}$ *and* $v_k^1(s) > v_j^1(s)$. *Moreover, at this equilibrium either* $1 \geq v_k(s) - [2\sigma_{ki} - (N+1)/2] \geq 0$ *or* $v_k^1(s) = n_{ki}$.

Proof. See Appendix. □

In summary, Propositions 3 and 4 together show that a robust equilibrium always exists.

5 Conclusions

In this paper, we analyze the implications of imperfections in the vote counting technology on the equilibrium sets of plurality and runoff voting games. In the context of voting, real life experience suggests that ballots are indeed occasionally lost or miscounted, so that our refinement method also has a straightforward intuitive interpretation. While our robustness concept is related to the notion of trembling-hand perfectness, the distribution of our miscounts cannot depend on the players' identities. In the context of vote-counting errors, this symmetry assumption appears reasonable and considerably strengthens the implications of our robustness refinement relative to trembling-hand perfection.

Under plurality rule, all robust equilibria satisfy Duverger's Law: That is, exactly two candidates receive all votes. Under very general conditions, all robust equilibria under runoff rule satisfy Duverger's Hypothesis: That is, first-round votes are dispersed over more than just two candidates.

In terms of equilibrium outcomes, robustness to miscounts generates sharper predictions under runoff rule than under plurality rule. Under plurality rule, the only outcome that can be excluded is a victory of the Condorcet loser.

In contrast, under runoff rule, the equilibrium outcome is unique for many constellations of voter preferences. If there is a Condorcet cycle, then all candidates whose core opponents (i.e., voters who rank this candidate lowest) are more than 1/3 of the population cannot be the outcome in a robust equilibrium. Generically, there is at least one such candidate, and often, there are two, in which case the unique robust equilibrium outcome is unique. Instead, if there is a Condorcet winner, there always is a robust equilibrium in which the Condorcet winner is elected. Moreover, conditions on the preference distribution that have to be satisfied for the second candidate in the Condorcet ranking to be the equilibrium election winner are very restrictive. Finally, the Condorcet loser cannot be the outcome in a robust equilibrium.

While our formal analysis is for three candidates, it is clear that several major results extend to settings with more than three candidates. As we have already pointed out Sections 3 and 4, Duverger's Law under plurality rule and Duverger's Hypothesis under runoff rule will continue to hold in settings with more than three candidates. In contrast, the characterization of which candidates can be the election outcome under runoff rule becomes considerably more complex as the number of candidates increases.

Appendix

Plurality Rule

Proof of Lemma 2

We have to show that no profile s with $v_i(s) > v_j(s) > v_k(s) > 0$ can be an equilibrium of a perturbed game $\mathcal{P}_\varepsilon$ for ε small enough.

So consider a profile s of the above mentioned type and notice that the probability that candidate C_i will win is $1 - o(v_i(s) - v_j(s))$ (where $o(x)$ is a term of order x in ε),[13] the probability that candidate C_j will win is $o(v_i(s) - v_i(s))$, and the probability that candidate C_k ends up as the winner is $o((v_i(s) - v_k(s)) + (v_j(s) - v_k(s)))$.

Consider player n who intends to vote for the third placed candidate, and fix all other individuals' vote intentions. The probability that n's vote is pivotal for a decision between C_k and C_i, or between C_k and C_j, is $o((v_i(s) - v_k(s)) + (v_j(s) - v_k(s)))$ in ϵ. On the other hand, the probability that n's vote is pivotal in a decision between Candidate 1 and Candidate 2 is $o(v_i(s) - w_j(s))$ in ϵ. In all other cases, voter n's vote does not matter. Since $v_i - v_j < (v_i - v_k) + (v_j - v_k)$, there exists an $\bar{\varepsilon} > 0$ such that the action which maximizes expected utility for all $\varepsilon \in (0, \bar{\varepsilon}]$ is to vote for C_i, if $u_n(C_i) > u_n(C_j)$, and to vote for C_j if $u_n(C_j) > u_n(C_i)$. Consequently, a vote for the third placed candidate cannot be optimal.

Proof of Proposition 1

First, it is easy to see that the profile in which just candidates C_i and C_j receive votes, and every voter who prefers C_i over C_j votes for C_i and vice versa, is an equilibrium. Suppose that $\Delta_{ij} > 0$. The probability that an individual's vote is pivotal between C_i and C_j is a term of order Δ_{ij}, while the probability that his vote, if cast for a third candidate, is pivotal between any other pair of candidates is a term of order $N-1 > \Delta_{ij}$. Hence, there is no point in deviating to vote for a third candidate. Similarly, it is also obvious that it cannot be optimal for a voter who prefers C_i to C_j to vote for C_j, and vice versa.

We now turn to the proof that there are no other equilibria. First, note that there cannot be an equilibrium in which only one candidate receives votes. This follows immediately from Lemma 1. Second, we will show that there cannot be an equilibrium in which three (or more) candidates receive a positive number of votes.

Lemma 2 already shows that we cannot have $v_i(s^*) \geq_j (s^*) > v_k(s^*) > 0$. The remaining possible configuration in which 3 candidates receive votes is $v_i(s^*) \geq v_j(s^*) = v_k(s^*)$.

First, can we have $v_i(s^*) = v_j(s^*) = v_k(s^*)$? In such an equilibrium, every core supporter of candidate C_i must vote for C_i (otherwise, a voter who prefers C_i, but votes for $C_j \neq C_i$ could switch to vote for C_i and secure C_i's victory). Thus we can rule out a three way tie if one of the three candidates has a strictly smaller core support than one of the other two as stated in condition i) of the proposition.

[13] The reason is that at least $v_i(s) - v_j(s)$ votes need to be not counted in order that there is a possibility that candidate C_i will not receive the most of the counted votes.

Suppose therefore that condition one is violated and observe that in an all over tie, every voter can secure a victory of his second ranked candidate (by switching his vote to this candidate). Since condition ii) guarantees that there is at least one voter who prefers this result to the even lottery over the three candidates, we again arrive at the conclusion that $v_i(s^*) = v_j(s^*) = v_k(s^*)$ is not possible.

Second, can we have $v_i(s^*) > v_j(s^*) = v_k(s^*)$? Again, it must be true that the respective core supporters vote for their favorite candidates: This is obvious for the core supporters of the winner, but it is also true for the core supporters of the runners-up: Suppose it were not so, but rather a core supporter of (say) C_j voted for C_k; in this case, switching the vote to C_j increases the probability that C_j wins by one order of magnitude (in ε), and in comparison to this effect, the change in C_k's winning probability is negligible.

However, consider the voting behavior of an individual who prefers C_j to C_k to C_i and who at s^* votes for C_j. Switching the vote to C_k increases the probability that C_k is elected from a term of order $v_i(s^*) - v_k(s^*)$ to a term of order $v_i(s^*) - v_k(s^*) - 1$ in ε and decreases the probability that C_j will be elected to a term of order $v_i(s^*) - v_k(s^*) + 1$, but the overall effect of the vote switch on this individual's expected utility must be positive for ε sufficiently small. Hence, there cannot be an equilibrium with $v_i(s^*) > v_j(s^*) = v_k(s^*)$.

Remark 1. *For the case of more than 3 candidates, the conditions of the proposition have to be adapted as follows:*

i) For every $\mathcal{K} \subset \mathcal{C}$ with $\#\mathcal{K} \geq 3$ $\min_{K \in \mathcal{K}} n_K^{\mathcal{K}} < \max_{K \in \mathcal{K}} n_K^{\mathcal{K}}$, where $n_K^{\mathcal{K}}$ is the core support of K if the set of candidates is restricted to $\mathcal{K}$.

ii) For every $\mathcal{K} \subset \mathcal{C}$ with $\#\mathcal{K} \geq 3$ there exists at least one voter who prefers his second ranked candidate from $\mathcal{K}$ to a lottery that gives all candidates in $\mathcal{K}$ with probability $1/\#\mathcal{K}$.

Runoff rule

Lemma 4 and Proposition 2 refer to situations in which in the absence of any miscounts the election is decided on the first round. Before presenting the details of the proofs of these two results it is convenient to recall the definitions of the miscount margins at a voting profile, s^*, where candidate C_i collects more than half of the first round vote intentions. Letting $\eta_{lm} = \max\{0, v_l(s^*) - v_m(s^*)\}$ we have

$$T_{ji}(s) = v_i(s) - \frac{N-1}{2} + \eta_{kj}(s) + \Delta_{ij} \qquad T_{jk}(s) = v_i(s) - \min\{v_j(s), v_k(s)\} + \Delta_{kj}$$

In what follows we write $C_i \to C_j$ for a deviation from C_i to C_j; changes in the miscount margins triggered by a deviation are denoted by ΔT_{ji} etc. The table below lists the changes in the miscount margins for the all possible types of deviations. Notice that the three possible types of deviations are the following: i) a deviation towards the candidate that obtains the majority; in the table this is represented by the deviation $C_j \to C_i$; ii) a deviation away from the candidate that obtains the majority of the votes ($C_i \to C_j$); iii) a deviation that does not involve the candidate that gets the majority of the votes ($C_j \to C_k$).

Deviation \ Miscount margin		ΔT_{ji}	ΔT_{jk}	ΔT_{ki}	ΔT_{kj}
$C_j \to C_i$					
1)	$v_k^1(s) < v_j^1(s)$	$+1$	$+1$	± 0	$+1$
2)	$v_k^1(s) \geq v_j^1(s)$	$+2$	$+2$	$+1$	$+2$
$C_j \to C_k$					
3)	$v_k^1(s) + 1 < v_j^1(s)$	± 0	-1	-2	-1
4)	$v_k^1(s) + 1 = v_j^1(s)$	$+1$	± 0	-1	± 0
5)	$v_k^1(s) \geq v_j^1(s)$	$+2$	$+1$	± 0	$+1$
$C_i \to C_j$					
6)	$v_k^1(s) > v_j^1(s)$	-2	-2	-1	-2
7)	$v_k^1(s) \leq v_j^1(s)$	-1	-1	± 0	-1

Table 2: The effects of deviations on the miscount margins

Proof of Lemma 4

i) We first show that $\Sigma_{ji} \cap V_i^1(s^*) = \emptyset$. $C_i \to C_j$ implies $\Delta T_j < 0$ and $\Delta T_j \leq \Delta T_k$ (see rows 6 and 7 of Table 2). Since $T_j(s^*) < T_k(s^*)$, it follows that $T_j(s^*) + \Delta T_j < T_(s^*), T_k(s^*) + \Delta T_k$. Thus, for small enough miscount probabilities the deviation is profitable for all individuals in Σ_{ji}.

   $\Sigma_{ij} \cap V_j^1(s^*) = \emptyset$: $C_j \to C_i$ implies $\Delta T_j > 0$ and $T_k \geq 0$ (see rows 1 and 2 of Table 2). Since $T_j(s^*) < T_k(s^*)$ this means that the deviation would be profitable for all players in Σ_{ij}.

$\Sigma_{ij} \cap V_k^1(s^*) = \emptyset$: If either $v_k(s^*) \leq v_j(s^*)$ or $[v_k(s^*) > v_j(s^*)$ and $T_{jk}(s^*) \leq T_{ji}(s^*)]$ then for $C_k \to C_i$ we have $\Delta T_j > 0$ and $\Delta T_k \geq 0$. Thus, in these cases the deviation is profitable for all individuals who rank C_i above C_j.

Finally, consider the case $v_k(s^*) > v_j(s^*)$ and $T_{jk}(s^*) > T_{ji}(s^*)$. From rows 3 and 4 of Table 2 we see that $C_k \to C_j$ implies $\Delta T_{ji} < 0$, and $\Delta T_{ji} \leq \Delta T_{ki}, \Delta T_{kj}, \Delta T_{jk}$. Thus, in order for s^* to be a robust equilibrium we must have that $\Sigma_{ji} \subset V_j^1(s^*)$. But then

$$(N+1)/2 \leq v_i(s^*) \leq \sigma_{ij} - v_k(s^*) \leq \sigma_{ij} - (v_j(s^*)+1) = \sigma_{ij} - \sigma_{ji} - 1 = \Delta_{ij} - 1,$$

which in turn implies that

$$\begin{aligned} & T_{ij}(s^*) - \min\{T_{jk}(s^*), T_{kj}(s^*)\} \\ = \quad & v_i^1(s^*) - (N-1)/2 + (v_k^1(s^*) - v_j^1(s^*)) + \Delta_{ij} - (v_i^1(s^*) - v_j^1(s^*)) \\ = \quad & \Delta_{ij} - (N-1)/2 + v_k^1(s^*) > 0. \end{aligned}$$

Since this contradicts our starting assumption that $T_{ji}(s^*) < T_{jk}(s^*), T_{jk}(s^*)$ we are done.

ii) $V_i^1(s^*) \cap \Sigma_{ji}$: $C_i \to C_j$ implies $\Delta T_j < 0$, $\Delta T_j < \Delta T_{ki}$. Since $\Delta_{jk} > 0$ implies that $T_{jk}(s) < T_{kj}(s)$ for all s, it therefore follows that the deviation is profitable for all players in Σ_{ji}.

$V_i^1(s^*) \cap \Sigma_{ji} = \emptyset$ and $v_i^1(s^*) \geq (N+1)/2)$ implies that $\Delta_{ij} > 0$. It remains to be shown that if C_i does not Condorcet dominate C_k then $L_i = \emptyset$.

Assume that $\Delta_{ki} > 0$. For $C_i \to C_k$ (see rows 6 and 7 of Table 2) we have $\Delta T_k < 0$ and $\Delta T_k < \Delta T_{ji}$. Thus, if in the initial situation $T_{jk}(s^*) > T_j(s^*) = T_{ji}(s^*) = T_{ki}(s^*)$ then the deviation benefits all players in Σ_{ki}. If no voter in $\Sigma_{ji} \cup \Sigma_{ki}$ is voting for C_i, then $V_i^1(s^*) \subset N_i$. But $n_i \geq v_i^1(s^*) \geq (N+1)/2$ contradicts the assumption $\Delta_{ki} > 0$.

Finally, we show that the remaining case $T_{jk}(s^*) = T_{ki}(s^*)$ can arise only if $L_i = 0$. In order to see this observe that

$$\begin{aligned} T_{jk}(s^*) - T_{ki}(s^*) \quad &= \quad v_i^1(s^*) - v_j^1(s^*) + \eta_{jk}(s^*) - [v_i^1(s^*) - (N-1)/2 + \eta_{jk}(s^*)] \\ &= \quad v_j^1(s^*) - (N-1)/2. \end{aligned}$$

If $v_j^1(s^*) = (N-1)/2$ then $v_k^1(s^*) = 0$. In such a case $C_j \to C_i$ implies $\Delta T_j > 0$ and $\Delta T_{ki} = 0$. We will argue now that that this means that for $t = T_{ki}(s^*) = T_{jk}(s^*)$ miscounts the deviation's only effect is a replacement of the outcome C_j with the outcome C_i. A $t-$configuration that before the deviation has delivered C_j via a runoff against C_i must have involved exactly $v_i^1(s^*) - (N-1)/2$ miscounts of first round votes of C_i. With this number of first round misounts the outcome in the post-deviation situation is always C_i. t-configurations that delivered C_j via a runoff against C_k must have involved all first round votes of C_i. In the post-deviation situation such configurations lead to a runoff between C_i and C_j that is won by C_i. Finally, a $t-$configuration that in the predeviation situation has produced C_k as outcome must have involved all votes in favor of C_j, including the one of the deviating player. But if the vote of the deviating player is not counted anyway, the deviation cannot change the outcome. So we can conclude that the deviation would benefit all players in Σ_{ij} and thus $\Sigma_{ij} \cap V_j^1(s^*) = \emptyset$.

Finally, if $v_j(s^*) > v_k(s^*) = 0$ then $C_j \to C_k$ implies that $\Delta T_{ki} < 0$ and $\Delta T_{ki} < \Delta T_{ji}, \Delta T_{jk}$ (see rows 3 and 4 of Table 2). This implies that voters in Σ_{ki} would benefit from the deviation. Combining this with the previous observation we can thus conclude that $V_k^1(s^*) = \Sigma_{ji} \cap \Sigma_{ki} = L_i = \emptyset$ and $V_j^1(s^*) = \Sigma_{ji} \backslash \Sigma_{ki} = \Sigma_{ji} = N_j$.

Proof of Lemma 5

We know already that $V_j^1(s^*) = V_k^1(s^*) = \Sigma_{ji}$. We have to argue that $V_k^1(s^*) = L_i$.

If $\Delta_{ik} = 0$ then $T_{ki}(s^*) - \min\{T_{jk}(s^*), T_{kj}(s^*)\} = v_i - (N-1)/2 + \eta_{jk}(s^*) - [v_i^1(s^*) - \min\{v_j^1(s^*) - v_k^1(s^*)\}] \leq 0$. Thus, in order to have $T_j(s^*) < T_k(s^*)$ it must be true that $T_{ji}(s^*) < T_{ki}(s^*)$. It is straightforward to verify that this latter condition can hold only if $v_j^1(s^*) - v_k^1(s^*) > \Delta_{ij} > 0$.

If $v_j^1(s^*) - v_k^1(s^*) > \Delta_{ij}$ then $C_j \to C_k$ implies $\Delta T_{ki} = -2$, $\Delta T_{ji} = 0$ and $\Delta T_{jk} = \Delta T_{kj} = -1$. If $T_{ki}(s^*) - 2 < T_{ji}(s^*)$ then the deviation is profitable for all individuals in L_i and we are done.

So suppose that $T_{ji}(s^*) \leq T_{ki}(s*) - 2$. Observe that a configuration of $t = T_{ki}(s^*) - 2$ miscounts, which after the deviation delivers C_k as outcome, does not include miscounts in round two. But then such a miscount configuration cannot have produced C_j as outcome in the initial situation but only C_i. Conversely, any $\tau-$configuration, with $T_{ji} \leq \tau \leq t$, that before the deviation has delivered C_j cannot deliver C_i afterwards. Whenever such a τ-configuration leads to a ji-runoff after the deviation then the outcome must be the same as before the deviation since the deviation does not affect second round votes. If instead the runoff changes to a ki-runoff then all miscounts must be concentrated on the first round; such a configuration cannot have resulted in C_j as outcome before the deviation. The runoff cannot change to a jk-runoff since $t < \min\{T_{jk}(s^*) - 1, T_{kj(s^*)-1}$.

Combining these observations allows us to conclude that (for a sufficiently small miscount probability) the relevant effect of $C_j \to C_k$ is an increase in the winning chances of C_k that goes entirely at the cost of the chances of C_i. Thus the deviation is profitable for all voters in L_i.

Next we show $C_k \to C_j$ benefits all voters in N_{ji}. First observe that we have to contemplate such a deviation only if $v_k^1(s^*) > 0$. But if this is the case then $T_{ki}(s^*) < \min\{T_{jk}(s^*), T_{kj}(s^*)\}$. Since $v_j^1(s^*) - v_k^1(s^*) > \Delta_{ij}$ we have that $\Delta T_{ji} = 0$ and $\Delta T_{ki}, \Delta T_{jk}, \Delta_{kj} > 0$. Configurations of $T_{ki}(s^*)$ miscounts which before the deviation delivered C_k as outcome must lead to a ij-runoff after the deviation. Given that such configurations only contain first round miscounts the ij-runoff must end with a win of C_i.

Other configurations with $t \leq T_{ki}(s^*)$ miscounts must produce both before and after the deviation an ij-runoff. The winner of this runoff depends only on second round votes which are not affected by the deviation.

Based on the preceding observations we can conclude that voters in N_{ji} profit from $C_k \to C_j$ and thus $N_{ji} = V_j^1(s^*)$.

We finally have to show that $n_{ji} > (N+1)/3 > l_i + 1$. Combining the facts $v_j(s) = n_{ji}$, $v_k(s) = n_{kj} + n_{jk}$ and $v_j(s) - v_k(s) > \Delta_{ij}$ and using the definition of Δ_{ji} delivers

$$n_{ji} - n_{kj} - n_{jk} > \sigma_{ij} - n_{ji} - n_{kj} - n_{jk} \quad \Leftrightarrow \quad 2n_{ji} > \sigma_{ij}.$$

Since $\sigma_{ij} \geq (N+1)/2$ it thus follows that $n_{ji} > (N+1)/4$. Finally, the sum of n_{ji} and $l_i + 1$ cannot exceed $(N+1)/2$ and so $l_i + 1 < (N+1)/4$.

Proof of Proposition 2

By Lemma 5, there are no robust equilibria such that i) votes are concentrated on two candidates only (so that the election is decided in the first round) and ii) the candidate who gets the most votes is not a Condorcet winner. Thus, we only have to show that votes must also be dispersed when the candidate who gets the most first round votes is a Condorcet winner.

Assume that C_i is the Condorcet winner. Let s^* be such that $v_i^1(s^*) \geq (N+1)/2$ and $v_k^1(s^*) = 0$. By Lemma 4, a necessary condition for robustness of s^* is that either $\Sigma_{ji} \subset V_j^1(s^*)$ or $\Sigma_{ki} \subset V_j^1(s^*)$. Therefore, $N_{kj} \subset \Sigma_{ji} \cap \Sigma_{ki} \subset V_j^1(s^*)$. We show now that, for individuals in N_{kj}, a deviation from C_j to C_k is profitable. In order to do so, we argue that the most likely outcome change triggered by the deviation is neither a switch from C_k to one of the other two candidates, nor a switch from C_j to C_i; all other outcome changes are profitable for individuals in N_{jk}.

Denote the post-deviation profile by s'. Since $v_j(s^*) \geq \min\{\sigma_{ji}, \sigma_{ki}\} \geq 3$ (there is at least one individual of each preference type) we have that $v_j(s') \geq 2 > 1 = v_k(s')$. From row 3 of Table 2 we see that $C_j \to C_k$ implies $T_{ji}(s^*) - T_{ji}(s') = 0$, $T_{jk}(s^*) - T_{jk}(s') = T_{kj}(s^*) - T_{kj}(s') = 1$ and $T_{ki}(s^*) - T_{ki}(s') = 2$.

Consider first outcome changes that involve candidate C_k. Since $T_k(s') < T_k(s^*)$ it follows that switches towards C_k are more likely than switches away from C_k, so that voters in N_{kj} are better off in expectation if the outcome change involves C_k.

It remains to be shown that outcome changes from C_j to C_i also cannot be relevant. Since the deviation only affects first round votes, any outcome change must be produced through a change in the runoff pair. Now observe that due to the deviation the number of miscounts that are required to obtain the runoff pair (C_j, C_k) decreases from $v_i^1(s^*)$ to $v_i^1(s^*) - 1$. Thus a switch toward this pair is more likely than any switch away from this pair. But then the only way to generate an outcome change from C_j to C_i that can be relevant is through a change in the runoff pair from (C_i, C_j) to (C_i, C_k). Any event where the deviation triggers such an outcome change must involve at least the following miscounts: a) $v_i^1(s^*) - (N-1)/2$ miscounts of first round C_i-votes (for, otherwise, C_i wins the election outright); b) $v_j^1(s^*) - 2$ miscounts of first-round C_j-votes (excluding the deviating player's ballot) for otherwise, the deviation cannot trigger a switch from the runoff pair (C_i, C_j) to (C_i, C_k); c) $\Delta_{ij} = N - 2\sigma_{ji}$ miscounts of second round votes of players that rank C_i above C_j, for otherwise C_j cannot win the runoff against C_i in the pre-deviation situation. Thus, the minimum number of miscounts necessary for a switch in outcome from C_j to C_i to occur is

$$v_i^1(s^*) - \frac{N-1}{2} + v_j^1(s^*) - 2 + N - 2\sigma_{ji} = v_i^1(s^*) - 1 + \frac{N-1}{2} - v_j^1(s^*).$$

Now remember that if at s^* candidate C_j is strictly more likely to benefit from miscounts than C_k (i.e. if $T_j(s^*) < T_k(s^*)$) then $V_j^1(s^*) = \Sigma_{ji}$. Since, by assumption, $\sigma_{ji} < (N-1)/2$ it follows that the above sum is strictly larger than $v_i^1(s^*) - 1$. Hence, the contemplated outcome change from C_j to C_i is less likely than a switch from a runoff between C_i and C_j that is won by C_i to a runoff between C_j and C_k. Thus, the deviation increases the expected utility of a type kj voter.

Finally, if $T_k(s^*) \leq T_j(s^*)$, then $T_k(s') < T_j(s')$ and thus the relevant effect the deviation is a replacement of the outcome C_i by the outcome C_k.

Proof of Lemma 6

Without loss of generality, assume that C_j Condorcet dominates C_k (so that C_k is the Condorcet loser). We show that at a voting profile s that satisfies the condition specified in the lemma, there must always be some player in $V_j^1(s)$ or $V_k^1(s)$ who benefits from a deviation towards C_i. In what follows we denote the corresponding post deviation voting profile by s'.

If $v_i^1(s) = \min\{v_j^1(s), v_k^1(s)\}$ then the deviation changes the outcome distribution even in the event that no miscounts occur. In particular, it increases the probability that C_i enters the second round (and thus wins the election). This increase of the Condorcet winner's chances to win go at the expense of C_j (C_k can never win when there are no miscounts). Thus the deviation benefits all in Σ_{ij}. Since Σ_{ij} comprises a majority of the electorate, not all of its members can be voting for C_i at the profile s.

If $v_i^1(s) < \min\{v_j^1(s), v_k^1(s)\}$ then the deviation strictly lowers T_i. A $T_i(s')-$configuration that delivers C_i after the deviation comprises only miscounts of first round votes. Consequently, the pre-deviation outcome for such configurations must be C_j. Conversely, there can be no switch from C_k to C_j or from C_j to C_k for configurations that comprise at most $T_i(s')$ miscounts. Since the deviation only affects first round votes, such changes could occur only if the miscounts lead to a runoff pair different from the pair jk that results if no miscounts happen. But miscount configurations for which the runoff pair changes to ik or ij either C_i either yield C_i as outcome, or they comprise more than $T_i(s')$ miscounts.

Proof of Lemma 7

Part i): Assume (wlog) that the Condorcet ranking is given by $C_i \succ C_j$, $C_j \succ C_k$ and $C_k \succ C_i$. We show that if $v_i^1(s) - 1 > v_j^1(k), v_k^1(s)$, then all players in $V_i^1(s)$ have an incentive to deviate. The basic idea is that while the deviation does not change the fact that C_i gets more votes than the opponents it changes the gap between v_j^1 and v_k^1 and thus determines the relative chances for the two candidates to meet C_i in the runoff. Voters in $V_i^1(s) \cap \Sigma_{ki}$ profit from $C_i \to C_k$ (it becomes more likely that C_i meets and loses against C_k instead of winning against C_j) while voters in $V_i^1(s) \cap \Sigma_{ik}$ benefit from $C_i \to C_j$ (making it less likely that C_i will meet C_k instead of C_j).

In what follows we spell out the details for the case $v_j^1(s) \geq v_k^1(s)$. The case $v_j^1(s) < v_k^1(s)$ is treated in a perfectly analogous way.

We first argue that the relevant effect of $C_i \to C_k$ is a replacement of the outcome C_i by the outcome C_k. This is easily seen if $v_j^1(s) - v_k^1(s) \leq 1$. In this case the deviation increases the probability of the outcome C_k even in the absence of miscounts. But when no miscounts occur C_j cannot have been winning the election in the predeviation situation and so the gain of C_k must go at the cost of C_i.

Consider the case $v_j^1(s) > v_k^1(s)+1$. $T_{ji} = \Delta_{ij}$, $T_{ki}(s) = v_j^1(s)-v_k^1(s)$, $T_{ik}(s) = T_{ki}(s)+\Delta_{ki}$, $T_{jk} = v_i^1(s) - v_k^1(s)$ and $T_{kj} = T_{jk} + \Delta_{jk}$. $C_i \to C_k$ implies $\Delta T_{ki} = -1$ and $\Delta T_{ji} = 0$. Changes in T_{ik}, T_{jk} and T_{kj} are irrelevant for these margins remain strictly larger than $T_{ki}(s) - 1$. If $T_{ki}(s)-1 < T_{ji}(s)$ then the relevant effect of the deviation is a replacement of C_i by C_k. In order to prove that the same is true also if $T_{ki}(s) - 1 \geq T_{ji}(s)$ it suffices to show that configurations of no more than $T_{ki}(s)-1$ miscounts yield C_j as outcome after the deviation if and only if they have delivered C_j in the pre-deviation situation. But this follows from the following observation: As long as miscounts do not trigger a change in the runoff pair, the outcome depends only on second round votes and miscounts, which are not affected by the deviation. The most likely change in the runoff pair is from ij to ik, which requires exactly $T_{ki}-1$ miscounts of first round votes. Any configuration which leads to this outcome does not involve second round miscounts and can thus in the pre-deviation situation cannot have delivered the outcome C_j.

By similar arguments it can be shown that the relevant effect of $C_i \to C_j$ is a replacement of C_k by C_i.

Part ii) We have to show that at if a voting profile s satisfies the conditions $v_i^1(s) \leq v_j^1(s)$ and $v_i^1(s) \leq v_k^1(s)$, with at least one of the two inequalities being strict, then voters in L_j have an incentive to deviate if they are not voting for C_i. Notice first that the assumptions that $C_i \succ C_j$ and that there is no CW imply that the Condorcet ranking is given by $C_i \succ C_j$, $C_j \succ C_k$ and $C_k \succ C_i$. The idea beneath the following proof is that a deviation towards C_i makes it more likely that instead of a jk-runoff (that is won by C_j) the first round election delivers a runoff pair that involves C_i. This either increases the winning chances of C_i (if C_i is paired with C_j) without hurting C_k or of C_k (if C_i is paired with C_k) without hurting C_i, or the chances of both C_i and C_k improve.

If $v_k^1(s) \geq v_j^1(s)$ then $C_j \to C_i$ closes the gap between C_j and C_i (C_i might even overtake C_j) while C_k still receives the most votes. This means that the chances of a ik-runoff increase at the cost of the jk-runoff.

Since the deviation does not affect second round votes, it cannot influence the outcome of the second round elections (in particular of the jk-runoff). Thus the deviation's relevant effect on the outcome distribution is an increase in the chances of C_k at the expense of C_j.

Proof of Proposition 3

The proof is constructive. We first define the two voting profiles s and s' and show that one of the two must give rise to the desired vote distribution. In a second step we then show that no player has an incentive to deviate from the profile which delivers the correct vote distribution

Step 1: Consider the following voting profiles:

s: Players in $L_j = N_{ki} \cup N_{ik}$ vote for C_i; players in N_{kj} vote for C_k. The votes of the remaining players (those in $\Sigma_{jk} = N_{ij} \cup N_{ji} \cup N_{jk}$) are distributed as follows:

If $n_{ij} \leq n_{kj} + n_{ji} + n_{jk} = \sigma_{ji}$ assign all votes of players in N_{ij} to C_j. Then distribute the votes in N_j over C_j and C_k such that $v_k^1(s) = v_j^1(s)$. If this is not possible because $N - l_j$ is an odd number, then take a ballot from a voter in N_{ij} who prefers a uniform lottery over all three candidates to having his second ranked candidate for sure and assign it to C_i.

If $\sigma_{ji} - l_j > n_{ij} - \sigma_{ji} > 0$ assign all votes of players in N_j to C_k. In describing how the remaining voters in N_{ij} are to be devided between C_j and C_i we have to distinguish two subcases. For this task it is convenient to introduce the following notation. Let X be the set of all players in N_{ij} who prefer a uniform lottery over all three candidates to having C_j for sure. Write $x = |X|$.

If $x \geq n_{ij} - \sigma_{ji}$ then assign $[n_{ij} - \sigma_{ji}]$ votes of players in X to C_i. The remaining votes go to C_j. Observe that the resulting profile s satisfies $v_j^1(s) - 1 = v_k^1(s) > v_i(s)$.

If $x < n_{ij} - \sigma_{ji}$ then assign all voters in X and sufficiently many voters in $N_{ij} - X$ to C_i so that the resulting profile s satisfies $v_j^1(s) - 1 = v_k^1(s) > v_i(s)$.

s': The only case which we have not considered in the construction of s is $n_{ij} - \sigma_{ji} \geq \sigma_{ji} - l_j > 0$. This condition implies that $\sigma_{ji} = n_{ji} + l_i \leq N/3$ and thus $l_i < N/3$. Since by assumption we have $l_j < N/3$ we therefore must have $l_k > N/3$. Next observe that $|n_{ji} - n_{ki}| \leq n_{ij} + n_{ik}$: $n_{ji} > n_{ij} + n_{ik} + n_{ki} = \sigma_{ij} \geq (N+1)/2$ contradicts the assumption that C_j is not a Condorcet winner. $n_{ki} > n_{ij} + n_{ik} + n_{ji} = l_k + n_{ik} > N/3$ is at odds with the condition $l_j < N/3 < l_k$.

From these observations it follows that we can construct s' as follows: All votes in L_i are assigned to C_k; votes in N_{ji} go to C_j. Votes in N_{ki} are assigned to C_i unless $N - l_i$ is odd; in this latter case C_k receives one of the votes of the players in N_{ki} who prefer a uniform lottery over all candidates to getting C_i for sure. The remaining votes in $N_{ij} \cup N_{ik}$ are distributed over the candidates C_i and C_j so that their final number of votes are equal.

Step 2: In this second step we show that if s and s' are constructed as described above, then no player has an incentive to deviate. Consider first the case where s is such that $v_j^1(s) = v_k^1(s) > v_i^1(s)$. Notice that at s we have $T_{ij}(s) = v_k^1(s) - v_i^1(s) = v_j^1(s) - v_i^1(s) = T_{ki}(s) < T_{ji}(s), T_{ik}(s)$. Since any deviation from s changes either T_{ij} or T_{ki} (or both) we can ignore the margins T_{ik} and T_{ji}. Finally, observe that not change the kj-margin, since $T_{kj} = \Delta_{jk}$. In the following we denote the post-deviation voting profile by $\hat{s}$.

i) $C_i \longrightarrow C_j, C_k$. Due to the deviation both the ij- and the ki-margin increase. Thus, if $T_{kj}(s) > T_{ki}(s) = T_{ij}(s) = t$ then the relevant change in the outcome distribution is an increase in the winning chances of C_j and an identical decrease of the chances of C_i and C_k. Such a change is detrimental for all individuals in $L_j \cup X$ which is exactly what we have to show. The same conclusion remains valid if $T_{kj}(s) \leq t$. In order to see this notice that the ij- and $ki-$margins comprise only miscounts of first round votes. A victory of C_k via a jk-runoff instead requires Δ_{jk} miscounts of second round votes. Thus, any $t-$configuration which delivers C_i through a ij-runoff or C_k through a $ki-$runoff must yield C_j in the post deviation situation. Moreover, there can be no other change in the outcomes for configurations which are composed of no more than t miscounts. That is, if such a configuration has delivered C_k via a kj-runoff before the deviation it must do so also after the deviation. In order for C_k to win against C_j the $t-$configuration must contain sufficiently many second round miscounts in favor of C_k. The deviation concerns only the first round voting behavior. Thus as long as the deviation does not change the runoff-pair C_k remains the winner. But in the post deviation situation a change of the runoff pair requires at least $t+1$ first round miscounts.

ii) $C_j \longrightarrow C_i, C_k$. Due to such deviations the ki-margin decreases by more than the ij-margin. It thus follows that if $T_{ki}(\hat{s}) < T_{kj}(s)$, then a player benefits from the deviation if and only if he ranks C_k above C_j. The increase in C_k's winning chances of order $T_{ki}(\hat{s})$ remains the relevant change in the outcome distribution also if $T_{ki}(\hat{s}) \geq T_{kj}(s)$. This follows from the fact that any miscount configuration of at most $T_{ki}(\hat{s})$ miscounts delivers C_k in the post deviation situation if and only if it delivers it also before the deviation. In order to show this the same arguments can be used which we have employed in i).

iii) $C_k \longrightarrow C_i, C_j$. This case is perfectly analogous to case ii). The relevant change in the outcome distribution triggered by the deviation is a decrease of the ij-margin (which decreases by more than the ki-margin). This change is detrimental for all individuals in Σ_{ji}.

s has been constructed such that candidate C_i only receives votes from players in $L_j \cup X$, C_j's votes come from players in Σ_{jk} and C_k is voted only by players in Σ_{ji}. Hence, from observations i) to iii) it follows that there at s there is no individual who has an incentive to deviate.

Also the voting profile s' is a profile where the two leading candidates obtain the same number of votes. We can therefore apply the same arguments in order to show that there are no voters who could profit by deviating from s'.

We are thus left with the case $v_j^1(s) - 1 = v_k^1(s) > v_i^1(s)$. Observe that $T_{ij}(s') < T_{ki}(s') \leq T_{ik}(s'), T_{ji}(s')$. The jk-margin does not change with any deviation. We can thus ignore it again for the same reasons which we have spelled out in the previous case. The deviation incentives for players who are voting for C_k are the same as in the previous case where C_j and C_k obtain the same number of votes. The relevant change in the outcome distribution generated by a deviation away from C_i is an increase of the winning probability of C_j at the cost of C_i's chances. Hence, the deviation is detrimental for all individuals in Σ_{ij}. Finally, consider deviations away from C_j. If a ballot is moved toward C_i then the ij-margin and the $ki-$margin decrease by one and two, respectively. Moreover, the two margins coincide in the post-deviation situation. The relevant change in the outcome distribution is therefore a decrease in C_j's winning probability which benefits in equal parts the other two candidates. Observe, that such a change is detrimental for all voters in $N_{ij} - X$. If the deviating vote goes to C_k than T_{ij} increase by one while T_{ki} decreases by one. This implies that $T_{ki}(\hat{s}) = T_{ij}(s) < T_{ij}(\hat{s}) = T_{ki}(s)$. Hence, the relevant outcome change triggered by the deviation is a replacement of C_i by C_j. We can therefore conclude again that the deviation is detrimental for all individuals in $N_{ij} - X$.

Proof of Proposition 4

Proof. Throughout the following proof we will denote vote profiles which result from a deviation from s by s'.

Part i): The proof of this statement relies on the following arguments: First observe that at s the smallest miscount margin is the jk-margin, i.e. $T_{jk}(s) < T_{ki}(s), T_{ji}(s)$. If σ_{ji} is an even number so that $v_k^1(s) = v_j^1(s)$, we have

$$\begin{aligned} T_{ki}(s) - T_{jk}(s) &= v_i^1(s) - \frac{N-1}{2} + N - 2\sigma_{ki} - [v_i^1(s) - \sigma_{ji}/2] \\ &= \frac{N+1}{2} + \frac{\sigma_{ji}}{2} - 2\sigma_{ki} > 0 \\ T_{ji}(s) - T_{jk}(s) &= v_i^1(s) - \frac{N-1}{2} + N - 2\sigma_{ji} - [v_i^1(s) - \sigma_{ji}/2] \\ &= \frac{N+1}{2} - \frac{3\sigma_{ji}}{2} > 0. \end{aligned}$$

It is not difficult to verify that a similar conclusion holds if σ_{ji} is odd so that $v_j^1(s) = v_k(s) + 1$. In particular, in this case we must have that $T_jk(s) < T_{ki}(s)$ and $T_{jk}(s) \leq T_{ji}(s)$.

Using Table 2 it can be shown that the jk- margin increases with any deviation from C_j or C_k towards C_i and for any deviation from C_k to C_j. Moreover such deviations do not reduce any other miscount margin. Thus the relevant change in the outcome distribution is a replacement of C_j by C_i and so it follows that no individual who is supposed to vote for C_j or C_k (i.e. those in Σ_{ji}) could ever benefit from them. Conversely, deviations from C_i towards C_j strictly lower the jk-margin by at least as many units as the reduce any other miscount margin. Consequently, they are profitable for a player if and only if he ranks C_j above C_i. If $v_j^1(s) = v_k^1(s)$ then a switch from C_j towards C_k increases the jk-margin and so again we can conclude that they are detrimental for players in Σ_{ji}. If $v_j^1(s) = v_k^1(s) + 1$ the jk-margin does not change when a player switches his vote from C_j to C_k. This case therefore requires a somewhat more involved argument.

Consider the ki-margin and ji-margin. From Table 2 we can see that due to the deviation the first decreases by one while the latter increases by one. Moreover, it is not difficult to verify that $T_{ki}(s') > T_{jk}(s) = T_{jk}(s')$. We therefore have to distinguish three cases: $T_{ki}(s') < T_{ji}(s)$, $T_{ki}(s') > T_{ji}(s)$ and $T_{ki}(s') = T_{ji}(s)$.

The first of these cases is possible only if $\sigma_{ji} < \sigma_{ki}$. Observe that in this case the relevant change in the outcome distribution which the deviation implies is a replacement of C_i by C_k. Therefore, s can be immune to the deviation only if all those voters in Σ_{ji} who rank C_k above C_i are already voting for C_k. Such a distribution of voters in Σ_{ji} is feasible only if $\Sigma_{ji} \cap \Sigma_{ki} = L_i$ comprises less than half of the members of Σ_{ji}. In order to see that this condition must indeed be satisfied under our assumptions observe that combining the assumption that $N - n_i = \sigma_{ki} + n_{ji} \geq (N+1)/2$ with $\sigma_{ki} < (N+1)/3$ implies $n_{ji} > (N+1)/2 - \sigma_{ki} > (N+1)/6$. At the same time $\sigma_{ji} = n_{ji} + l_i < n_{ki} + l_i = \sigma_{ki}$ means that $n_{ji} < n_{ki}$. But then from $l_i \geq n_{ki}$ we get that $\sigma_{ji} = l_i + n_{ji} > 2n_{ji} > (N+1)/3$ which contradicts our assumption that $\sigma_{ji} < \sigma_{ki}$.

If $T_{ki}(s') = T_{ji}(s)$ the deviation implies a replacement of C_j by C_k (or more precisely, a replacement of C_j by C_i and a replacement of C_i by C_k, with a zero net effect for C_i). Hence the deviation is profitable only for those players who rank C_k above C_j. But again that is only a minorty of Σ_{ji} and hence there is a way to distribute players over C_j and C_k such that no voter can gain by deviating from C_j to C_k.

Finally if $T_{ki}(s') > T_{ji}(s)$ then the relevant change regards situations where in the pre-deviation situation C_j has won the election through a runoff against C_i. Since no voter who is supposed to vote for C_j or C_k wants C_j be replaced by C_i we are done.

Part ii) In a first step we show that under our assumptions it must be true that $v_k^1(s) > v_j^1(s)$ and $T_{ki}(s) < T_{jk}(s), T_{ji}(s)$. The first condition is satisfied if $n_{ki} > l_i$. The second one instead requires that the two differences

$$\begin{aligned} T_{ji}(s) - T_{ki}(s) &= v_i^1(s) - (N-1)/2 + n_{ki} - l_i + N - 2\sigma_{ji} - [v_i^1(s) - (N-1)/2 + N - 2\sigma_{ki}] \\ &= 2\sigma_{ki} - 2\sigma_{ji} + n_{ki} - l_i \qquad \text{and} \\ T_{jk}(s) - T_{ki}(s) &= v_i^1(s) - l_i - [v_i^1(s) - (N-1)/2 + N - 2\sigma_{ki}] = \sigma_{ki} + n_{ki} - (N+1)/2 \end{aligned}$$

are both strictly positive. In order to show this it is sufficient to prove that the following three inequalities must hold: $n_{ki} > l_i$, $\sigma_{ki} > \sigma_{ji}$ and $\sigma_{ki} + n_{ki} > (N+1)/2$.

The second inequality is implied by the assumptions $\sigma_{ki} < (N+1)/3$ and $\sigma_{ji} < 4\sigma_{ki} - N$. Of course, $\sigma_{ji} < \sigma_{ki}$ is equivalent to $n_{ji} < n_{ki}$. But then $\sigma_{ki} + n_{ji} \geq (N+1)/2$ implies $\sigma_{ki} + n_{ki} > (N+1)/2$. Finally, Using $n_{ji} + \sigma_{ki} \geq (N+1)/2$ and $\sigma_{ki} = n_{ki} + l_i < (N+1)/3$ once more, we get $n_{ji} > (N+1)/6$. But then $n_{ki} > n_{ji}$ and $\sigma_{ki} < (N+1)/3$ are compatible only if $l_i < (N+1)/6 < n_{ji} < n_{ki}$.

At s voters are split between C_i and the other two candidates according to how they rank C_i versus C_k. With the help of Table 2 it is easy to verify that any deviation away from C_i lowers the ki-margin (and no other margin decreases stronger) while any deviation towards C_i increases this margin without decreasing any of the other margins. Consequently, at s no voter could profit from such deviations.

It remains to be shown that no player wants to switch from C_j to C_k or vice versa. In order for the three conditions $n_{ki} > n_{ji} > l_i$, $n_{ki} + n_{ji} + l_i \geq (N+1)/2$ and $n_{ki} + l_i < (N+1)/3$ to hold it must be true that $n_{ki} - l_i > 2$. But then deviations between C_j and C_k leave the ki-margin unchanged. They do change however the $jk-$margin (a deviation towards C_k increases it while a deviation in opposite direction decreases it). They also change the chances of getting a ji-runoff when there are enough second round miscounts so that C_k would win against C_i (the $ji-$margin must be larger than $R_{ij}(s) + \Delta_{ki}$). But since

$$R_{ij}(s) + \Delta_{ki} - T_{jk}(s) = (N+1)/2 - \sigma_{ki} - l_i > 0$$

it follows that the $jk-$margin is always the (weakly) smallest margin after the $ki-$margin. Consider a deviation from C_k to C_k. If $R_{ij}(s') + \Delta_{ki} > T_{jk}(s')$ then the relevant effects of the deviation is a replacement of C_i by C_j Hence, it is profitable for a player if and only if he ranks C_j above C_i. But at s only voters who prefer C_i over C_j are supposed to vote for C_k. What if $R_{ij}(s') + \Delta_{ki} = T_{jk}(s')$. In this case there is a replacement of C_i by C_j and a replacement of C_k by C_i. Both changes are detrimental for players in N_{ki}.

The last case to consider is a deviation from C_j to C_k. This increases T_{jk} and $R_{ij} + \Delta_{ki}$. The relevant effect of the deviation is thus a replacement of C_j by C_i. But since voters of C_j belong to L_i, no such player would benefit from the switch.

Part iii) The assumptions $\sigma_{ji} \geq (N+1)/3$ and $\sigma_{ji} > \sigma_{ki}$ guarantee that $\min\{T_{jk}, T_{kj}, T_{ki}\} - T_{ji} > 0$, for all vote profiles where $v_i^1 = \sigma_{ij}$ and $v_j^1 \geq v_k^1$. Denote the term $2\sigma_{ji} - (N+1)/2$ by $\bar{v}$. It is straightforward to show that $\sigma_{ji} > \bar{v} > \sigma_{ji}/2$. Thus the condition $v_j^1 \geq v_k^1$ is satisfied at s. Since the ji-margin increases with a deviation towards C_i while no other margin decreases, we can immediately conclude that it is detrimental for all voters in $\Sigma_{ji} = V_j^1(s) \cup V_k^1(s)$ ($V_i^1(s) = \Sigma_{ij}$). Similarly, deviations away from C_i lower the ji-margin at least as strongly as they lower any other margin. Hence, they benefit only voters who rank C_j above C_i of which there are none among the players who are supposed to vote for C_i. It remains to be shown that no player in Σ_{ji} has an incentive to deviate from C_j to C_k or vice versa.

Observe that $\bar{v}$ has been chosen such that $v_j^1(s) = \bar{v}$ implies that the the number of miscounts which are necessary for a victory of C_j through a jk-runoff ($T_{jk} = v_i^1(s) - v_k^1(s) = N - \sigma_{ji} - (\sigma_{ji} - \bar{v}) = (N-1)/2$) is the same as the minimal number of miscounts which are necessary to trigger an ik-runoff on round one (notation: R_{ik}) and a win of C_j in an ij-runoff ($v_i^1(s) - (N-1)/2 + v_j^1(s) - v_k^1(s) + \Delta_{ij} = T_{ki}(s) - (\Delta_{ik} - \Delta_{ij}) = N - \sigma_{ji} - (N-1)/2 + 2\bar{v} - \sigma_{ji} + N - 2\sigma_{ji} = (N-1)/2$). This also implies that at s we have $T_{jk}(s) < T_{ki}(s)$.

Now consider a deviation from C_j to C_k. If $v_j^1(s) - v_k^1(s) = 1$ then the deviation increases the ji-margin and thus is not profitable for the voters of C_j. If $v_j^1(s) - v_k^1(s) > 1$ then the deviation leaves the ji-margin unchanged, decreases T_{jk} by one and $R_{ik} + \Delta_{ij}$ by two. Thus there are miscount configurations with $T_{jk}(s) - 2$ miscounts for which the deviation changes the outcome from C_j (via a ij-runoff) to C_i (via a ik-runoff). In order to complete the argument we have to prove that there are no miscount configurations with at most $T_{jk} - 2$ miscounts for which C_i is replaced by C_j. With at most $T_{jk}(s) - 2$ miscounts C_j can win only via a ij-runoff. But any admissible miscount configuration which yields an ij-runoff after the deviation and contains enough second round miscounts in favor of C_j must have delivered C_j already before the deviation.

Next consider a deviation from C_k to C_j. Such a deviation increases both T_{jk} and $R_{ik} + \Delta_{ij}$. This means that there are $T_{jk}(s)$-configurations (containing only first round miscounts) for which C_j is replaced by C_i. On the other hand there are also $T_{jk}(s)$-configurations which in the pre-deviation situation produce a ik-runoff which ends with a win of C_i while after the deviation they result in a win of C_j via a ij-runoff. For any miscount configuration with strictly less than $T_{jk}(s)$ miscounts, there can be no change in the outcome distribution. Hence, depending on whether $T_{jk}(s)$-configurations for which C_j replaces C_i are more likely than T_{jk}-configurations for which C_i replaces C_j either every voter of C_k has an incentive to deviate or all of them want to stick to their strategy. In the latter case we are done.

If instead all voters want to deviate from C_k to C_j when $v_j^1(s) = \bar{v}$ then none of them would want to move in the opposite direction if $v_j^1(s) = \bar{v} + 1$. Moreover, at any such profile voters would still not want to move away from/towards C_i. Finally, $v_j^1(s) = \bar{v} + 1$ implies that $T_{jk}(s) < R_{ik}(s) + \Delta_{ij}$. This means that a deviation from C_k to C_j is unambiguously detrimental for all players in σ_{ji} since its relevant effect is the increase of T_{jk} (which amounts to a replacement of C_j by C_i).

Part iv) Consider first situations where $v_k^1(s) = \bar{v}$ or $v_k^1(s)+1$. The assumptions $\sigma_{ki} \geq (N+1)/3$ and $\sigma_{ki} > \sigma_{ji}$ guarantee that $\min\{T_{jk}, T_{kj}, T_{ji}\} - T_{ki} > 0$, for all vote profiles where $v_i^1 = \sigma_{ij}$

and $v_k^1 \geq v_j^1$. Denote the term $2\sigma_{ki} - (N+1)/2$ by $\bar{v}$. It is straightforward to show that $\sigma_{ki} > \bar{v} > \sigma_{ki}/2$. Thus the condition $v_j^1 \geq v_k^1$ is satisfied at s. Just as in part iii) one can show that voters cannot benefit from deviations toward and away from C_i. In what follows we can therefore concentrate our attention on deviations from C_j to C_k or vice versa.

$v_j^1(s) = \bar{v}$ defines the vote distribution for which the jk-miscount margin coincides with the number of miscounts which are necessary to trigger an ij-runoff and to guarantee a victory of C_k in a ik-Runoff. That is, $T_{jk}(s) = v_i^1(s) - \bar{v} = v_i^1(s) - (N-1)/2 + \sigma_{ki} - 2\bar{v} + N - 2\sigma_{ki} = R_{ij}(s) + \Delta_{ik}$. So when $v_j^1(s) = \bar{v}$ then we have $T_{ki}(s) < T_{jk}(s) = R_{ij}(s) + \Delta_{ik} < T_{ji}(s)$. At any such profile no voter of C_k has an incentive to deviate towards C_j. If the difference in the votes of C_k and C_j is smaller than two then due to such a deviation C_j would either overtake C_k or catch up with C_k. In either case the relevant change in the outcome distribution would be a reduction of C_k's chances in favor of C_i (of order T_{ki}). In all other cases the deviation leaves T_{ki} unchanged but reduces $R_{ij} + \Delta_{ik}$ by two while T_{jk} decreases by only one. So the relevant change in the outcome distribution would again be a replacement of C_k by C_i (there are miscount configurations with $R_{ij} + \Delta_{ki} - 2$ miscounts which before the deviation deliver a ik runoff which C_k wins but end up in a ij-runoff after the deviation which can be won only by C_i because Δ_{ki} second round miscounts can never suffice for a victory of C_k). That there are no other changes in the outcome distribution of a lower order is straightforward to see.

What about deviations in the opposite direction, from C_j to C_k? Any such deviation leaves T_{ki} unchanged and increases both T_{jk} (by one) and $R_{ij} + \Delta_{ik}$ (by two). What are the implied (relevant) changes in the outcome distribution? First, there is a replacement of C_j by C_i (when instead of a jk-runoff we get an ik-runoff which ends with a win of C_i because there are no second round miscounts). Second, there is a replacement of C_i by C_k (instead of a win of C_i via a ij-runoff, we get a ik-runoff which is won by C_k). Overall, C_k's chances must increase and C_j's must decrease. Notice that this implies that individuals in N_{ki} must benefit from the deviation no matter what the net effect on C_i's chances are (this is not crucial for the immediately following argument but will be important later on). Let X be the set of players in Σ_{ki} who prefer a vote profile with $v_k^1(s) = \bar{v} + 1$ over a profile with $v_k^1(s) = \bar{v}$; moreover, let x be the number of elements in X. If $x \leq \bar{v}$ then it is feasible to assign the votes in Σ_{ki} in a way to C_j and C_k such that C_k receives exactly $\bar{v}$ votes and $X \subset V_k^1$. By construction any such profile is robust.

So consider next the case $x > \bar{v}$. We will argue now that in this case there must either exist a robust equilibrium where C_k receives $\bar{v}+1$ votes or there must be an equilibrium where C_k gets all votes of players in N_{ki} (and no other votes). In order to see this, consider a situation where $\bar{v}+1$ players vote for C_k and ballots are distributed such that $V_k^1 \subset X$ (such a distribution must exist since $x > \bar{v}$). By construction no player of C_k then has an incentive to deviate to C_j. So again the only question is if there are voters of C_j who might want to switch to C_k.

By doing so they further increase T_{jk} (by one) and $R_{ji} + \Delta_{ik}$ (by two). Since in the starting situation T_{jk} is strictly smaller than $R_{ij} + \Delta_{ik}$ it follows that the relevant change in the outcome distribution is a replacement of C_j by C_i. The deviation is therefore profitable only for individuals who rank C_i above C_j. This means that if it is possible to distribute voters in Σ_{ki} in such a way between C_j and C_k so that on top of the conditions i) $v_k^1 = \bar{v}+1$ votes and ii) $V_k^1 \subset X$ are satisfied but also iii) $V_j^1 \cap \Sigma_{ij} = \emptyset$ holds, then no player has an incentive to deviate and we are done.

Assume therefore that the third condition cannot be met when the first two hold. Since $\Sigma_{ki} \cap \Sigma_{ji} = N_{ki}$ and $N_{ki} \subset X$ this means that $n_{ki} > \bar{v}$. Take a vote profile where the electorate of C_k is composed by exactly N_{ki}. By construction at such a profile we must have that $T_{jk} < R_{ji} + \Delta_{ki}$. A deviation towards C_j is detrimental for ki-types since it leads to a replacement of C_i by C_j (T_{jk} increases). A deviation towards C_k is detrimental for all voters of C_j (i.e. those in L_i) since it leads to a replacement of C_j by C_i (T_{jk} decreases and remains strictly smaller than $R_{ij} + \Delta_{ik}$). $\square$

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