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Put–Call Parity and Market Frictions*

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Abstract

We extend the Fundamental Theorem of Finance and the Pricing Rule Representation Theorem of Cox and Ross (see Ross [35] and [37] and Cox and Ross [9]) to the case in which market frictions are taken into account but the Put–Call Parity is still assumed to hold. In turn, we obtain a representation of the pricing rule as a discounted expectation with respect to a *nonadditive* risk neutral probability. As a further contribution, in so doing we endogenize the state space structure and the contingent claim representation usually assumed to represent assets and markets.

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1 Introduction

We extend the Fundamental Theorem of Finance and the Pricing Rule Representation Theorem (see Ross [35] and [37] and Cox and Ross [9]) to the case in which market frictions are taken into account.[1] We still assume the Put–Call Parity and the absence of arbitrage opportunities. In turn, we obtain a representation of the pricing rule as a discounted expectation with respect to a *nonadditive* risk neutral probability. In other words, the market prices contingent claims as an ambiguity averse decision maker. As a further contribution, we remove the state space structure and the contingent claim representation that are usually assumed exogenously to model assets and markets. In particular, this allows us to provide a unique mathematical framework where we can both discuss the Fundamental Theorem of Finance and the Pricing Rule Representation Theorem.

Most of the fundamental theory of asset pricing relies on two main hypotheses: frictionless markets and absence of arbitrage (see, e.g., Ross [35] and [37], Cox and Ross [9], and, in a dynamic setting, Harrison and Kreps [23], Harrison and Pliska [24], and Delbaen and Schachermayer [11]).[2] On the other hand, frictions and transaction costs are present in financial markets and play an important role. Important evidence of these facts is the existence of bid–ask spreads (see, e.g., Amihud and Mendelson [4] and [5]). As a consequence, the Finance literature developed models that incorporate transaction costs and taxes (see, e.g., Garman and Ohlson [17], Prisman [34], Ross [38], Jouini and Kallal [26], and Luttmer [31]). In particular, [34], [26], and [31] observe how taxes/transaction costs generate pricing rules that are not linear but still can be compatible with the no arbitrage assumption. In particular, [34] shows that convex transaction costs or taxes generate convex pricing rules. Furthermore, if transaction costs are different among securities but proportional to the volumes dealt then the respective pricing rules are sublinear, as in [26] and [31].

In a standard framework, the no friction assumption paired with the Law of One Price delivers the fundamental Put–Call Parity, first discovered by Stoll [43] (see also Kruizenga [29]). Moreover, the no friction assumption also implies that when a risk free position is added to an existing portfolio the price of the resulting portfolio is equal to the price of the original portfolio plus the price of the position on the risk free. This last implication is basically equivalent to say that the price on the risk free market is linear and in particular the bid–ask spread is zero on this market. From an applied point of view, the absence of frictions on the risk free market and the Put–Call Parity are two particularly important assumptions since they can be empirically tested (see, e.g., Stoll [43] and [44], Gould and Galai [20], Klemkosky and Resnick [28], Amihud and Mendelson [5], and Kamara and Miller [27]).

In this paper, we study price functionals and pricing rules that satisfy the Put–Call Parity and exhibit no frictions in the risk free market. These two no frictions assumptions are, at the same time, conceptually much weaker than the standard ones and much easier to test empirically. As in the standard case, we further retain the no arbitrage postulate. We show that these pricing rules can be characterized as discounted expectations with respect to a nonadditive probability (Choquet expectations). When such a probability is concave, the associated pricing rule is sublinear. In this way, we provide testable conditions under which transaction costs can generate a sublinear pricing rule (as in Jouini and Kallal [26] and Luttmer [31]) which is further a nonadditive expectation (see also Chateauneuf, Kast, and Lapied [7]).

Choquet expected value and Choquet Expected Utility (henceforth, CEU) were introduced in Economics and Decision Theory to account for deviations from the standard model of Subjective Expected Utility

[1] The combination of these two results is also known in the literature as the Fundamental Theorem of Asset Pricing (see, e.g., [12] and [16]).

[2] For an introduction to the topic see Dybvig and Ross [14], Ross [39], [16], and [12].

(henceforth, SEU) as formulated by Savage [40] and Anscombe and Aumann [1]. Starting from the seminal paper of Ellsberg [15], the SEU paradigm began to falter. In fact, the Ellsberg paradox is a thought experiment where the choices that seem natural cannot be rationalized by the existence of a probability over the state space that further summarizes the decision maker preferences through a SEU criterion. Schmeidler [42] (see also Gilboa [18]) showed instead how the CEU model could accommodate such pattern of choices. At the same time, the CEU model could also account for Knightian Uncertainty and Ambiguity Aversion, that is, the decision maker ignorance about the true probability distribution over the state space and the possible aversion to such uncertainty. The key difference between the CEU and the SEU model is that the decision maker, in the first case, considers a *nonadditive* probability, rather than an additive one, as a measure of likelihood. Moreover, expectations are computed through a Choquet integral (see Choquet [8] and Schmeidler [41]) which naturally generalizes the usual notion of integral.[3]

The rest of the paper is organized as follows. We first discuss, in Section 2, our contribution in a finite dimensional setting: Theorem 1. There, we also briefly review the famous Fundamental Theorem of Finance and the Pricing Rule Representation Theorem. This allows us to introduce and discuss the extension we are after. Section 3 contains our main result, Theorem 2, in a very general setting where we dispense with the assumptions of finite dimensionality, as well as, the assumption of existence of an underlying state space. Appendix A provides the mathematical tools and the representation theorem behind our results. The proofs are relegated to Appendix B.

2 The Finite Dimensional Case

2.1 Mathematical Preliminaries

In this section, we provide the mathematical preliminaries that are necessary for Section 2 and all the examples in the paper which refer to this section. We refer the reader to Appendix A for all the other relevant mathematical notions. Consider a finite state space $\Omega = \{\omega_1, ..., \omega_m\}$. We endow Ω with the σ-algebra coinciding with the power set. A nonadditive probability is a set function ν such that $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B \subseteq \Omega$. We say that ν is a concave nonadditive probability if and only if for each A and B

$$\nu(A \cap B) + \nu(A \cup B) \leq \nu(A) + \nu(B).$$

A probability μ is instead an additive set function such that $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Clearly, an additive probability can be identified with a vector in $\mathbb{R}^m$. Finally, a nonadditive probability ν is balanced if and only if there exists a probability μ such that $\mu \leq \nu$. We denote by $core(\nu)$ the set of all probabilities μ such that $\mu \leq \nu$.

Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $(\boldsymbol{x})^+ = \max\{\boldsymbol{x}, 0\}$, $(\boldsymbol{x})^- = \max\{-\boldsymbol{x}, 0\}$, and $\min\{\boldsymbol{x}, \boldsymbol{y}\}$ denote, respectively, the positive part of a vector in $\mathbb{R}^m$, the negative part of a vector in $\mathbb{R}^m$, and the minimum between vectors in $\mathbb{R}^m$. With a small abuse of notation, we sometimes denote by k both the real number k and the constant vector that takes value k. With such a notation, $\boldsymbol{x} \wedge k$ and $\boldsymbol{x} \vee k$ denote the minimum and the maximum between vector $\boldsymbol{x}$ and the constant vector k. We say that a function $\tilde{\pi} : \mathbb{R}^m \to \mathbb{R}$ is

- *positive* if and only if $\boldsymbol{x} \geq 0$ implies $\tilde{\pi}(\boldsymbol{x}) \geq 0$;[4]
- *monotone* if and only if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\tilde{\pi}(\boldsymbol{x}) \geq \tilde{\pi}(\boldsymbol{y})$;

[3] For a comprehensive recent survey on the literature of choice under Ambiguity see Gilboa and Marinacci [19].

[4] Notice that equivalently, with $\tilde{\pi}$ linear, we could say $\tilde{\pi}$ is positive if and only if $0 \geq \boldsymbol{x}$ implies $0 \geq \tilde{\pi}(\boldsymbol{x})$. See also Section 3.3.

- *translation invariant* if and only if for each $\boldsymbol{x} \in \mathbb{R}^m$ and $k \in \mathbb{R}$

$$\tilde{\pi}(\boldsymbol{x}+k) = \tilde{\pi}(\boldsymbol{x}) + k\tilde{\pi}(1);$$

- *constant modular* if and only if for each $\boldsymbol{x} \in \mathbb{R}^m$ and $k \in \mathbb{R}$

$$\tilde{\pi}(\boldsymbol{x} \vee k) + \tilde{\pi}(\boldsymbol{x} \wedge k) = \tilde{\pi}(\boldsymbol{x}) + k\tilde{\pi}(1);$$

- *subadditive* if and only if for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$

$$\tilde{\pi}(\boldsymbol{x}+\boldsymbol{y}) \leq \tilde{\pi}(\boldsymbol{x}) + \tilde{\pi}(\boldsymbol{y}).$$

Finally, a nonadditive probability ν induces a function on $\mathbb{R}^m$ via the Choquet integral $\int_\Omega \boldsymbol{x} d\nu$. If $\boldsymbol{x}$ is such that $x_1 > ... > x_m$, then we have that

$$\int_\Omega \boldsymbol{x} d\nu = \sum_{i=1}^{m} x_i \mu^\nu(\omega_i)$$

where μ^ν is the probability defined by

$$\mu^\nu(\omega_1) = \nu(\omega_1) \text{ and } \mu^\nu(\omega_i) = \nu\left(\bigcup_{j=1}^{i}\{\omega_j\}\right) - \nu\left(\bigcup_{j=1}^{i-1}\{\omega_j\}\right) \qquad \forall i \in \{2, ..., m\}.$$ [5]

2.2 The Classical Framework

In illustrating the results of this section, we follow Dybvig and Ross [14] and Ross [39]. We consider a two periods market where all tradings happen at time 0. Let n be the number of primary assets. We denote by $\Omega = \{\omega_1, ..., \omega_m\}$ the state space that we eventually use to represent the uncertainty behind any asset evaluation at time 1. In this case, an asset or a portfolio of assets can also be represented as a vector $\boldsymbol{x} \in \mathbb{R}^m$. We denote by G the Arrow-Debreu tableau of securities' payoffs which is a matrix with m rows and n columns. Each row i denotes the payoff/evaluation of each primary asset in ω_i while each column j is the primary asset j in its contingent claim form. In other words, the entry g_{ij} of G represents the payoff/evaluation of primary asset j in state ω_i.[6] Thus, the market of all tradable portfolios can be represented by the vector space $P = \mathbb{R}^n$, where a vector $\boldsymbol{\eta}$ represents the portfolio consisting of η_i units of each primary asset i.[7] Each portfolio $\boldsymbol{\eta} \in P$ has also a representation as a contingent claim which is $G\boldsymbol{\eta} \in \mathbb{R}^m$ and the space of portfolios has a contingent claim representation as $C = \{G\boldsymbol{\eta} : \boldsymbol{\eta} \in \mathbb{R}^n\}$. In this subsection, we assume that

Assumption $C \cap \mathbb{R}^m_{++} \neq \emptyset$.

The two spaces are connected via a linear function $T : \mathbb{R}^n \to C \subseteq \mathbb{R}^m$ that associates to each portfolio $\boldsymbol{\eta}$ the contingent claim $T(\boldsymbol{\eta}) = G\boldsymbol{\eta}$. A further datum in this problem are the prices of the n primary assets: $p_1, ..., p_n$. If there are no frictions in the market (NF), these prices induce a price functional $p : P \to \mathbb{R}$ such that

$$p(\boldsymbol{\eta}) = \sum_{i=1}^{n} p_i \eta_i \qquad \forall \boldsymbol{\eta} \in \mathbb{R}^n. \tag{NF}$$

[5] In general, in a finite setting, the probability μ^ν_σ used to compute $\tilde{\pi}(\boldsymbol{x})$ depends on the permutation σ that orders the payoffs of $\boldsymbol{x}$ decreasingly. In defining μ^ν, we implicitly considered the trivial permutation given by the identity. See also Appendix A.

[6] We assume that the rows of G are different from each other, that is, there are not redundant states of the world in Ω.

[7] The analysis of our paper is done by assuming that there are not short-sale constraints. We can dispense with such an assumption as outlined in Section 2.4.2.

The value $p(\boldsymbol{\eta})$ captures the market value of $\boldsymbol{\eta}$. Another assumption that is fundamental in Asset Pricing is the no arbitrage assumption (NA):[8]

$$p(\boldsymbol{\eta}) < 0 \quad \Rightarrow \quad G\boldsymbol{\eta} \not\geq \mathbf{0}. \tag{NA}$$

It amounts to impose that there do not exist portfolios that have negative price and deliver a nonnegative payment in each contingency. It is immediate to see that under NF the no arbitrage assumption delivers the Law of One Price, that is, if two portfolios $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ induce the same contingent claim, then $p(\boldsymbol{\eta}_1) = p(\boldsymbol{\eta}_2)$. Thus, given the map T, each price functional p induces a well defined pricing rule $\tilde{\pi} : C \to \mathbb{R}$ over contingent claims. In fact, the price of a contingent claim $\boldsymbol{x} \in C$ is

$$\tilde{\pi}(\boldsymbol{x}) = p(\boldsymbol{\eta})$$

where $\boldsymbol{\eta} \in \mathbb{R}^n$ is such that $\boldsymbol{x} = G\boldsymbol{\eta}$. We thus have:

Fundamental Theorem of Finance *Let $p : P \to \mathbb{R}$ be a price functional such that $p \neq 0$. The following statements are equivalent:*

(i) *there are no frictions in the market (NF) and no arbitrage opportunities (NA);*

(ii) *$\tilde{\pi}$ is a positive linear pricing rule.*

On the other hand, we can also characterize positive linear pricing rules:

Representation Theorem *Let $\tilde{\pi} : C \to \mathbb{R}$ be a pricing rule such that $\tilde{\pi} \neq 0$. The following statements are equivalent:*

(i) *$\tilde{\pi}$ is a positive linear pricing rule;*

(ii) *there exist a risk neutral probability μ and a riskless rate $r > -1$ such that*

$$\tilde{\pi}(\boldsymbol{x}) = \frac{1}{1+r}\mathbb{E}_\mu \boldsymbol{x} = \frac{1}{1+r}\sum_{i=1}^{m} x_i \mu_i \qquad \forall \boldsymbol{x} \in C. \tag{1}$$

From a mathematical point of view, the NF assumption translates into an assumption of linearity of both: the price functional p and the pricing rule $\tilde{\pi}$. The NA assumption is an assumption of positivity of the pricing rule $\tilde{\pi}$. At first sight, the NA condition does not seem to have a clear mathematical counterpart for p since it is based on the contingent claim representation of portfolios. On the other hand, the NA assumption is a positivity condition of the price functional p whenever the space of portfolios P is endowed with the preorder induced by G, that is,

$$\boldsymbol{\eta}_1 \geq_G \boldsymbol{\eta}_2 \quad \iff \quad G\boldsymbol{\eta}_1 \geq G\boldsymbol{\eta}_2.$$

In other words, a portfolio $\boldsymbol{\eta}_1$ is declared at least as good as a portfolio $\boldsymbol{\eta}_2$ if and only if each possible evaluation at time 1 of $\boldsymbol{\eta}_1$ is greater than the one of portfolio $\boldsymbol{\eta}_2$.[9] In light of this observation, p satisfies the NA condition if and only if

$$\boldsymbol{\eta} \geq_G \mathbf{0} \quad \Rightarrow \quad p(\boldsymbol{\eta}) \geq 0$$

which is a condition of positivity for p.

In both cases, given the linearity of p and $\tilde{\pi}$, the positivity assumption contained in the NA condition is equivalent to a monotonicity assumption on both p and $\tilde{\pi}$.

[8] A stronger version of the NA assumption (see also [30] and [39]) is $p(\boldsymbol{\eta}) \leq 0$ implies $G\boldsymbol{\eta} \ngtr 0$. This is a condition of strict monotonicity, while NA is a condition of monotonicity.

[9] Notice that this order is typically very different from the pointwise order with which $\mathbb{R}^n$ is naturally endowed.

2.3 Our Main Result

In the Representation Theorem, the probability measure μ takes the name of risk neutral probability. It is unique when C is complete, that is, when all contingent claims are available and $C = \mathbb{R}^m$. In the rest of the section, we assume that

Assumption $C = \mathbb{R}^m$ and the columns of G are linearly independent.

The goal of our paper is to characterize price functionals and pricing rules which satisfy, among the others, the Put–Call Parity. Thus, it is natural to consider all possible calls and puts with nonnegative strike price. It is well known that completeness of markets can be achieved in different ways. One way that plays a fundamental role in our result is the closure with the respect to option contracts (see Ross [36] and Green and Jarrow [22]). For example, in a finite dimensional setting, by Ross [36], it follows that if C contains the risk free asset, an efficient fund, and all call options contracts written on it with nonnegative strike price, then $C = \mathbb{R}^m$.[10] For simplicity, we further assumed that the columns of G are linearly independent. This amounts to impose that in the market of primary assets there are not redundant securities (see also LeRoy and Werner [30], Follmer and Schied [16], and Section 3.1). Mathematically, this translates into T being injective.

One important consequence of the market being complete is that the risk free asset $\boldsymbol{x}_{rf}$ is available, that is, the constant unit vector belongs to C. Given a contingent claim $\boldsymbol{x}$ in C, if we denote by $\boldsymbol{c}_{x,k}$ the call option (resp., $\boldsymbol{p}_{x,k}$ the put option) on $\boldsymbol{x}$ with strike price $k \geq 0$, then we have the following well known equality

$$\boldsymbol{c}_{x,k} - \boldsymbol{p}_{x,k} = \boldsymbol{x} - k\boldsymbol{x}_{rf}.$$

If $\tilde{\pi}$ were linear, this equality would deliver a version of the famous Put–Call Parity (henceforth, PCP) for European options, that is,

$$\tilde{\pi}\left(\boldsymbol{c}_{x,k}\right) + \tilde{\pi}\left(-\boldsymbol{p}_{x,k}\right) = \tilde{\pi}\left(\boldsymbol{x}\right) - k\tilde{\pi}\left(\boldsymbol{x}_{rf}\right). \tag{PCP}$$

At time 0, PCP alone requires that the difference between the ask price for the call option $\boldsymbol{c}_{x,k}$ and the bid price for the put option $\boldsymbol{p}_{x,k}$ coincides with the difference between the ask price of the underlying asset $\boldsymbol{x}$ and the price of k units of the risk free asset. In other words, the two (payoff equivalent) strategies must have the same cost.[11]

In this paper, in terms of pricing rule $\tilde{\pi}$, we assume that:

Assumption (nf) The PCP holds for all $\boldsymbol{x}$ and all $k \geq 0$ and there are no frictions on the risk free market, that is,

$$\tilde{\pi}\left(\boldsymbol{x} + k\boldsymbol{x}_{rf}\right) = \tilde{\pi}\left(\boldsymbol{x}\right) + k\tilde{\pi}\left(\boldsymbol{x}_{rf}\right) \qquad \text{for all } \boldsymbol{x} \in C \text{ and all } k \in \mathbb{R}.$$

Assumption (na) There are no arbitrage opportunities, that is, $\tilde{\pi}\left(\boldsymbol{x}\right) \geq \tilde{\pi}\left(\boldsymbol{y}\right)$ whenever $\boldsymbol{x} \geq \boldsymbol{y}$.

Notice that both conditions can be stated in terms of the primitive price functional $p : P \to \mathbb{R}$ through the order $\geq_G$ and then stated in terms of pricing rule $\tilde{\pi}$ through the formula $p = \tilde{\pi} \circ T$.[12] In particular, it is immediate to observe that na for p amounts to assume that

$$p\left(\boldsymbol{\eta}_2\right) < p\left(\boldsymbol{\eta}_1\right) \quad \Rightarrow \quad G\boldsymbol{\eta}_2 \not\geq_G G\boldsymbol{\eta}_1.$$

[10] Recall that an efficient fund is defined to be a contingent claim $\boldsymbol{x}$ such that if $i \neq j$, then $x_i \neq x_j$.

[11] Notice that if there are frictions, a violation of the PCP might not lead to an immediately available arbitrage opportunity. Nevertheless, if one of the two strategies was costing more than the other, in an economy with rational agents, we could think that no agent would buy the most expensive portfolio, thus yielding an equivalent equilibrium price where the PCP is satisfied.

[12] Or, equivalently, $\tilde{\pi} = p \circ T^{-1}$ since we assumed T injective.

Indeed, both assumptions require the price functional p to preserve certain lattice equalities and inequalities.[13] The key insight is that those conditions for p can so be stated in terms of the preorder $\geq_G$ (see Examples 1 and 3). In light of these observations, clearly nf is weaker than NF. In terms of $\tilde{\pi}$, it is a requirement of linearity of the pricing rule for two specific decompositions of $\boldsymbol{x} - k\boldsymbol{x}_{rf}$.[14] On the other hand, assumption na is stronger than NA, but under NF they coincide, as already argued.[15]

Thus, we have all the ingredients to state our nonlinear versions of the Fundamental Theorem of Finance and the Representation Theorem in the finite dimensional case:

Theorem 1 *Let $C = \mathbb{R}^m$, let $p : P \to \mathbb{R}$ be a price functional such that $p \neq 0$, and let $\tilde{\pi} : C \to \mathbb{R}$ be the associated pricing rule. The following statements are equivalent:*

(i) p satisfies nf and na;

(ii) $\tilde{\pi}$ is monotone, translation invariant, constant modular, and such that $\tilde{\pi} \neq 0$;

(iii) there exist a nonadditive risk neutral probability ν and a riskless rate $r > -1$ such that

$$\tilde{\pi}(\boldsymbol{x}) = \frac{1}{1+r} \int_\Omega \boldsymbol{x} d\nu \qquad \text{for all } \boldsymbol{x} \in C \tag{2}$$

where the integral in (2) is a Choquet integral.

Moreover,

1. *r and ν are unique.*
2. *If ν is balanced, then $\tilde{\pi}(\boldsymbol{x}) \geq -\tilde{\pi}(-\boldsymbol{x})$ for all $x \in C$, that is, there exists a positive bid-ask spread.*
3. *ν is a (additive) risk neutral probability if and only if p satisfies NF.*

Notice that the equivalence between (i) and (ii) is the generalized version of the Fundamental Theorem of Finance, while the equivalence of (ii) and (iii) is the generalized version of the Representation Theorem.[16] Completeness of the market delivers uniqueness of the representation in point 1. At the same time, point 2 provides, in terms of the representation, a sufficient condition for the presence of a positive bid-ask spread. Finally, the standard Fundamental Theorem of Asset Pricing is derived as a particular case in point 3.

2.4 Some Extension

2.4.1 The Subadditive Case

One contribution of our main result is to characterize among sublinear pricing rules the ones that are Choquet pricing rules.

One way in which nonlinear price functionals and pricing rules can arise, even in complete markets and without short-sale constraints, is through the existence of different prices for selling and buying primary assets (see also [30]). In other words, if there exist positive bid-ask spreads in the market of primary assets, then the NF fails to hold. In fact, if for each primary asset $i \in \{1, ..., n\}$ there exists a (ask) price, p_i^a, for

[13] See also the proof of Theorem 1 for a more formal statement of these facts.

[14] In particular, it implies linearity on the subspace generated by the risk free asset.

[15] Notice that if there are frictions, a violation of assumption na might not lead to an immediately available arbitrage opportunity. Nevertheless, if $\boldsymbol{x} \geq \boldsymbol{y}$ and $\tilde{\pi}(\boldsymbol{x}) < \tilde{\pi}(\boldsymbol{y})$, in an economy with rational agents, we could think that no agent would buy $\boldsymbol{y}$, thus yielding an equivalent equilibrium price where the na is satisfied.

[16] It should be observed that the equivalence between (ii) and (iii) stands also if $\tilde{\pi}$ is a pricing rule that satisfies the assumptions in (ii) but it is not necessarily associated to a price functional p.

buying asset i which is greater than the (bid) price, p_i^b, for selling asset i, it follows that $p : P \to \mathbb{R}$ is such that

$$p(\boldsymbol{\eta}) = \sum_{i=1}^{n} \eta_i^+ p_i^a - \sum_{i=1}^{n} \eta_i^- p_i^b \qquad \forall \boldsymbol{\eta} \in \mathbb{R}^m.$$

It is then immediate to see that p is a genuine sublinear price functional, that is, $p(\lambda\boldsymbol{\eta}) = \lambda p(\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in P$ and all $\lambda \geq 0$ and $p(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2) \leq p(\boldsymbol{\eta}_1) + p(\boldsymbol{\eta}_2)$ for all $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in P$. At the same time, under the Law of One Price, the associated pricing rule satisfies the same properties, that is, $\tilde{\pi}(\lambda \boldsymbol{x}) = \lambda \tilde{\pi}(\boldsymbol{x})$ and $\tilde{\pi}(\boldsymbol{x} + \boldsymbol{y}) \leq \tilde{\pi}(\boldsymbol{x}) + \tilde{\pi}(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and for all $\lambda \geq 0$. Starting from the paper of Jouini and Kallal [26], transaction costs and frictions have been considered and modelled by imposing sublinearity of the pricing rule. The first property is called positive homogeneity and it is satisfied by Choquet integrals while the second property is called subadditivity. Under the na assumption, a sublinear pricing rule takes the following characterization

$$\tilde{\pi}(\boldsymbol{x}) = \frac{1}{1+r} \max_{\mu \in R} \mathbb{E}_\mu \boldsymbol{x} \qquad \forall \boldsymbol{x} \in C$$

where R is a closed and convex set of risk neutral probabilities. An important example of sublinear pricing rules are Choquet pricing rules with concave ν. In that case, we have that R coincides with the core of ν. Our paper shows that these latter rules are characterized among the pricing rules considered by Jouini and Kallal [26] and Luttmer [31] as the ones that further satisfy the PCP. In the case of sublinear pricing we can also weaken the na assumption. In fact, the NA assumption we first considered stated that

$$p(\boldsymbol{\eta}) < 0 \quad \Rightarrow \quad G\boldsymbol{\eta} \not\geq \mathbf{0}.$$

In a context with no frictions, this is equivalent to assume that

Assumption (NA') If $G\boldsymbol{\eta} \leq \mathbf{0}$, then $p(\boldsymbol{\eta}) \leq \mathbf{0}$.

It is immediate to see that it is weaker than the na assumption.

Corollary 1 *Let $C = \mathbb{R}^m$, let $p : P \to \mathbb{R}$ be a price functional such that $p \neq 0$, and let $\tilde{\pi} : C \to \mathbb{R}$ be the associated pricing rule. The following statements are equivalent:*

(i) p satisfies nf, NA', and it is such that $p(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2) \leq p(\boldsymbol{\eta}_1) + p(\boldsymbol{\eta}_2)$ for all $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in P$;

(ii) $\tilde{\pi}$ is monotone, translation invariant, constant modular, subadditive, and such that $\tilde{\pi} \neq 0$;

(iii) there exist a concave nonadditive risk neutral probability ν and a riskless rate $r > -1$ such that

$$\tilde{\pi}(\boldsymbol{x}) = \frac{1}{1+r} \int_\Omega \boldsymbol{x} d\nu = \frac{1}{1+r} \max_{\mu \in core(\nu)} \mathbb{E}_\mu \boldsymbol{x} \qquad \text{for all } \boldsymbol{x} \in C$$

where the first integral in (2) is a Choquet integral.

Moreover, r and ν are unique.

Subadditive Choquet pricing rules have also been characterized by Chateauneuf, Kast, and Lapied [7]. This latter work offers a generalized version of the Representation Theorem in terms of Choquet pricing. The assumptions characterizing $\tilde{\pi}$ in [7] are monotonicity (our na assumption), subadditivity, and comonotonic additivity. Although this is a legitimate mathematical representation result, comonotonic additivity is a difficult property to test since it requires a contingent claim representation for the assets considered. In fact, two assets are comonotonic if and only if the associated contingent claims $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy

$$(x_i - x_j)(y_i - y_j) \geq 0 \qquad \forall i, j \in \{1, ..., m\}$$

and comonotonic additivity requires additivity of the price functional over pairs of comonotonic assets which is a difficult property to test.[17] On the other hand, our characterization of Choquet pricing relies on the PCP for *European options* which is easier to test. In fact, in the literature there are several studies testing the validity of the PCP (see Stoll [43] and [44], Gould and Galai [20], Klemkosky and Resnick [28], and Kamara and Miller [27]).

2.4.2 Discount Certificates and Call Options

An important class of derivatives are discount certificates (see [16]). A discount certificate on a claim $\boldsymbol{x}$ with cap $k \geq 0$ is a contingent claim that in state ω_i pays x_i if $x_i \leq k$ and k if $x_i > k$. We will denote this derivative security by $\boldsymbol{d}_{x,k}$. It is immediate to see that for each $\boldsymbol{x} \in C$ and $k \geq 0$

$$\boldsymbol{c}_{x,k} + \boldsymbol{d}_{x,k} = \boldsymbol{x}. \tag{3}$$

When $\tilde{\pi}$ is linear, this latest equality delivers the following relation

$$\tilde{\pi}\left(\boldsymbol{c}_{x,k}\right) + \tilde{\pi}\left(\boldsymbol{d}_{x,k}\right) = \tilde{\pi}\left(\boldsymbol{x}\right). \tag{4}$$

We say that $\tilde{\pi}$ satisfies nf' if and only if

Assumption (nf') $\tilde{\pi}$ satisfies (4) for all $\boldsymbol{x}$ and all $k \geq 0$ and there are no frictions on the risk free market, that is,

$$\tilde{\pi}\left(\boldsymbol{x} + k\boldsymbol{x}_{rf}\right) = \tilde{\pi}\left(\boldsymbol{x}\right) + k\tilde{\pi}\left(\boldsymbol{x}_{rf}\right) \qquad \text{for all } \boldsymbol{x} \in C \text{ and all } k \in \mathbb{R}.$$

As seen before for the PCP, also this condition can be stated in terms of the primitive price functional $p : P \to \mathbb{R}$ through the order $\geq_G$ and then stated in terms of pricing rule $\tilde{\pi}$ through the formula $p = \tilde{\pi} \circ T$. Even in this case, the assumption in (4) is a condition on the price functional p to preserve a lattice equality (see Examples 1 and 3).

Proposition 1 *Let $C = \mathbb{R}^m$ and let $p : P \to \mathbb{R}$ be a price functional. The following statements are equivalent:*

(i) p satisfies nf and na;

(ii) p satisfies nf' and na.

In light of Theorem 1, in order to characterize the associated pricing rule $\tilde{\pi}$ in terms of Choquet pricing, we can drop the PCP relation and replace it with the relation contained in (4). Notice that also in this case we provide conditions for Choquet pricing that can be empirically tested (see Jarrow and O'Hara [25]). We conclude with two remarks:

Remark 1 *Chateauneuf, Kast, and Lapied [7] argue that Choquet pricing can account for violations of both the PCP as well as the relation contained in (4) when dividends are taken into account. In terms of PCP, the key difference is that the PCP version they consider is the following one:*

$$\tilde{\pi}\left(\boldsymbol{p}_{x,k}\right) = \tilde{\pi}\left(\boldsymbol{c}_{x,k}\right) + \tilde{\pi}\left(-\boldsymbol{x}\right) + k\tilde{\pi}\left(\boldsymbol{x}_{rf}\right).$$

This condition is different from ours since, under transaction costs, we have that, typically, $-\tilde{\pi}\left(\boldsymbol{c}_{x,k}\right) \neq \tilde{\pi}\left(-\boldsymbol{c}_{x,k}\right)$. On the other hand, if $\tilde{\pi}$ can be represented as a discounted Choquet integral and dividends are not taken into account, it is immediate to see that (4) must hold.[18]

[17] That is, $\pi\left(\boldsymbol{x} + \boldsymbol{y}\right) = \pi\left(\boldsymbol{x}\right) + \pi\left(\boldsymbol{y}\right)$ for all pairs of comonotonic $\boldsymbol{x}$ and $\boldsymbol{y}$ in C.

[18] In fact, $\boldsymbol{c}_{x,k}$ and $\boldsymbol{d}_{x,k}$ are comonotonic for all $\boldsymbol{x} \in C$ and $k \geq 0$ (see, for a proof, [32, Lemma 4.6]). Since Choquet pricing rules are comonotonic additive, it follows that

$$\pi\left(\boldsymbol{x}\right) = \pi\left(\boldsymbol{c}_{x,k} + \boldsymbol{d}_{x,k}\right) = \pi\left(\boldsymbol{c}_{x,k}\right) + \pi\left(\boldsymbol{d}_{x,k}\right).$$

Remark 2 *In the literature, in order to consider transaction costs, often the space of marketed portfolios has been considered to be just a cone (see, e.g., Luttmer [31]). For example, this is the case if short-sale constraints are assumed. In such a case, we would have that* $P = \mathbb{R}^n_+$ *and, given the Arrow and Debreu tableau* G*, the space of marketed contingent claims would be the cone* $C_+ = \{G\boldsymbol{\eta} : \boldsymbol{\eta} \in \mathbb{R}^n_+\}$*. In this case, we could still provide the equivalence between points (i) and (iii) of Theorem 1 given three caveats: (a)* $\boldsymbol{x}_{rf} \in C_+$*, (b) for each* $\boldsymbol{x} \in C_+$ *and* $k \geq 0$ *we must have that* $\boldsymbol{c}_{x,k}, \boldsymbol{d}_{x,k} \in C_+$*, and (c) the nf condition is replaced with the nf' condition.*

3 The General Case

3.1 A Generalized Market Model

We consider a market and we model it as a vector space M. Each element x in M is interpreted as a financial asset or a portfolio. Given a set of weights $\{\lambda_i\}_{i=1}^l \subseteq \mathbb{R}$ and a set $\{x_i\}_{i=1}^l \subseteq M$, we interpret

$$\sum_{i=1}^{l} \lambda_i x_i$$

as the portfolio constructed by buying/selling x_i using the quantities $|\lambda_i|$, with an interpretation of buying if λ_i is positive and of selling if λ_i is negative. The goal of this section is to study a price functional π defined over the market M when all tradings take place at time 0 and then the value of each asset is revealed at time 1. We remove the hypothesis that there exists an agreed state space Ω or, in other words, we do not necessarily represent the market as a space of random variables/contingent claims. Instead, we consider a set of evaluations maps $\mathcal{V}$. That is, the value of each asset x at time 1 is determined by an evaluation map $v \in \mathcal{V}$. We make three assumptions on $\mathcal{V}$:

1. Each $v \in \mathcal{V}$ is a linear mapping from M to $\mathbb{R}$.

2. For each $x \in M$ the interval $[\inf_{v\in\mathcal{V}} v(x), \sup_{v\in\mathcal{V}} v(x)]$ is bounded.

3. If $x, y \in M$, then $v(x) = v(y)$ for all $v \in \mathcal{V}$ implies that $x = y$.

Given an element $x \in M$ and an element $v \in \mathcal{V}$, $v(x)$ is the value that asset x will take at period 1 under the evaluation map v. From a practical point of view, assumption 1. is justified in the following way: given a portfolio $x = \sum_{i=1}^{l} \lambda_i x_i$ and a brokerage account, at the end of a trading day the value of x, $v(x)$, is typically computed/approximated to be $\sum_{i=1}^{l} \lambda_i v(x_i)$. In other words, the portfolio is marked to market. This does not mean that if the portfolio x had to be sold the realized proceedings would be $\sum_{i=1}^{l} \lambda_i v(x_i)$ but it provides an estimate for a future and uncertain evaluation. From a theoretical point of view, the linearity assumption contained in 1. is in line with Debreu [10] and the fact that the market is modelled to have just two periods. Condition 2. implies that, at time 1, the value of each asset will be in a bounded range. Condition 3. imposes that there are no redundancies. In fact, there do not exist two securities which are not equal but, in terms of their value at time 1, are indistinguishable.[19]

[19] We could dispense with Condition 3. by declaring two elements x and y in M equivalent, $x \sim y$, if and only if $v(x) = v(y)$ for all $v \in \mathcal{V}$. Given this equivalence relation, we could then take the quotient $M/\sim$. This mathematical step would be reasonable from a financial point of view since evaluating an asset x should be based just on the future evaluations of x itself and nothing else.

Given the set of evaluations $\mathcal{V}$, we endow this set with the σ-algebra generated by the class of the following subsets:

$$(\{v \in \mathcal{V} : v(x) > t\})_{x \in M, t \in \mathbb{R}} .$$

We denote this σ-algebra by $\mathcal{B}$. This class of subsets is not unusual in Finance and the σ-algebra generated by it is natural in Mathematics. In fact, given $x \in M$ and $t \in \mathbb{R}$, if there exists $z \in M$ such that

$$v(z) = 1_{\{w \in \mathcal{V} : w(x) > t\}}(v) \qquad \forall v \in \mathcal{V},$$

then, following Nachman [33], z is a *simple* call option on x with exercise price t. On the other hand, if $\mathcal{V}$ is endowed with a natural topology that makes it compact, in some case, $\mathcal{B}$ turns out to be the Baire σ-algebra.

Definition 1 *Given a vector space M, a price functional $\pi : M \to \mathbb{R}$, and a set of linear functions $\mathcal{V}$ on M, we will say that $(M, \mathcal{V}, \pi)$ is a market if and only if $\mathcal{V}$ satisfies Conditions 1.–3.*

Example 1 *In Section 2, we represented the market of tradable assets as the vector space of all portfolios P. Each vector $\boldsymbol{\eta} \in P$ represented the portfolio where each primary asset i is held in quantity η_i. In such a case, the price functional we considered was called p. At the same time, given the Arrow-Debreu tableau, the set of evaluations $\mathcal{V}$ can be seen as the set of m linear functionals induced by the rows of G, that is, for each $i = 1, ..., m$*

$$v_i(\boldsymbol{\eta}) = \sum_{j=1}^{n} g_{ij}\eta_j \qquad \forall \boldsymbol{\eta} \in P.$$

Conditions 1. and 2. are then satisfied by construction and the finite dimensionality of P. On the other hand, Condition 3. is satisfied whenever it is imposed that the columns of G are linearly independent, that is, there are not redundant securities.

Example 2 *In Section 2, another way we represented the market was through the space of contingent claims C. In such a case, the price functional (pricing rule) we considered was called $\tilde{\pi}$. At the same time, given the state space Ω, the set of evaluations $\mathcal{V}$ can be seen as the set of Dirac measures $\{\delta_{\omega_i}\}_{i=1}^{m}$ and the linear functionals induced by each of these measures, that is, for each $i = 1, ..., m$*

$$v_i(\boldsymbol{x}) = x_i \qquad \forall \boldsymbol{x} \in C.$$

Conditions 1., 2., and 3. are then satisfied by construction.

3.2 Put–Call Parity and Nonlinear Pricing

Given a market $(M, \mathcal{V}, \pi)$ and an asset/portfolio $x \in M$, notice that x defines a function over $\mathcal{V}$, that is,

$$v \mapsto v(x) \qquad v \in \mathcal{V}.$$

One way in which the market could form a price for x, $\pi(x)$, could be by discounting and averaging the possible evaluations of x at time 1 under a measure of likelihood $\nu : \mathcal{B} \to [0, 1]$ and a risk free rate $r \in (-1, \infty)$. This is what happens in a market with no frictions and no arbitrages. In such a case, ν is additive. This reasoning could be extended to the nonadditive case where the integrals are going to be defined using the concept of Choquet integration (see Appendix A.1). In particular, we could have that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M. \tag{5}$$

Our purpose is to characterize a price functional $\pi : M \to \mathbb{R}$ like the one in (5) when minimal assumptions of no arbitrage are made and market frictions are also taken into account.

When modelling a market, the existence of a risk free asset is often assumed. Such an asset has the fundamental feature of having a constant value at time 1. In particular, this value is independent of what might happen between time 0 and time 1. Therefore, in our context, we will define the risk free asset in the following way:

Definition 2 *An asset x_{rf} in M is risk free if and only if*

$$v\left(x_{rf}\right) = 1 \qquad \forall v \in \mathcal{V}.$$

Another important type of financial assets are European options: call and put. Those are derivative contracts since their value at time 1 is strictly related to the value of the underlying asset x. Those are securities that give the right to trade an asset x at time 1 at a fixed strike price k. The call option gives the right to buy while the put option the right to sell. The value at time 1 of a call option $c_{x,k}$ on x with strike price k depends on the value of x at time 1. For each valuation $v \in \mathcal{V}$, it is $v\left(x\right) - k$ if $v\left(x\right) \geq k$ and 0 otherwise. The value at time 1 of a put option $p_{x,k}$ on x with strike price k depends on the value of x at time 1. For each valuation $v \in \mathcal{V}$, it is $k - v\left(x\right)$ if $v\left(x\right) \leq k$ and 0 otherwise. Formally, we have that:

Definition 3 *Let x be an asset in M. Then,*

(i) $c_{x,k}$ is a call option on x with strike price k if and only if

$$v\left(c_{x,k}\right) = \left(v\left(x\right) - k\right)^{+} \qquad \forall v \in \mathcal{V}.$$

(ii) $p_{x,k}$ is a put option on x with strike price k if and only if

$$v\left(p_{x,k}\right) = \left(k - v\left(x\right)\right)^{+} \qquad \forall v \in \mathcal{V}.$$

Given an asset x and a strike price k, if M allows tradings on x_{rf}, $c_{x,k}$, and $p_{x,k}$, then an important relationship connects x, x_{rf}, $c_{x,k}$, and $p_{x,k}$:

Proposition 2 *Let $\left(M, \mathcal{V}, \pi\right)$ be a market that contains the risk free asset and let $x \in M$ and $k \in \mathbb{R}$. If $c_{x,k}$ and $p_{x,k}$ belong to M, then*

$$c_{x,k} - p_{x,k} = x - kx_{rf}. \tag{6}$$

In other words, the portfolio obtained by buying a call option on asset x with exercise price k and selling a put option on the same asset x with exercise price k is equal to the portfolio obtained by buying a unit of asset x and selling k units of the risk free asset.

Proposition 3 *Let $\left(M, \mathcal{V}, \pi\right)$ be a market that contains the risk free asset. M contains all call options with nonnegative strike price if and only if M contains all put options with nonnegative strike price.*

Remark 3 *It is important to notice that, given a market $\left(M, \mathcal{V}, \pi\right)$, we do not assume that M is a vector lattice. On the contrary, by defining the primitive notions of call and put options, we prove in Appendix B that a market $\left(M, \mathcal{V}, \pi\right)$ which contains the risk free asset and all call options is a vector lattice with the respect to a natural order, $\geq_{\mathcal{V}}$, that we later define.*

In the next few definitions, we introduce some properties of the price functional π:

Definition 4 *Let $\left(M, \mathcal{V}, \pi\right)$ be a market that contains the risk free asset. The price functional π is said to be cash additive if and only if*

$$\pi\left(x + \lambda x_{rf}\right) = \pi\left(x\right) + \lambda\pi\left(x_{rf}\right) \qquad \forall x \in M, \forall \lambda \in \mathbb{R}. \tag{7}$$

The above assumption is equivalent to state that there are no frictions in the market when it comes to trading the risk free asset. In particular, we have that the risk free asset market is frictionless and $\pi(\lambda x_{rf}) = \lambda\pi(x_{rf})$ for all λ in $\mathbb{R}$.

Definition 5 *Let $(M, \mathcal{V}, \pi)$ be a market that contains the risk free asset and all possible call options with $k \geq 0$. The price functional π is said to satisfy the Put–Call Parity if and only if*

$$\pi(c_{x,k}) + \pi(-p_{x,k}) = \pi(x) - k\pi(x_{rf}) \qquad \forall x \in M, \forall k \in \mathbb{R}_+.$$

In other words, the two equivalent trading strategies contained in (6) must have the same price. It is immediate to see that the set of evaluations $\mathcal{V}$ induces a preorder on M. In fact, it is reasonable to declare x at least as good as y if and only if the value of x at time 1 is greater than the value of y at time 1, irrespective of the evaluation function v chosen in $\mathcal{V}$.

Definition 6 *Let $(M, \mathcal{V}, \pi)$ be a market. We say that x is at least as good as y if and only if*

$$v(x) \geq v(y) \qquad \forall v \in \mathcal{V}.$$

In this case, we write $x \geq_{\mathcal{V}} y$.

Definition 7 *Let $(M, \mathcal{V}, \pi)$ be a market. The price functional π is said to be monotone if and only if*

$$x \geq_{\mathcal{V}} y \quad \Rightarrow \quad \pi(x) \geq \pi(y).$$

Notice that this latter condition on π is simply a generalization of a no arbitrage condition. In fact, if the price functional is cash additive, then $\pi(0) = 0$ and the previous condition implies that $\pi(x) \geq 0$ whenever $x \geq_{\mathcal{V}} 0$.

Example 3 *In Section 2, we represented the market of tradable assets as the vector space of all portfolios P and also as the vector space of all tradable contingent claims C. In the first case, Example 1, we have that $\mathcal{V}$ is the set of linear evaluations induced by the rows of the Arrow-Debreu tableau G. It follows that a portfolio $\boldsymbol{\eta}_{rf}$ corresponds to the risk free asset if and only if $G\boldsymbol{\eta}_{rf}$ is the constant vector with each component equal to 1. Similarly, given a portfolio $\boldsymbol{\eta}$, a call (resp., a put) option on $\boldsymbol{\eta}$ with strike price k is the portfolio $\boldsymbol{c}_{\eta,k}$ (resp., $\boldsymbol{p}_{\eta,k}$) such that*

$$G\boldsymbol{c}_{\eta,k} = \left(G\,\boldsymbol{\eta} - kG\boldsymbol{\eta}_{rf}\right)^+ = \max\left\{G\,\boldsymbol{\eta} - kG\boldsymbol{\eta}_{rf}, \boldsymbol{0}\right\} \text{ (resp., } G\boldsymbol{p}_{\eta,k} = \left(kG\boldsymbol{\eta}_{rf} - G\boldsymbol{\eta}\right)^+ = \max\left\{kG\boldsymbol{\eta}_{rf} - G\boldsymbol{\eta}, \boldsymbol{0}\right\}\text{).}$$

Finally, a discount certificate on a portfolio $\boldsymbol{\eta}$ with cap $k \geq 0$ is the portfolio $\boldsymbol{d}_{\eta,k}$ such that

$$G\boldsymbol{d}_{\eta,k} = \min\left\{G\boldsymbol{\eta}, kG\boldsymbol{\eta}_{rf}\right\}.$$

Moreover, we have that $\geq_{\mathcal{V}}$ is equal to $\geq_G$. Along the same lines, we have that $\boldsymbol{c}_{\eta,k}$ (resp., $\boldsymbol{p}_{\eta,k}$) is the positive part, with respect to $\geq_G$, of the vector $\boldsymbol{\eta} - k\boldsymbol{\eta}_{rf}$ (resp., $k\boldsymbol{\eta}_{rf} - \boldsymbol{\eta}$). On the other hand, $\boldsymbol{d}_{\eta,k}$ is the minimum, with respect to $\geq_G$, between $\boldsymbol{\eta}$ and $k\boldsymbol{\eta}_{rf}$. The price functional p satisfies the PCP if and only if

$$p(\boldsymbol{c}_{\eta,k}) + p\left(-\boldsymbol{p}_{\eta,k}\right) = p(\boldsymbol{\eta}) - kp\left(\boldsymbol{\eta}_{rf}\right) \qquad \forall \boldsymbol{\eta} \in P, \forall k \geq 0. \tag{8}$$

Similarly, p is cash additive if and only if

$$p\left(\boldsymbol{\eta} + k\boldsymbol{\eta}_{rf}\right) = p(\boldsymbol{\eta}) + kp\left(\boldsymbol{\eta}_{rf}\right) \qquad \forall \boldsymbol{\eta} \in P, \forall k \in \mathbb{R}. \tag{9}$$

Finally, p satisfies (4) if and only if

$$p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k}) = p(\boldsymbol{\eta}) \qquad \forall \boldsymbol{\eta} \in P, \forall k \geq 0. \tag{10}$$

Observe that p satisfies condition nf, as stated in Section 2, if and only if it satisfies (8) and (9) and it satisfies nf' if and only if it satisfies (9) and (10). In the second case, Example 2, we have that $\mathcal{V}$ is the set of linear functionals induced by the Dirac measures on Ω. In this case, we have that a contingent claim $\boldsymbol{x}_{rf}$ corresponds to the risk free asset if and only if it is the constant vector with each component equal to 1. Similarly, given a contingent claim $\boldsymbol{x}$, a call (resp., a put) on $\boldsymbol{x}$ with strike price k is the portfolio $\boldsymbol{c}_{x,k}$ (resp., $\boldsymbol{p}_{x,k}$) such that

$$\boldsymbol{c}_{x,k} = (\boldsymbol{x} - k\boldsymbol{x}_{rf})^+ \ \ (\text{resp., } \boldsymbol{p}_{x,k} = (k\boldsymbol{x}_{rf} - \boldsymbol{x})^+ \text{ }).$$

Moreover, we have that $\geq_{\mathcal{V}}$ coincides with the usual pointwise order.

Before stating the main result, we need to introduce two new objects $\nu_\star, \nu^\star : \mathcal{B} \to [0,1]$ defined by

$$\nu_\star (A) = \frac{\sup \{\pi (x) : \hat{x} \leq_{\mathcal{V}} 1_A\}}{\pi (x_{rf})} \text{ and } \nu^\star (A) = \frac{\inf \{\pi (x) : \hat{x} \geq_{\mathcal{V}} 1_A\}}{\pi (x_{rf})}$$

where $\hat{x} : \mathcal{V} \to \mathbb{R}$ is such that $\hat{x}(v) = v(x)$ for all $v \in \mathcal{V}$ and $\pi (x_{rf})$ is assumed to be different from zero 0.

Theorem 2 *Let $(M, \mathcal{V}, \pi)$ be a market, with $\pi \neq 0$, that contains the risk free asset and all call options with $k \geq 0$. The following statements are equivalent:*

(i) π is monotone, cash additive, and satisfies the Put–Call Parity;

(ii) there exist a nonadditive probability $\nu : \mathcal{B} \to [0,1]$ and a risk free rate $r > -1$ such that

$$\pi (x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu (v) \qquad \forall x \in M. \tag{11}$$

Moreover,

1. *r is unique.*
2. *Each nonadditive probability $\nu : \mathcal{B} \to [0,1]$ that satisfies $\nu_\star \leq \nu \leq \nu^\star$ represents π as in (11).*
3. *If $B(\mathcal{V}, \mathcal{B}) = \{\hat{x} : x \in M\}$, then ν is unique.*
4. *If ν is balanced, then $\pi (x) \geq -\pi(-x)$ for all $x \in C$, that is, there exists a positive bid-ask spread.*

Notice that Theorem 2 naturally delivers the equivalence of points (ii) and (iii) in Theorem 1. On the other hand, the equivalence between (i) and (ii) in Theorem 1 basically follows from applying Theorem 2 to the price functional p and noticing three facts:

(a) p has a representation as in (11) where $\mathcal{V}$ is the set of rows of the Arrow and Debreu tableau G;

(b) thus, p satisfies monotonicity, translation invariance, and constant modularity with the respect to $\geq_G$;

(c) $\pi = p \circ T$ shares the same properties since T maintains these vector lattice properties.

Finally, we have that points 1. and 3. (resp., 4.) of Theorem 2 deliver point 1. (resp., 2.) of Theorem 1.

3.3 The Subadditive Case

One way in which the literature introduced transaction costs in pricing has been by considering subadditive price functionals and pricing rules (see Jouini and Kallal [26] and Luttmer [31]).

Definition 8 *Let $(M, \mathcal{V}, \pi)$ be a market. The price functional π is said to be subadditive if and only if*

$$\pi(x+y) \leq \pi(x) + \pi(y) \qquad \forall x, y \in M.$$

If π is cash additive, this assumption is particularly important since it implies the existence of a positive bid–ask spread. In fact, we have that $\pi(x) + \pi(-x) \geq \pi(0) = 0$ for all $x \in M$. In this context, the assumption of monotonicity contained in Definition 7 can be weakened to be the following notion of positivity:

Definition 9 *Let $(M, \mathcal{V}, \pi)$ be a market. The price functional π is said to be positive if and only if*

$$0 \geq_{\mathcal{V}} x \quad \Rightarrow \quad 0 \geq \pi(x).$$

Corollary 2 *Let $(M, \mathcal{V}, \pi)$ be a market where $\pi \neq 0$ and that contains the risk free asset and all call options with $k \geq 0$. The following statements are equivalent:*

(i) π is positive, cash additive, subadditive, and satisfies the Put–Call Parity;

(ii) there exist a concave nonadditive probability $\nu : \mathcal{B} \to [0,1]$ and a risk free rate $r > -1$ such that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x) \, d\nu(v) \qquad \forall x \in M.$$

Moreover, r is unique and ν can be chosen to be $\nu^\star$.

Remark 4 *The two above conditions are also equivalent to the following one:*

(i') π is positive, such that $\pi(kx_{rf}) = k\pi(x_{rf})$ for all $k \in \mathbb{R}$, sublinear, and satisfies the Put–Call Parity.

Notice that, in (i'), we weakened cash additivity to a genuine assumption of no frictions on the risk free asset and we strengthened subadditivity to sublinearity.[20]

Our last result allows us to discuss uniqueness of the nonadditive probability ν also when the market does not span the entire space of all contingent claims, that is, $\{\hat{x} : x \in M\} \neq B(\mathcal{V}, \mathcal{B})$. In order to do so, we need to introduce a notion of continuity for the price functional π.[21]

Definition 10 *Let $(M, \mathcal{V}, \pi)$ be a market. The price functional π is said to be continuous if and only if for each sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ we have that*

$$\lim_n v(x_n) = 0 \quad \forall v \in \mathcal{V} \quad \Rightarrow \quad \lim_n \pi(x_n) = 0,$$

provided there exists $l \in \mathbb{R}$ such that $|v(x_n)| \leq l$ for all $v \in \mathcal{V}$ and for all $n \in \mathbb{N}$.

Corollary 3 *Let $(M, \mathcal{V}, \pi)$ be a market where $\pi \neq 0$ and that contains the risk free asset and all call options with $k \geq 0$. The following statements are equivalent:*

(i) π is positive, cash additive, subadditive, continuous, and satisfies the Put–Call Parity;

(ii) there exist a continuous and concave nonadditive probability $\nu : \mathcal{B} \to [0,1]$ and a risk free rate $r > -1$ such that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x) \, d\nu(v) \qquad \forall x \in M.$$

Moreover, r and ν are unique.

[20] Recall that π is sublinear if and only if $\pi(\lambda x + \mu y) \leq \lambda\pi(x) + \mu\pi(y)$ for all $\lambda, \mu \geq 0$ and for all $x, y \in M$.

[21] In reading Definition 10, notice that continuity is required with respect to the bounded weak convergence induced by $\mathcal{V}$.

A Nonlinear Integration

A.1 Choquet Integral

In this paper, Choquet integrals play a fundamental role. Given a measurable space (S,Σ) and a set function $\nu:\Sigma\to\mathbb{R}$ we will say that ν is:

(i) *a nonadditive probability* if $\nu(\emptyset)=0$, $\nu(S)=1$, and $\nu(A)\leq\nu(B)$ provided $A\subseteq B$;

(ii) *concave* if $\nu(A\cup B)+\nu(A\cap B)\leq\nu(A)+\nu(B)$ for all A and B;

(iii) *continuous* if $\lim_{n\to\infty}\nu(A_n)=\nu(A)$ whenever either $A_n\downarrow A$ or $A_n\uparrow A$;

(iv) *a probability* if it is a nonadditive probability such that $\nu(A\cup B)=\nu(A)+\nu(B)$ for all A and B which are pairwise disjoint;

(v) *a probability measure* if it is a probability such that $\lim_n\nu(A_n)=0$ whenever $A_n\downarrow\emptyset$;

(vi) *balanced* if ν is a nonadditive probability and there exists a probability μ such that

$$\mu(B)\leq\nu(B)\qquad\forall B\in\Sigma.$$

We denote by $\Delta(S)$ the set of all probabilities on Σ. Given a nonadditive probability ν, we define

$$core(\nu)=\{\mu\in\Delta(S):\mu\leq\nu\}.$$

It is well known that a concave nonadditive probability is such that

$$\nu(B)=\max_{\mu\in core(\nu)}\mu(B)\qquad\forall B\in\Sigma$$

and a nonadditive probability ν is balanced if and only if $core(\nu)\neq\emptyset$. Given a bounded and Σ-measurable function $f:S\to\mathbb{R}$, the Choquet integral of f with respect to a nonadditive probability ν is defined as the quantity

$$\int_S fd\nu=\int_S f(s)\,d\nu(s)=\int_0^\infty\nu(f>t)\,dt+\int_{-\infty}^0[\nu(f>t)-\nu(S)]\,dt$$

where the integrals on the right hand side are Riemann integrals and $(f>t)=\{s\in S:f(s)>t\}$ for all $t\in\mathbb{R}$. Thus, the Choquet integral defines a functional on the space of bounded, real valued, and Σ-measurable functions: $B(S,\Sigma)$. It is well known that when ν is concave

$$\int_S f(s)\,d\nu(s)=\max_{\mu\in core(\nu)}\int_S f(s)\,d\mu(s).$$

A.2 A Representation Result

Consider a nonempty set S. We define by $B(S)$ the set of all bounded and real valued functions on S. By L we denote a Stone vector lattice contained in $B(S)$, that is, L is a Riesz subspace of $B(S)$ which further contains all constant functions. L is endowed with the pointwise order. Object of our study is a functional $I:L\to\mathbb{R}$. We say that:

(i) I is monotone if and only if $I(f)\geq I(g)$ provided $f\geq g$.

(ii) I is translation invariant if and only if $I(f+k1_S)=I(f)+kI(1_S)$ for all $f\in L$ and all $k\in\mathbb{R}$.

(iii) I is constant modular if and only if $I(f\vee k1_S)+I(f\wedge k1_S)=I(f)+kI(1_S)$ for all $f\in L$ and all $k\in\mathbb{R}$.

(iv) I is subadditive if and only if $I(f+g) \leq I(f) + I(g)$ for all $f, g \in L$.

(v) I is (bounded) pointwise continuous at 0 if and only if $\lim_n I(f_n) = 0$ whenever $\{f_n\}_{n\in\mathbb{N}} \subseteq L$ is uniformly bounded and $f_n \longrightarrow 0$ pointwise.

(vi) I is comonotonic additive if and only if $I(f+g) = I(f) + I(g)$ whenever $f, g \in L$ are such that

$$[f(s) - f(s')][g(s) - g(s')] \geq 0 \qquad \forall s, s' \in S.$$

We denote by $\sigma(L)$ the smallest σ-algebra on S which makes measurable all the functions contained in L. Given a monotone $I : L \to \mathbb{R}$ such that $I(1) = I(1_S) > 0$, we define $\nu_*, \nu^* : \sigma(L) \to [0,1]$ by

$$\nu_*(A) = \frac{\sup\{I(f) : L \ni f \leq 1_A\}}{I(1)} \text{ and } \nu^*(A) = \frac{\inf\{I(f) : L \ni f \geq 1_A\}}{I(1)} \qquad \forall A \in \sigma(L).$$

Theorem 3 *Let L be a Stone vector lattice and $I : L \to \mathbb{R}$ such that $I \neq 0$. The following statements are equivalent:*

(i) I is monotone, translation invariant, and constant modular;

(ii) there exists a nonadditive probability $\nu : \sigma(L) \to [0,1]$ and a number $\delta \in (0,\infty)$ such that

$$I(f) = \delta\left[\int_0^{\infty} \nu(f > t)\, dt + \int_{-\infty}^{0} [\nu(f > t) - \nu(S)]\, dt\right] \qquad \forall f \in L. \tag{12}$$

Moreover,

1. *δ is unique.*
2. *Each nonadditive probability $\nu : \sigma(L) \to [0,1]$ that satisfies $\nu_* \leq \nu \leq \nu^*$ represents I as in (12).*
3. *I is subadditive if and only if ν^* is concave.*
4. *I is subadditive and pointwise continuous at 0 if and only if there exists a concave and continuous nonadditive probability ν satisfying (12).*
5. *If ν is continuous and concave, then ν is unique among the nonadditive probabilities satisfying (12) and with such properties.*

Proof. (i) implies (ii). Define $\bar{I} : L \to \mathbb{R}$ by $\bar{I}(f) = I(f)/I(1)$ for all $f \in L$. Since $I(1) > 0$, it is immediate to see that $\bar{I}$ is monotone, translation invariant, and constant modular. By Greco [21], it follows that there exists a nonadditive probability $\nu : \sigma(L) \to [0,1]$ such that

$$\bar{I}(f) = \int_0^{\infty} \nu(f > t)\, dt + \int_{-\infty}^{0} [\nu(f > t) - \nu(S)]\, dt \qquad \forall f \in L. \tag{13}$$

Moreover, each nonadditive probability $\hat{\nu} : \sigma(L) \to [0,1]$ that satisfies

$$\nu_*(A) = \sup\left\{\bar{I}(f) : L \ni f \leq 1_A\right\} \leq \hat{\nu}(A) \leq \inf\left\{\bar{I}(f) : L \ni f \geq 1_A\right\} = \nu^*(A) \qquad \forall A \in \sigma(L),$$

represents $\bar{I}$ as in (13). If we define $\delta = I(1)$, then we have that $I = \delta \bar{I}$. Given (13), the statement follows.

(ii) implies (i). Define $\bar{I} : L \to \mathbb{R}$ by

$$\bar{I}(f) = \int_0^{\infty} \nu(f > t)\, dt + \int_{-\infty}^{0} [\nu(f > t) - \nu(S)]\, dt \qquad \forall f \in L.$$

By [41] and [32, Proposition 4.8] and since $I = \delta \bar{I}$, it is immediate to see that I is monotone and comonotonic additive. By [32, Proposition 4.11], it follows that I is translation invariant. Consider $f \in L$ and $k \in \mathbb{R}$. By [32, Lemma 4.6], $f \wedge k1_S$ and $f \vee k1_S$ are comonotonic. Since I is comonotonic additive and translation invariant, it follows that I is constant modular.

1. By (12), we have that $I(1_S) = \delta$ delivering the uniqueness of δ.

2. The statement follows from Greco [21].

3. Assume that I is further subadditive. Define $I^* : B(\sigma(L)) \to \mathbb{R}$ by $I^*(f) = \inf \{I(g) : L \ni g \geq f\}$ for all $f \in L$. Define $J : B(\sigma(L)) \to \mathbb{R}$ by $J(f) = \delta \int_S f d\nu^*$ for all $f \in B(\sigma(L))$. It is routine to prove that I^* and J are monotone, translation invariant, constant modular, and such that $I^*(f) = I(f) = J(f)$ for all $f \in L$. By the main statement, it follows that there exists $\eta : \sigma(L) \to [0,1]$ and $\delta' > 0$ such that

$$I^*(f) = \delta' \left[\int_0^\infty \eta(f > t) \, dt + \int_{-\infty}^0 [\eta(f > t) - \eta(S)] \, dt \right] \qquad \forall f \in B(\sigma(L)).$$

By [32, Theorem 4.6] and since I is subadditive, I^* is subadditive implying that η is concave. It is immediate to see that $\delta = \delta'$. By construction, we have that $I^* \geq J$. These latter two facts prove that $\eta \geq \nu^*$. On the other hand, by point 2., we have that $\eta \leq \nu^*$. We can conclude that $\nu^* = \eta$. The opposite implication follows from [32, Theorem 4.6].

4. and 5. Both points follow from [6, Theorem 22]. ■

B Proofs

Before proving the main statements of our paper, we first introduce some piece of notation. We will define $\langle \cdot, \cdot \rangle : \mathcal{V} \times M \to \mathbb{R}$ by $\langle v, x \rangle = v(x)$ for all $v \in \mathcal{V}$ and for all $x \in M$. Given an element $x \in M$, we denote by $\hat{x}$ the function from $\mathcal{V}$ to $\mathbb{R}$ such that $\hat{x}(v) = \langle v, x \rangle$ for all $v \in \mathcal{V}$. Since $\mathcal{V}$ satisfies Condition 2., we have that

$$L = \left\{ f \in \mathbb{R}^{\mathcal{V}} : f = \hat{x} \text{ for some } x \in M \right\} \subseteq B(\mathcal{V})$$

where the latter is the set of all real valued bounded functions defined over $\mathcal{V}$. Given a market $(M, \mathcal{V}, \pi)$, we study the ordered space $(M, \geq_{\mathcal{V}})$. In such a context, given two elements x and y in M, we define, if they exist,

$$x \wedge y = \inf \{x, y\} \text{ and } x \vee y = \sup \{x, y\}.$$

Proposition 4 *Let $(M, \mathcal{V}, \pi)$ be a market. The following statements are true:*

1. *If M contains the risk free asset x_{rf}, then x_{rf} is an order unit for $(M, \geq_{\mathcal{V}})$.*

2. *If M contains the risk free asset x_{rf} and all call options with $k \geq 0$, then $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit. In particular, this implies that*

$$c_{x,k} + kx_{rf} = x \vee kx_{rf} \text{ and } c_{x,k} = (x - kx_{rf}) \vee 0 \qquad \forall x \in M, \forall k \in \mathbb{R}$$
$$\text{and}$$
$$kx_{rf} - p_{x,k} = x \wedge kx_{rf} \text{ and } p_{x,k} = (kx_{rf} - x) \vee 0 \qquad \forall x \in M, \forall k \in \mathbb{R}.$$

3. *If M contains the risk free asset x_{rf} and all put options with $k \geq 0$, then $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit.*

4. *If M contains the risk free asset x_{rf} and all call (resp., put) options with $k \geq 0$, then L is a Stone vector lattice.*

5. *If M contains the risk free asset x_{rf} and all call (resp., put) options with $k \geq 0$, then the map $T : M \to L$, defined by*

$$x \mapsto \hat{x},$$

is a lattice isomorphism.

Proof. We proceed by Steps.

Step 1. $\geq_{\mathcal{V}}$ is a partial order relation.

Proof of the Step.

First, notice that for each $x \in M$

$$\langle v, x\rangle \geq \langle v, x\rangle \qquad \forall v \in \mathcal{V},$$

that is, $x \geq_{\mathcal{V}} x$, thus $\geq_{\mathcal{V}}$ is reflexive. Similarly, consider $x, y, z \in M$ and assume that $x \geq_{\mathcal{V}} y$ and $y \geq_{\mathcal{V}} z$. It follows that

$$\langle v, x\rangle \geq \langle v, y\rangle \geq \langle v, z\rangle \qquad \forall v \in \mathcal{V},$$

that is, $x \geq_{\mathcal{V}} z$, thus, $\geq_{\mathcal{V}}$ satisfies transitivity. Finally, since $\mathcal{V}$ satisfies Condition 3., it follows that $\geq_{\mathcal{V}}$ is antisymmetric and so it is a partial order. □

Step 2. $\geq_{\mathcal{V}}$ is such that for each $x, y \in M$

$$\begin{aligned} &x \geq_{\mathcal{V}} y \quad \Rightarrow \quad x + z \geq_{\mathcal{V}} y + z \qquad \forall z \in M \\ &\quad \textit{and} \\ &x \geq_{\mathcal{V}} y \quad \Rightarrow \quad \lambda x \geq_{\mathcal{V}} \lambda y \qquad \forall \lambda \geq 0. \end{aligned}$$

Proof of the Step.

Consider $x, y, z \in M$ and assume that $x \geq_{\mathcal{V}} y$. It follows that for each $v \in \mathcal{V}$

$$\langle v, x\rangle \geq \langle v, y\rangle \quad \Rightarrow \quad \langle v, x\rangle + \langle v, z\rangle \geq \langle v, y\rangle + \langle v, z\rangle \quad \Rightarrow \quad \langle v, x + z\rangle \geq \langle v, y + z\rangle\,,$$

that is, $x + z \geq_{\mathcal{V}} y + z$. Similarly, consider $x, y \in M$, $\lambda \geq 0$, and assume that $x \geq_{\mathcal{V}} y$. It follows that

$$\langle v, x\rangle \geq \langle v, y\rangle \qquad \forall v \in \mathcal{V} \quad \Rightarrow \quad \lambda \langle v, x\rangle \geq \lambda \langle v, y\rangle \qquad \forall v \in \mathcal{V} \Rightarrow \quad \langle v, \lambda z\rangle \geq \langle v, \lambda y\rangle \qquad \forall v \in \mathcal{V},$$

that is, $\lambda x \geq_{\mathcal{V}} \lambda y$, proving the statement. □

Step 3. If M contains the risk free asset x_{rf}, then x_{rf} is an order unit for $(M, \geq_{\mathcal{V}})$.

Proof of the Step.

Consider $x \in M$. Since $\mathcal{V}$ satisfies Condition 2., we have that there exists $\lambda \in \mathbb{R}$ such that $\sup_{v \in \mathcal{V}} |\langle v, x\rangle| \leq \lambda < \infty$. If x_{rf} is the risk free asset, then this implies that

$$\langle v, \lambda x_{rf}\rangle \geq \langle v, x\rangle \geq \langle v, -\lambda x_{rf}\rangle \qquad \forall v \in \mathcal{V},$$

that is, $\lambda x_{rf} \geq_{\mathcal{V}} x \geq_{\mathcal{V}} -\lambda x_{rf}$ proving that x_{rf} is an order unit. □

Step 4. If M contains the risk free asset x_{rf} and all call options with $k \geq 0$, then $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit. In particular, this implies that

$$\begin{aligned} &c_{x,k} + kx_{rf} = x \vee kx_{rf} \ \textit{and} \ c_{x,k} = (x - kx_{rf}) \vee 0 \qquad \forall x \in M, \forall k \in \mathbb{R} \\ &\qquad\qquad \textit{and} \\ &kx_{rf} - p_{x,k} = x \wedge kx_{rf} \ \textit{and} \ p_{x,k} = (kx_{rf} - x) \vee 0 \qquad \forall x \in M, \forall k \in \mathbb{R}. \end{aligned}$$

Proof of the Step.

Consider $x, y \in M$ and $c_{x-y,0}$. Define $z = x - c_{x-y,0}$. We have that

$$\hat{z}(v) = \langle v, z\rangle = \langle v, x\rangle - \langle v, c_{x-y,0}\rangle = \langle v, x\rangle - (\langle v, x-y\rangle)^{+} = \langle v, x\rangle - (\langle v, x\rangle - \langle v, y\rangle)^{+}$$
$$= \hat{x}(v) - (\hat{x}(v) - \hat{y}(v))^{+} = (\hat{x} \wedge \hat{y})(v) \leq \hat{x}(v), \hat{y}(v) \qquad \forall v \in \mathcal{V}.$$

The first equality follows by definition of z and the linearity of each v in $\mathcal{V}$. The second equality follows by the definition of $c_{x-y,0}$. The third equality follows by the linearity of each v in $\mathcal{V}$. The fourth equality follows by definition of $\hat{x}$ and $\hat{y}$. The fifth equality follows from a well known lattice equality (see [3, Theorem 1.7]).

We can conclude that $x, y \geq_{\mathcal{V}} z$. Next, assume that $w \in M$ is such that $x, y \geq_{\mathcal{V}} w$. By definition of $\geq_{\mathcal{V}}$ and the previous part, it follows that

$$\hat{x}(v), \hat{y}(v) \geq \hat{w}(v) \qquad \forall v \in \mathcal{V} \quad \Rightarrow \quad \langle v, z\rangle = (\hat{x} \wedge \hat{y})(v) \geq \hat{w}(v) = \langle v, w\rangle \qquad \forall v \in \mathcal{V}.$$

This implies that $z \geq_{\mathcal{V}} w$, that is, z is the greatest lower bound for x and y and $z = x \wedge y$. By Steps 1, 2, and 3, $(M, \geq_{\mathcal{V}})$ is an ordered vector space with unit. This fact matched with the previous one implies that $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit. Next, consider $x \in M$ and $k \in \mathbb{R}$. It is immediate to check that $c_{x,k} = c_{x-kx_{rf},0}$. It follows that

$$c_{x,k} + kx_{rf} = c_{x-kx_{rf},0} + kx_{rf} = (x - x \wedge kx_{rf}) + kx_{rf}$$
$$= x + kx_{rf} - x \wedge kx_{rf} = x \vee kx_{rf}.$$

where the first equality is trivial, the second one follows from the previous part of the proof, that is $c_{x-y} = x - x \wedge y$, the third follows from a simple rearrangement, and the fourth one is a well known lattice equality (see [2, Theorem 8.6]). On the other hand, from above we have that $c_{x,k} = x \vee kx_{rf} - kx_{rf} = (x - kx_{rf}) \vee 0$. A similar argument delivers the equalities

$$kx_{rf} - p_{x,k} = x \wedge kx_{rf} \text{ and } p_{x,k} = (kx_{rf} - x) \vee 0 \qquad \forall x \in M, \forall k \in \mathbb{R}.$$

□

Step 5. If M contains the risk free asset x_{rf} and all put options with $k \geq 0$, then $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit.

Proof of the Step.

Consider $x, y \in M$ and $p_{y-x,0}$. Define $z = x - p_{y-x,0}$. We have that

$$\hat{z}(v) = \langle v, z\rangle = \langle v, x\rangle - \langle v, p_{y-x,0}\rangle = \langle v, x\rangle - (-\langle v, y-x\rangle)^{+} = \langle v, x\rangle - (\langle v, x\rangle - \langle v, y\rangle)^{+}$$
$$= \hat{x}(v) - (\hat{x}(v) - \hat{y}(v))^{+} = (\hat{x} \wedge \hat{y})(v) \leq \hat{x}(v), \hat{y}(v) \qquad \forall v \in \mathcal{V}.$$

By the same arguments contained in the proof of Step 4, it follows that $(M, \geq_{\mathcal{V}})$ is a Riesz space with unit. □

Step 6. The map $T : M \to L$, defined by

$$x \mapsto \hat{x},$$

is a bijective linear operator. In particular, L is a vector space and if M contains the risk free asset x_{rf}, then L contains the constant functions.

Proof of the Step.

By construction, the map T is surjective. On the other hand, if we have that $T(x_1) = T(x_2)$, then it follows that

$$\langle v, x_1\rangle = \hat{x}_1(v) = \hat{x}_2(v) = \langle v, x_2\rangle \qquad \forall v \in \mathcal{V}. \tag{14}$$

Since $\mathcal{V}$ satisfies Condition 3., (14) implies that $x_1 = x_2$, proving that T is injective. Next, given $x, y \in M$ and $\lambda, \mu \in \mathbb{R}$, we have that

$$T(\lambda x + \mu y)(v) = \langle v, \lambda x + \mu y \rangle = \lambda \langle v, x \rangle + \mu \langle v, y \rangle = \lambda T(x)(v) + \mu T(y)(v) \qquad \forall v \in \mathcal{V},$$

that is, $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$, proving that T is linear. Since M is a vector space and T is linear and bijective, L is a vector space. At the same time, if x_{rf} is the risk free asset and $x_{rf} \in M$, then $1_{\mathcal{V}} = T(x_{rf}) \in L$. Since L is a vector space, it follows that L contains all the constant functions. □

Step 7. If M contains the risk free asset x_{rf} and all call (resp., put) options with $k \geq 0$, then L is a Stone vector lattice.

Proof of the Step.

By Step 6, we have that L is a vector space which contains all the constant functions. We are left to prove it is closed under finite pointwise suprema or, equivalently, infima. Let us consider two elements $f, g \in L$. By definition, there exist $x, y \in M$ such that $f = T(x)$ and $g = T(y)$. By the proof of Step 4 (resp., Step 5), we have that $z = x - c_{x-y,0}$ (resp., $z = x - p_{y-x,0}$) is such that

$$f \wedge g = T(x) \wedge T(y) = \hat{x} \wedge \hat{y} = \hat{z} = T(z) \in L,$$

proving the statement. □

Step 8. If M satisfies the hypotheses of 5., then the map T is a lattice isomorphism.

Proof of the Step.

By Step 6, we have that T is a bijective linear operator. We are left to show that T preserves the lattice operations. Consider $x, y \in M$. By the proof of Step 4, we have that $x \wedge y = x - c_{x-y,0} \in M$ and

$$T(x \wedge y) = T(x) \wedge T(y),$$

proving the statement. □

Step 3 proves Statement 1. Step 4 proves Statement 2. Step 5 proves Statement 3. Step 7 proves Statement 4. Step 8 proves Statement 5. ■

Proposition 5 *Let $(M, \mathcal{V}, \pi)$ be a market that contains the risk free asset. The following statements are equivalent:*

(i) M contains all call options;

(ii) M contains all call options with $k \geq 0$;

(iii) M contains all put options with $k \geq 0$;

(iv) M contains all put options.

Proof. (i) implies (ii). It is trivial.

(ii) implies (iv). Consider $x \in M$ and $k \in \mathbb{R}$. Since M contains the risk free asset and all call options with $k \geq 0$, it follows that $c_{kx_{rf}-x,0} \in M$. This implies that

$$\langle v, c_{kx_{rf}-x,0} \rangle = (\langle v, kx_{rf} - x \rangle - 0)^+ = (k - \langle v, x \rangle)^+ \qquad \forall v \in \mathcal{V},$$

that is, $p_{x,k} = c_{kx_{rf}-x,0} \in M$.

(iv) implies (iii). It is trivial.

(iii) implies (i). Consider $x \in M$ and $k \in \mathbb{R}$. Since M contains the risk free asset and all put options with $k \geq 0$, it follows that $p_{kx_{rf}-x,0} \in M$. This implies that

$$\left\langle v, p_{kx_{rf}-x,0} \right\rangle = (0 - \langle v, kx_{rf} - x \rangle)^+ = (\langle v, x \rangle - k)^+ \qquad \forall v \in \mathcal{V},$$

that is, $c_{x,k} = p_{kx_{rf}-x,0} \in M$. ■

Proof of Proposition 2. Consider $x \in M$ and $k \in \mathbb{R}$. Assume that $c_{x,k}$ and $p_{x,k}$ belong to M. Consider $z = c_{x,k} - p_{x,k} \in M$. Since $x_{rf} \in M$, it follows that

$$\begin{aligned} v(z) &= v(c_{x,k}) - v(p_{x,k}) = (\langle v, x \rangle - k)^+ - (k - \langle v, x \rangle)^+ \\ &= (\langle v, x - kx_{rf} \rangle)^+ - (\langle v, kx_{rf} - x \rangle)^+ \\ &= \max\{\langle v, x - kx_{rf} \rangle, 0\} - \max\{\langle v, kx_{rf} - x \rangle, 0\} \\ &= \langle v, x - kx_{rf} \rangle \qquad \forall v \in \mathcal{V}. \end{aligned}$$

Since $\mathcal{V}$ satisfies Condition 3., we have that $c_{x,k} - p_{x,k} = z = x - kx_{rf}$, proving the statement. ■

Proof of Proposition 3. It is an immediate consequence of Proposition 5. ■

Proposition 6 *Let $(M, \mathcal{V}, \pi)$ be a market that contains the risk free asset and all call options with $k \geq 0$ and let $\pi : M \to \mathbb{R}$ be a cash additive price functional. The following conditions are equivalent:*

(i) $\pi(c_{x,k}) + \pi(-p_{x,k}) = \pi(x) - k\pi(x_{rf})$ *for all $x \in M$ and all $k \in \mathbb{R}_+$;*

(ii) $\pi(x \vee kx_{rf}) + \pi(x \wedge kx_{rf}) = \pi(x) + k\pi(x_{rf})$ *for all $x \in M$ and all $k \in \mathbb{R}_+$;*

(iii) $\pi(x \vee kx_{rf}) + \pi(x \wedge kx_{rf}) = \pi(x) + k\pi(x_{rf})$ *for all $x \in M$ and all $k \in \mathbb{R}$.*

Proof. (i) implies (iii). Consider $x \in M$ and $k \in \mathbb{R}$. By Proposition 4 point 2., we have that

$$c_{x,k} + kx_{rf} = x \vee kx_{rf} \text{ and } kx_{rf} - p_{x,k} = x \wedge kx_{rf}.$$

Since π is cash additive, $c_{x,k} = c_{x-kx_{rf},0}$, and $p_{x,k} = p_{x-kx_{rf},0}$, we have the following chain of implications

$$\begin{aligned} \pi\left(c_{x-kx_{rf},0}\right) + \pi\left(-p_{x-kx_{rf},0}\right) &= \pi(x - kx_{rf}) \\ &\Longrightarrow \\ \pi(c_{x,k}) + \pi(-p_{x,k}) &= \pi(x) - k\pi(x_{rf}) \\ &\Longrightarrow \\ \pi(c_{x,k}) + k\pi(x_{rf}) + \pi(-p_{x,k}) + k\pi(x_{rf}) &= \pi(x) + k\pi(x_{rf}) \\ &\Longrightarrow \\ \pi(c_{x,k} + kx_{rf}) + \pi(kx_{rf} - p_{x,k}) &= \pi(x) + k\pi(x_{rf}) \\ &\Longrightarrow \\ \pi(x \vee kx_{rf}) + \pi(x \wedge kx_{rf}) &= \pi(x) + k\pi(x_{rf}), \end{aligned}$$

proving the statement.

(iii) implies (ii). It is trivial.

(ii) implies (i). Consider $x \in M$ and $k \in \mathbb{R}_+$. By Proposition 4 point 2., M is a Riesz space with unit. This implies that

$$x \vee kx_{rf} - kx_{rf} = (x - kx_{rf}) \vee 0 = c_{x,k} \text{ and } x \wedge kx_{rf} - kx_{rf} = (x - kx_{rf}) \wedge 0 = -p_{x,k}.$$

Since π is cash additive, we have the following chain of implications

$$\begin{aligned}
\pi(x \vee kx_{rf}) + \pi(x \wedge kx_{rf}) &= \pi(x) + k\pi(x_{rf}) \\
&\Longrightarrow \\
\pi(x \vee kx_{rf}) - k\pi(x_{rf}) + \pi(x \wedge kx_{rf}) - k\pi(x_{rf}) &= \pi(x) - k\pi(x_{rf}) \\
&\Longrightarrow \\
\pi(x \vee kx_{rf} - kx_{rf}) + \pi(x \wedge kx_{rf} - kx_{rf}) &= \pi(x) - k\pi(x_{rf}) \\
&\Longrightarrow \\
\pi(c_{x,k}) + \pi(-p_{x,k}) &= \pi(x) - k\pi(x_{rf}),
\end{aligned}$$

proving the statement. ■

Proof of Theorem 2. Before starting, observe that $\sigma(L) = \mathcal{B}$.

(i) implies (ii). Define $I : L \to \mathbb{R}$ by $I = \pi \circ T^{-1}$. Since T is a lattice isomorphism such that $T(x_{rf}) = 1_{\mathcal{V}}$ and $\pi \neq 0$, it follows that I is well defined, monotone, translation invariant, and such that $I \neq 0$. Since π satisfies the PCP and by Proposition 6, we have that

$$\pi(x \wedge kx_{rf}) + \pi(x \vee kx_{rf}) = \pi(x) + k\pi(x_{rf}) \qquad \forall x \in M, \forall k \in \mathbb{R}. \tag{15}$$

Since T is a lattice isomorphism and $T(x_{rf}) = 1_{\mathcal{V}}$, this implies that I is constant modular. Since L is a Stone vector lattice and by Theorem 3, we have that there exist $\delta > 0$ and a nonadditive probability $\nu : \mathcal{B} \to [0,1]$ such that

$$I(f) = \delta \int_{\mathcal{V}} f(v)\, d\nu(v) \qquad \forall f \in L.$$

Define $r = 1/\delta - 1 > -1$. Since $\pi = I \circ T$, we have that

$$\pi(x) = I(T(x)) = \frac{1}{1+r} \int_{\mathcal{V}} \langle v, x \rangle\, d\nu(v) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M,$$

proving the statement.

(ii) implies (i). Consider $r > -1$ and a nonadditive probability $\nu : \mathcal{B} \to [0,1]$ such that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M.$$

Define $I : L \to \mathbb{R}$ by

$$I(f) = \delta \int_{\mathcal{V}} f(v)\, d\nu(v) \qquad \forall f \in L \tag{16}$$

where $\delta = 1/(1+r)$. By Theorem 3, we have that I is monotone, translation invariant, and constant modular. It is immediate to see that $\pi = I \circ T$. Since T is a lattice isomorphism and $T(x_{rf}) = 1_{\mathcal{V}}$, it follows that π is monotone, cash additive, and it satisfies (15). By Proposition 6, it follows that π also satisfies the PCP.

1. Consider $r_1, r_2 > -1$ and $\nu_1, \nu_2 : \mathcal{B} \to [0,1]$ such that

$$\pi(x) = \frac{1}{1+r_i} \int_{\mathcal{V}} v(x)\, d\nu_i(v) \qquad \forall x \in M, \forall i \in \{1,2\}.$$

It follows that $\frac{1}{1+r_1} = \pi(x_{rf}) = \frac{1}{1+r_2}$, proving that $r_1 = r_2$ and so the statement.

2. Consider a nonadditive probability $\nu : \mathcal{B} \to [0,1]$ such that $\nu_\star \leq \nu \leq \nu^\star$. Define $I = \pi \circ T^{-1}$. It follows that $\nu_\star = \nu_*$ and $\nu^\star = \nu^*$. This implies that $\nu_* \leq \nu \leq \nu^*$. By Theorem 3, we have that $I(f) = I(1_{\mathcal{V}}) \int_{\mathcal{V}} f d\nu$ for all $f \in L$. Since $\pi = I \circ T$, the statement follows.

3. Assume that $B(\mathcal{V},\mathcal{B}) = \{\hat{x} : x \in M\}$. It follows that for each $A \in \mathcal{B}$ there exists $x_A \in M$ such that $v(x) = 1_A(v)$ for all $v \in \mathcal{V}$. Consider (r_1, ν_1) and (r_2, ν_2) representing π as in (11). By point 1., we have that $r_1 = r_2$ and

$$\frac{1}{1+r_1}\nu_1(A) = \pi(x_A) = \frac{1}{1+r_2}\nu_2(A) \qquad \forall A \in \mathcal{B}.$$

This implies that $\nu_1 = \nu_2$, proving the statement.

4. Consider (r, ν) representing π as in (11). Define $\bar{\nu} : \mathcal{B} \to [0,1]$ by

$$\bar{\nu}(A) = \nu(\mathcal{V}) - \nu(A^c) \qquad \forall A \in \mathcal{B}.$$

Assume that ν is balanced. It follows that there exists $\mu \in \Delta(\mathcal{V})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. This implies that $\bar{\nu}(A) \leq \mu(A)$ for all $A \in \mathcal{B}$. By [32, Proposition 4.12], if we define I as in (16), then

$$-I(-f) = \frac{1}{1+r}\int_{\mathcal{V}} f(v)\, d\bar{\nu}(v) \qquad \forall f \in L.$$

By the definition of Choquet integral and since $\bar{\nu} \leq \mu \leq \nu$, it follows that $-I(-f) \leq \frac{1}{1+r}\int_{\mathcal{V}} f(v)\, d\mu(v) \leq I(f)$ for all $f \in L$. Since $\pi = I \circ T$ and T is linear, we have that

$$-\pi(-x) = -I(-T(x)) \leq I(T(x)) = \pi(x) \qquad \forall x \in M,$$

proving the statement. ■

Proof of Corollary 2. Before starting the proof, we introduce a third point:

(iii) π is monotone, cash additive, subadditive, and satisfies the Put–Call Parity.

(i) implies (iii). Consider $x, y \in M$ such that $x \geq_{\mathcal{V}} y$. Define $z = y - x$. Notice that $0 \geq_{\mathcal{V}} z$ and $x + z = y$. Since π is subadditive and π is positive, we have that

$$\pi(x) \geq \pi(x) + \pi(z) \geq \pi(x+z) = \pi(y),$$

proving that π is also monotone and so the statement.

(iii) implies (ii). Define $I : L \to \mathbb{R}$ by $I = \pi \circ T^{-1}$. By the same argument contained in the proof of Theorem 2, we have that I is well defined, monotone, translation invariant, constant modular, and such that $I \neq 0$. Since π is subadditive and T is a lattice isomorphism, we have that I is also subadditive. Since L is a Stone vector lattice and by Theorem 3, we have that there exist $\delta > 0$ such that

$$I(f) = \delta \int_{\mathcal{V}} f(v)\, d\nu^*(v) \qquad \forall f \in L.$$

Since I is subadditive and by Theorem 3, we have that $\nu^* = \nu^\star$ is concave. Define $r = 1/\delta - 1$. Since $\pi = I \circ T$, we have that

$$\pi(x) = \frac{1}{1+r}\int_{\mathcal{V}} \langle v, x\rangle\, d\nu^*(v) = \frac{1}{1+r}\int_{\mathcal{V}} v(x)\, d\nu^\star(v) \qquad \forall x \in M,$$

proving the statement.

(ii) implies (i). Consider $r > -1$ and a concave nonadditive probability $\nu : \mathcal{B} \to [0,1]$ such that

$$\pi(x) = \frac{1}{1+r}\int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M.$$

Define $I : L \to \mathbb{R}$ as in (16) where $\delta = 1/(1+r)$. By Theorem 3, we have that I is monotone, translation invariant, and constant modular. Since ν is concave and by [32, Theorem 4.6], I is also subadditive. It is immediate to see that $\pi = I \circ T$. Since T is a lattice isomorphism and $T(x_{rf}) = 1_{\mathcal{V}}$, it follows that π is monotone, cash additive, subadditive, and it satisfies (15). By Proposition 6, it follows that π also satisfies the PCP.

Uniqueness of $r > -1$ follows from Theorem 2 and the fact that ν can be chosen to be $\nu^\star$ follows from the previous part of the proof. ■

Proof of Corollary 3. Before starting the proof, we introduce a third point:

(iii) π is monotone, cash additive, subadditive, continuous, and satisfies the Put–Call Parity.

(i) implies (iii). The statement follows by the same argument used in the proof of Corollary 2.

(iii) implies (ii). Define $I : L \to \mathbb{R}$ by $I = \pi \circ T^{-1}$. By the same argument contained in the proof of Corollary 2, we have that I is well defined, monotone, translation invariant, constant modular, subadditive, and such that $I \neq 0$. Since π is continuous and T is a lattice isomorphism, we have that I is also pointwise continuous at 0. Since L is a Stone vector lattice and by Theorem 3, we have that there exist a continuous and concave nonadditive probability $\nu : \mathcal{B} \to [0, 1]$ and $\delta > 0$ such that

$$I(f) = \delta \int_{\mathcal{V}} f(v)\, d\nu(v) \qquad \forall f \in L.$$

Define $r = 1/\delta - 1$. Since $\pi = I \circ T$, we have that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} \langle v, x \rangle\, d\nu(v) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M,$$

proving the statement.

(ii) implies (i). Consider $r > -1$ and a continuous and concave nonadditive probability $\nu : \mathcal{B} \to [0, 1]$ such that

$$\pi(x) = \frac{1}{1+r} \int_{\mathcal{V}} v(x)\, d\nu(v) \qquad \forall x \in M.$$

Define $I : L \to \mathbb{R}$ as in (16) where $\delta = 1/(1+r)$. By Theorem 3 and [32, Theorem 4.6], we have that I is monotone, translation invariant, constant modular, subadditive, and pointwise continuous at 0. It is immediate to see that $\pi = I \circ T$. Since T is a lattice isomorphism and $T(x_{rf}) = 1_{\mathcal{V}}$, it follows that π is monotone, cash additive, subadditive, continuous, and it satisfies (15). By Proposition 6, it follows that π also satisfies the PCP.

Uniqueness of $r > -1$ and ν follow from Theorem 3. ■

Proof of Theorem 1. We use the notation of Examples 1 and 3. Notice that $(P, \geq_G, p)$ is a market according to Definition 1. Since $C = \mathbb{R}^m$, we also have that it contains the risk free asset and all call options with nonnegative strike price. In turn, this delivers that $(P, \geq_G)$ is a Riesz space with unit $\boldsymbol{\eta}_{rf}$. Notice that in this case L is isomorphic to C and so $\mathcal{B}$ can be considered to be the power set of Ω.

(i) implies (ii). $p : P \to \mathbb{R}$ is assumed to satisfy nf and na. The first assumption amounts to impose that

$$p\left(\boldsymbol{\eta} + k\boldsymbol{\eta}_{rf}\right) = p(\boldsymbol{\eta}) + kp\left(\boldsymbol{\eta}_{rf}\right) \qquad \forall \boldsymbol{\eta} \in P, \forall k \in \mathbb{R}$$

and

$$p\left(\boldsymbol{c}_{\eta,k}\right) + p\left(-\boldsymbol{p}_{\eta,k}\right) = p(\boldsymbol{\eta}) - kp\left(\boldsymbol{\eta}_{rf}\right) \qquad \forall \boldsymbol{\eta} \in P, \forall k \in \mathbb{R}_+.$$

The second assumption means that $\boldsymbol{\eta}_1 \geq_G \boldsymbol{\eta}_2$ implies $p(\boldsymbol{\eta}_1) \geq p(\boldsymbol{\eta}_2)$. Thus, p is monotone, cash additive, and it satisfies the PCP as in Definition 5. By Proposition 6, we have that p is monotone, cash additive, and constant modular. Next, recall that $\tilde{\pi} : C \to \mathbb{R}$ is defined by $\tilde{\pi} = p \circ T^{-1}$ where $T(\boldsymbol{\eta}) = G\boldsymbol{\eta}$. Since $p \neq 0$, we have that $\tilde{\pi} \neq 0$. Since it is immediate to see that T is the operator of Proposition 4, we have that

T is a lattice isomorphism. This implies that $\tilde{\pi}$ is monotone, cash additive/translation invariant, constant modular, and such that $\tilde{\pi} \neq 0$.

(ii) implies (iii). Since $C = \mathbb{R}^m$, $\tilde{\pi}$ is monotone, translation invariant, constant modular, and such that $\tilde{\pi} \neq 0$ and by Theorem 3, we have that there exists a nonadditive probability on Ω and $\delta > 0$ such that

$$\tilde{\pi}(\boldsymbol{x}) = \delta \int_{\Omega} \boldsymbol{x} d\nu = \frac{1}{1+r} \int_{\Omega} \boldsymbol{x} d\nu \qquad \forall \boldsymbol{x} \in C \tag{17}$$

where $r = \frac{1}{\delta} - 1$.

(iii) implies (i). Assume $\tilde{\pi}$ is represented as in (17). In light of Theorem 3, we have that $\tilde{\pi}$ is monotone, cash additive/translation invariant and constant modular. Recall that p is such that $p = \tilde{\pi} \circ T$. Since T is a lattice isomorphism, we have that p satisfies the same properties. By Proposition 6, it follows that p satisfies nf and na.

1. Uniqueness of r and ν follow from points 1. and 3. of Theorem 2.
2. It follows from point 4. of Theorem 2. ■

Proof of Corollary 1. In light of the proof of Theorem 1 and Corollary 2, the proof is immediate.

Proof of Proposition 1. As already observed, $(P, \geq_G, p)$ is a market according to Definition 1. Since $C = \mathbb{R}^m$, we also have that it contains the risk free asset and all call options with nonnegative strike price. In turn, this delivers that $(P, \geq_G)$ is a Riesz space with unit $\boldsymbol{\eta}_{rf}$. Moreover, in light of these facts, it is immediate to check that $\boldsymbol{c}_{\eta,k} = (\boldsymbol{\eta} - k\boldsymbol{\eta}_{rf}) \vee 0$, $\boldsymbol{p}_{\eta,k} = (k\boldsymbol{\eta}_{rf} - \boldsymbol{\eta}) \vee 0$, and $\boldsymbol{d}_{\eta,k} = \boldsymbol{\eta} \wedge k\boldsymbol{\eta}_{rf}$ for all $\boldsymbol{\eta} \in P$ and $k \geq 0$ where suprema and infima are with respect to $\geq_G$.

(i) implies (ii). We just need to show that nf and na imply condition (4). Consider $\boldsymbol{\eta} \in P$ and $k \geq 0$. Since p satisfies nf, we have that

$$\begin{aligned}
p(\boldsymbol{c}_{\eta,k}) + p(-\boldsymbol{p}_{\eta,k}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k} - k\boldsymbol{\eta}_{rf}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k}) - kp(\boldsymbol{\eta}_{rf}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k}) &= p(\boldsymbol{\eta}),
\end{aligned}$$

proving the implication.

(ii) implies (i). We just need to show that nf' and na imply the PCP. Consider $\boldsymbol{\eta} \in P$ and $k \geq 0$. Since p satisfies nf', we have that

$$\begin{aligned}
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k}) &= p(\boldsymbol{\eta}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k}) - kp(\boldsymbol{\eta}_{rf}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(\boldsymbol{d}_{\eta,k} - k\boldsymbol{\eta}_{rf}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}) \\
&\Longrightarrow \\
p(\boldsymbol{c}_{\eta,k}) + p(-\boldsymbol{p}_{\eta,k}) &= p(\boldsymbol{\eta}) - kp(\boldsymbol{\eta}_{rf}),
\end{aligned}$$

proving the implication. ■

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