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Ambiguity Aversion and Under-diversification

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Abstract

We examine asset allocation decisions under smooth ambiguity aversion when an investor has a prior degree of belief in an asset pricing model (e.g., the domestic CAPM). Different from the Bayesian portfolio approach, in our model the investor separately relies on the conditional distribution of returns and on the posterior over uncertain parameters to make asset allocation decisions, rather than on the predictive distribution of returns that integrates priors and likelihood information in a single distribution. This is a key feature implied by smooth ambiguity preferences. We find that in the perspective of US investors, ambiguity aversion can generate strong home bias in their equity holdings, regardless of their belief in the domestic CAPM or of their degree of risk aversion. Our results extend and become stronger under regime-switching investment opportunities.

JEL Classification: C61; D81; G11.

Keywords: Ambiguity aversion, Bayesian portfolio analysis, CAPM, Smooth ambiguity.

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1 Introduction

Empirical evidence documents that US investors tend to hold a substantially larger proportion of their equity portfolios in domestic stocks than is suggested either by the weight of the US market in the value-weighted world equity portfolio or by its sample return moments used in standard mean-variance calculations. Both mean-variance analysis and the composition of the value-weighted world equity portfolio suggest that a US investor should hold approximately between 50 to 60 percent in foreign stocks in his equity portfolio (see, for example, Lewis (1999)). However, by all existing records and methodologies, foreign stock holdings of a US investor remain as low as 10 percent (see e.g., Ahearne et al. (2004) and Warnock (2002)).[1] Previous studies such as Pastor (2000) have explored the home bias puzzle in a Bayesian framework where a US investor holds uncertain beliefs over the domestic capital asset pricing model (CAPM). The major finding is that for the investor to persistently hold a significantly under-diversified portfolio, his confidence in the domestic CAPM must be extremely strong. In practice, the commonly observed degree of home bias in international equity holdings would rule out mispricing of foreign stocks larger than two percent in absolute value from rather naive domestic CAPM models. Since there is little evidence in the empirical literature supporting such a strong belief, the extent to which incorporating mispricing uncertainty concerning the foreign stock portfolio helps explaining the observed home bias remains limited. Yet, understanding the sources of the home bias puzzle remains a priority to all finance scholars. The welfare costs of under-diversification in international stock and bond portfolios are conjectured to be massive: for instance, in the case of an average US investor, Lewis (1999) estimates the costs of the forgone gains from diversification to be in the range of 20% to almost double of lifetime (permanent) consumption.

In this paper, we propose a new approach that takes into account uncertain beliefs over an asset pricing model (e.g., the CAPM), parameter uncertainty, and more importantly, ambiguity aversion. We use the smooth ambiguity utility model proposed by Klibanoff et al. (2005) to characterize ambiguity attitudes. Our quantitative analysis reveals that an ambiguity-averse US investor with a *moderate* prior degree of belief in the domestic CAPM may rationally hold a *substantially* under-diversified portfolio relative to the standard mean-variance portfolio. The extent of such under-diversification can fully justify the home bias observed in the data. This

[1] Bekaert and Wang (2009) have recently extended this original evidence based on US data to a rather large worldwide sample of 27 countries finding that, as reported since the 1990s, the home bias is a persistent (at most, slowly declining) phenomenon involving all countries. However, Bekaert and Wang document that when home bias indicators are adjusted by market size, Japan is the most home biased developed country with a bias of close to 90%, whereas the Netherlands is the least biased; emerging markets are more biased than developed markets on average; Europe is the least biased continent, and Asia the most biased. Therefore, after adjusting for the underlying scale of the countries involved, US data may even under-estimate the average, planetary extent of this phenomenon.

result holds true regardless of the precise strength of an investor's prior belief about the mispricing of foreign stocks, i.e., whether the level of mispricing uncertainty is high or low, and whether the investor believes *a priori* that the level of mispricing is negative or positive. In addition, we find that under-diversification in the investor's portfolio holdings cannot be obtained by simpy increasing the degree of (constant, relative) risk aversion to somehow surrogate an increase in the aversion to ambiguity. As a result, and differently from existing studies (e.g., Maenhout (2004) and Liu (2011)), in our model the effect of ambiguity aversion is fundamentally different from that of risk aversion.

The intuition for how smooth ambiguity aversion preferences may explain the home bias is simply explained. Different from the standard Bayesian approach, which fully relies on the predictive distribution of returns, our approach has that the conditional distribution of returns cannot be integrated over the posterior of uncertain parameters to produce a predictive distribution. In other words, optimal portfolio choice depends on *both* the posterior of the parameters as well as the conditional distribution of returns, instead of a predictive distribution that amalgamates the two as prescribed by standard statistical theory. This distinctive feature is implied by smooth ambiguity preferences, which distinguish the attitude toward pure risk from the attitude toward ambiguity and hence imply that it would be optimal to keep separate the parameters' posteriors from the conditional distribution of returns. Nevertheless, the Bayesian approach implicitly assumes that these two attitudes are intertwined and both governed by the curvature of a von Neumann-Morgenstern (vNM) utility function. In our model, ambiguity is explicitly represented by uncertainty about primitive parameters determining investment opportunities. As a result, ambiguity aversion can be alternatively interpreted as an aversion toward parameter uncertainty and asset pricing model uncertainty.

Yet, even under smooth ambiguity preferences, investors are risk averse in that they dislike any mean-preserving spread of the end-of-period wealth given a portfolio allocation in the beginning of the period. However, investors are also ambiguity averse in the sense that they are averse to any mean-preserving spread of conditional expected utility value induced by the posterior distribution of the parameters. In fact, under smooth ambiguity preferences, a concave transformation is imposed on conditional expected utility values to reflect an inherent degree of pessimism.[2] This concave function results in "irreducibility of compound lotteries". As a consequence, unlike the Bayesian approach, it is infeasible to hinge asset allocation decisions directly on the predictive distribution of returns: Under ambiguity aversion, investors effectively attach a higher weight on

[2] However, such pessimism may concern to an equal degree both domestic and foreign securities, i.e., no differentials of pessimism is exogeneously imposed. In fact, our results would go through even if investors were to be more pessimistic on domestic stocks than on foreign ones, as long as they remain averse to mean-preserving spreads of conditional expected utility values.

states with lower conditional expected utility values.

A number of papers provide empirical evidence that stock markets often experience distinct regimes and also show that accounting for regime switching in asset allocation decisions could be important, also to explain the home country bias puzzle (see, for example, Ang and Bekaert (2002) and Guidolin and Timmermann (2008)). However, these papers do not explicitly incorporate parameter uncertainty and asset pricing model uncertainty. Guidolin and Timmermann (2008) use a Markov switching model with two regimes to characterize the time series dynamics (predictability) in international stock markets and find that distinguishing regimes in asset allocation is helpful in explaining the home bias puzzle. However, at least for short investment horizons, in their paper a strong difference between the strength of under-diversification in the bear (when it is strong) and bull (when it is weaker) regimes remains. In our paper, we extend our analysis to regime switching investment opportunities, using the approach developed by Tu (2010). In particular, we embed the smooth ambiguity preferences into a portfolio choice model that simultaneously takes into account parameter uncertainty, mispricing uncertainty and regime switching. We find that with the addition of parameter uncertainty and asset pricing model uncertainty, the effect of regime switching on international asset allocation is remarkable and stronger than what had been previously reported under simpler expected utility maximization. Consistent with Guidolin and Timmermann (2008), investors prefer a substantially under-diversified portfolio in the bear regime, which features lower mean returns and higher volatilities and correlations. Moreover, ambiguity aversion results in a highly under-diversified portfolio also in the bull regime. Based on a sensibly calibrated degree of ambiguity aversion, our model produces a relative weight in US stocks that is close to standard empirical measures documented in the literature (see Bekaert and Wang (2009) and Tesar and Werner (1995)).

This paper is related to a number of others studying portfolio choice under ambiguity aversion. However, our focus on the mixed effects of aversion to parameter as well as model uncertainty on under-diversification, is unique. One strand of literature has examined the impact of a preference for robustness (see, for example, Maenhout (2004), Maenhout (2006), Garlappi et al. (2007), and Liu (2010)). Garlappi et al. (2007) and Liu (2011) analyze dynamic portfolio choice under incomplete information and ambiguity in a multiple priors framework. Campanale (2011) studies life-cycle portfolio choice when investors engage in learning under multiple priors. Chen et al. (2011) assume that investors have recursive smooth ambiguity preferences and are confronted with model uncertainty. In a simple application to a portfolio problem with a single risky asset, they discuss the implications ambiguity aversion for the economic value of return predictability. They find that an ambiguity-averse investor slants his beliefs towards the submodel of returns that delivers the lowest continuation value so that both the myopic and the hedging demands implied by his strategy can be quite different from those implied by Bayesian strategies. However, all of these papers assume that there is only one risky asset and that there is neither parameter uncertainty nor asset pricing model uncertainty.

The closest paper to ours, at least in terms of empirical objective and underlying storyline is Uppal and Wang (2003), who study asset allocation with multiple risky assets in an independently and identically distributed (i.i.d.) setting where different levels of ambiguity are attached to the return distributions of domestic vs. foreign assets and the parameters are assumed to be perfectly known. They find that differences in the level of ambiguity concerning alternative returns distributions are capable to generate international portfolios that are significantly under-diversified relative to those obtained in the standard mean-variance model. As an application, Uppal and Wang rely on this mechanism to address the home bias of US investors. Our model differs from Uppal and Wang (2003) in several respects. First, Uppal and Wang (2003) postulate that ambiguity arises as a result of model misspecification pertaining to the returns processes of the risky assets, while in our model ambiguity is captured by uncertainty directly concerning the assumed asset pricing model and the parameters of the distributions of returns. Second, in order to address the home bias puzzle, Uppal and Wang (2003) explicitly assume that a US investor has more knowledge of domestic stock returns than of foreign stocks returns. This feature is achieved in the model by giving additional weight to the marginal distribution of US returns beyond what is revealed by the joint distribution of all the stocks in her portfolio. In contrast to Uppal and Wang (2003), our model does not explicitly assume a different degree of ambiguity on US vs. to foreign stocks. Third, in order to derive an analytical solution, Uppal and Wang (2003) assume a form of "homothetic robustness". In particular, in their model the degree of ambiguity must be scaled by some function of the investor's lifetime utility to preserve homogeneity of the value function. Even though "homothetic robustness" simplifies calculations, it is imposed *ad hoc* and fails to result directly from primitive assumptions concerning the vNM utility function and the smooth ambiguity-inducing transformation. As a result, in our paper the solution to the asset allocation problem is based on numerical methods and does not rely on homogeneity.

The remainder of this paper is organized as follows. Section 2 introduces the smooth ambiguity preferences framework. Section 3 describes the investment opportunities set and the investor's portfolio choice problem. Section 4 calibrates our model to the data and discusses quantitative results. Section 5 extends the analysis by incorporating regime switching. Section 6 concludes. The numerical algorithms and the calibration of the degree of ambiguity aversion are presented in the Appendix.

2 Smooth Ambiguity Preferences

Consider an investor with a one-period investment horizon who allocates wealth between a risk-free asset and a portfolio of risky assets. The investor is assumed to make asset allocation decisions based on available information containing historical returns on all the investable assets and prior information. The investor is "small" in the sense that his portfolio decisions cannot influence the probability distribution of asset returns. In particular, let W denote the investor's current wealth, and ψ the proportion of her wealth invested in the risky portfolio. The investor's next period wealth is given by

$$W_{t+1} = W_t \left[(1-\psi) R_f + \psi \omega' \left(\mathbf{R}^e_{t+1} + \iota R_f \right) \right]$$

where R_f is the (net) rate of return on the risk-free asset, $\mathbf{R}^e_{t+1}$ is a vector of the next-period returns on all the investable assets in excess of R_f, and ω stands for a vector of weights in the risky portfolio (such that $\omega'\iota = 1$, where ι is an appropriately sized vector of ones).

In the smooth ambiguity utility model, the investor's preferences are characterized by the following functional that includes a double expectation operator:

$$V(W_{t+1}) \equiv \int_\Theta \phi \left(\int_S u(W_{t+1})\, d\pi_\theta \right) d\mu(\theta) \equiv \mathbb{E}_\mu \phi \left(\mathbb{E}_\pi u(W_{t+1}) \right) \tag{1}$$

where $u(\cdot)$ is a standard von Neumann-Morgenstern utility function, and $\phi(\cdot)$ is a monotonically increasing function.[3] Due to parameter uncertainty, the investor suffers from subjective uncertainty about the probability distribution of asset returns. In a Bayesian framework, this uncertainty is captured by the multiplicity of the set Θ, within which each element induces a probability distribution π_θ over returns. The investor's subjective uncertainty is represented by the posterior $\mu(\theta)$ over the parameter space Θ. As usual, attitude toward pure, quantifiable risk is then characterized by the shape of $u(\cdot)$: the investor is risk averse if and only if $u(\cdot)$ is concave. However, (1) features an additional form of non-quantifiable uncertainty, commonly called ambiguity: the investor's attitude towards ambiguity are captured by the shape of the function $\phi(\cdot)$. In particular, a concave ϕ implies ambiguity aversion, which is defined to be an aversion to mean preserving spreads in the distribution over expected utility values induced by the posterior $\mu(\theta)$.

A key feature of the class of preferences defined in (1) is that it achieves a separation between ambiguity, identified as a characteristic of the investor's subjective beliefs, and ambiguity attitude, identified as a characteristic of the investor's tastes. Ambiguity is characterized by

[3] Klibanoff et al. (2009) extend this class of preferences to a dynamic setting. Ju and Miao (2011) further generalize this decision framework by distinguishing risk aversion from the elasticity of intertemporal substitution.

uncertainty about the parameter space Θ, while ambiguity attitude by the shape of the function $\phi(\cdot)$. Ambiguity aversion is defined to be an aversion to mean preserving spreads in the distribution over expected utility values induced by the posterior $\mu(\theta)$. This distribution gives us the probabilities of different evaluations of a portfolio choice $\{\psi, \omega\}$ under different probability measures induced by elements in Θ that are deemed to be relevant. In the ambiguity literature, this induced distribution is called a "second order probability" distribution (see, e.g., Segal (1987), Segal and Spivak (1990), and Gollier (2011)).

An ambiguity averse investor displays different attitudes toward risk and uncertainty embodied, respectively, in the first and second order probabilities. Intuitively, she prefers choices the outcomes of which are more robust to possible variation in second-order probabilities. In this paper, we show that this preference for robustness (in a loose sense, not necessarily in the definition proposed by Hansen and Sargent (2007)) is able to generate important implications for international portfolio diversification that go well beyond what has been shown in the existing literature. As we shall show later on, such important empirical implications useful to explain observed diversification behavior (or a lack thereof) derives from the fact that the smooth ambiguity utility model displays a property of *irreducibility of compound distributions*. That is, the model does not impose or implies a compound reduction involving the posterior distribution $\mu(\theta)$ and the conditionals π_θs in the support of $\mu(\theta)$ (see Gollier (2011)). Only in the special case of $\phi(\cdot)$ being linear, such reduction is feasible, and the investor displays ambiguity neutrality and she becomes observationally equivalent to a subjective expected utility maximizer. Thus, this model includes a Bayesian investor who simply maximizes her expected utility as a special case when $\phi(\cdot)$ is linear: therefore Bayesian portfolio analysis imposes the reduction of the first and second order probabilities to unique, single convolution which implies that the aversion attitude towards pure risk and ambiguity (Knightian uncertainty, using the terminology of Epstein and Wang (1994)) are tightly intertwined. In particular, a Bayesian investor will be characterized by the following expected utility functional:

$$\mathbb{E}_t\left(u\left(W_{t+1}\right)\right) = \int\int ... \int_{\mathbf{R}^e_{t+1}} u\left(W_{t+1}\right) p\left(\mathbf{R}^e_{t+1}|\Phi_t\right) d\mathbf{R}^e_{t+1}$$

where $p\left(\mathbf{R}^e_{t+1}|\Phi_t\right)$ is the (joint, multivariate) predictive density of excess asset returns and Φ_t is the information set available to the portfolio optimizer.[4] In Bayesian portfolio analysis, the predictive density can be obtained by integrating the conditional density of excess asset returns

[4] We assume that the modeled, " domestic" investor has no informational advantage over domestic stocks vs. foreign stocks, thus ruling out information-based explanations of the home country bias (see Ahearne et al. (2004) for a discussion) arising from differential information flows (e.g., Brennan et al. (2005)), differential memory capacity constraints (e.g., van Nieuwerburgh S. and Veldkamp (2009)), or under-diversification caused by long-lasting private information (see Liu et al. (2010)).

onto the posterior distribution:

$$p\left(\mathbf{R}_{t+1}^{e}|\Phi_{t}\right)=\int\int...\int_{\Theta}p\left(\mathbf{R}_{t+1}^{e}|\theta,\Phi_{t}\right)p\left(\theta|\Phi_{t}\right)d\theta \tag{2}$$

where $p\left(\mathbf{R}_{t+1}^{e}|\theta,\Phi_{t}\right)$ is the density of returns conditional on the (vector of) parameters θ and information up to time t, and $p\left(\theta|\Phi_{t}\right)$ is the posterior density over the parameter space. The smooth ambiguity utility model precludes the reduction of compounded probabilities in (2), as the ambiguity-averse investor displays different attitudes toward risk embodied in $p\left(\mathbf{R}_{t+1}^{e}|\theta,\Phi_{t}\right)$ and the uncertainty captured by $p\left(\theta|\Phi_{t}\right)$. In particular, she dislikes mean preserving spreads applied to the distribution of conditional expected utility values induced by $p\left(\theta|\Phi_{t}\right)$ and any selected portfolio choice $\{\psi,\omega\}$. On the contrary, because a linear $\phi(\cdot)$ implies that in (1) $V\left(W_{t+1}\right)\propto\int_{\Theta}\int_{S}u\left(W_{t+1}\right)d\pi_{\theta}d\mu\left(\theta\right)=\int_{\Theta}\int_{S}u\left(W_{t+1}\right)dq(\theta)=\mathbb{E}_{q}[u\left(W_{t+1}\right)]$, where $q(\theta)$ is the reduced, compounded distribution obtained as a convolution of π_{θ} and $\mu\left(\theta\right)$, Bayesian investors will be neutral to mean preserving spreads over expected utility values induced by the posteriors $\mu\left(\theta\right)$.

3 The Portfolio Choice Problem

We consider a problem in which there are $N+K$ risky assets, K of which are benchmark portfolios, and N of which are nonbenchmark assets. The benchmark portfolios mimic K priced sources of risk in an asset pricing model, for instance, the CAPM. Our model specification for investable assets returns follows Pastor (2000) and Tu (2010): the multivariate conditional mean (regression) of the non-benchmark returns as a function of the benchmark returns is given by

$$\mathbf{R}^{e}=\mathbf{XB}+\mathbf{U}, \tag{3}$$

where $\mathbf{X}\equiv\left[\begin{array}{cc}\iota_{T} & \mathbf{F}^{T}\end{array}\right]$, ι_{T} denotes a $T\times1$ vector of ones, $\mathbf{F}^{T}$ denotes a $T\times K$ matrix of excess returns on the K benchmark portfolios, the regression coefficient matrix $\mathbf{B}$ is $\mathbf{B}\equiv\left[\begin{array}{cc}\alpha & \mathbf{B}_{2}\end{array}\right]'$ in which α is $N\times1$ and $\mathbf{B}_{2}$ is $N\times K$. If an asset pricing model is correctly specified, then the non-benchmark assets' vector of mispricings, α, is equal to zero. We assume that the rows of the disturbance matrix $\mathbf{U}$ are jointly distributed as identically and independently (i.i.d.) distributed normal variates, with a vector of zero means and positive definite $N\times N$ covariance matrix $\mathbf{\Sigma}$. The benchmark returns are assumed to be i.i.d. normal $\mathbf{F}_{t}\sim N\left(\mathbf{E}_{F},V_{F}\right)$ and independent of the rows of the disturbance matrix $\mathbf{U}$.

To allow for parameter uncertainty, the parameters $\{\mathbf{B},\mathbf{\Sigma},\mathbf{E}_{F},\mathbf{V}_{F}\}$ are assumed to be unknown. Since our main focus is on mispricing uncertainty, the prior distribution is specified to be uninformative about all the parameters except for the vector of mispricings, α.[5] The regression coefficients in the multivariate regression (3) and the moments of the benchmark returns are assumed to be independent in the prior:

$$p(\mathbf{B}, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F) = p(\mathbf{B}, \mathbf{\Sigma})\, p(\mathbf{E}_F, \mathbf{V}_F)\,.$$

In particular, the priors on $\mathbf{B}$ and $\mathbf{\Sigma}$ are Normal-Inverted Wishart:

$$\left.\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right| \mathbf{\Sigma} \sim N\left(\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}, \begin{bmatrix} \frac{\sigma_\alpha^2}{s^2}\mathbf{\Sigma} & 0 \\ 0 & \mathbf{\Omega} \end{bmatrix}\right) \quad (4)$$

$$\mathbf{\Sigma} \sim IW\left(s^2(v-N-1)\,\mathbf{I}_N, v\right), \quad v > N+1 \quad (5)$$

where $\beta \equiv \mathrm{vec}(\mathbf{B}_2)$ ("vec" denotes the operator that stacks the columns of a matrix into a vector), the mispricing vector α and the non-benchmark assets' betas in β are assumed to be independent in the prior, and $\bar{\alpha}$ and $\bar{\beta}$ are the prior means of α and β, respectively. The prior on β is made uninformative by assuming that $\mathbf{\Omega}$ is a diagonal matrix with extremely large diagonal terms. The prior on $\mathbf{\Sigma}$ is made uninformative by setting the degrees of freedom parameter v to be a large number. In the prior (5), the properties of the inverted-Wishart distribution imply that the prior mean of $\mathbf{\Sigma}$ is $s^2\mathbf{I}_N$. Thus, the marginal prior distribution of the mispricing vector α has a mean of $\bar{\alpha}$ and a covariance matrix of $s^2\mathbf{I}_N$.

In the specification described so far, the only informative prior is about the mispricing of the non-benchmark assets, which can be allowed for by setting σ_α to small values. If the benchmark assets are zero-investment portfolios, a correctly specified asset pricing model implies that α is a vector of zeros. When the investor believes *a priori* that possible values of α center around zero (i.e., $\bar{\alpha}$ is a vector of zeros), the magnitude of σ_α represents the investor's prior degree of belief in the pricing model. A large value of σ_α indicates a highly dispersed, and therefore weak, belief in the validity of the pricing model. As noted by Pastor (2000), the prior specification above also accommodates the case in which the prior mean of α deviates from zero, $\bar{\alpha} \neq \mathbf{0}$. In such a case, the value of σ_α reflects the investor's belief in the accuracy of the forecast of the mispricing parameters/indices. Finally, the prior on the benchmark moments $\mathbf{E}_F$ and $\mathbf{V}_F$ is a standard diffuse prior:

$$p(\mathbf{E}_F, \mathbf{V}_F) \propto |\mathbf{V}_F|^{-(K+1)/2}\,.$$

Note that the benchmark returns $\mathbf{F}$ are normal conditional on $\mathbf{E}_F$ and $\mathbf{V}_F$ and that the non-

[5] Tu and Zhou (2010) have shown that imposing informative (and implicit) priors on the solution to the problem rather than on the primitive parameters may consistently improve realized portfolio performances. Such priors directly concerning sensible portfolio weights can be backed out from the Euler equation for the portfolio problem. In order to focus on the role of ambiguity, we do not address the issue of the role of informative, economic priors in our paper, apart from the priors on the vector of mispricings, α.

benchmark returns $\mathbf{R}^e$ are normal conditional on $\{\mathbf{B}, \mathbf{F}, \mathbf{\Sigma}\}$. Thus, in what follow we take the set of all possible posterior values of $\{\alpha, \beta, \mathbf{F}, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F\}$ as the set Θ in the decision model (1). It follows that given a sample of data with length T, the inner expectation in (1) can be rewritten as:

$$\mathbb{E}_\pi \left[u \left(W_{T+1} \right) | \theta \right] = \int_{\mathbf{R}^e_{T+1}} \int_{\mathbf{F}_{T+1}} u \left(\left(1 - \psi \right) R_f + \psi \omega' (\tilde{\mathbf{R}}^e_{T+1} + R_f) \right) d\mathbf{F}_{T+1} \left(\mathbf{E}_F, \mathbf{V}_F \right) d\mathbf{R}^e_{T+1} \left(\mathbf{B}, \mathbf{F}, \mathbf{\Sigma} \right)$$

where $\theta \equiv \{\alpha, \beta, \mathbf{F}, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F\}$, the second integral is over possible values of $\mathbf{F}_{T+1}$ given the posterior parameters $\mathbf{E}_F$ and $\mathbf{V}_F$, and the first integral is over possible values of $\mathbf{R}^e_{T+1}$ conditional on $\{\alpha, \beta, \mathbf{F}, \mathbf{\Sigma}\}$. $\tilde{\mathbf{R}}^e$ denotes the returns on all investable assets $\tilde{\mathbf{R}}^e \equiv \left[\begin{array}{cc} \mathbf{R}^e & \mathbf{F} \end{array} \right]$.

The investor chooses portfolio weights $\{\psi, \omega\}$ to maximize his expected utility in (1). For concreteness, we select the vNM utility function $u(\cdot)$ to have a standard power, isoelastic form:

$$u \left(W \right) = \frac{W^{1-\gamma}}{1-\gamma}, \quad 0 < \gamma \neq 1,$$

where γ is the coefficient of relative risk aversion. We also follow examples in Klibanoff et al. (2005) and conveniently assume that ϕ has a negative exponential form (see also Collard et al. (2011)):

$$\phi \left(x \right) = -\frac{1}{\eta} e^{-\eta x}, \quad \eta > 0. \tag{6}$$

As Klibanoff et al. (2005) have emphasized, this function displays constant ambiguity aversion where η is interpreted as the coefficient of ambiguity aversion. Thus, the investor solves the following optimization problem:

$$\max_{\psi,\omega} - \int_\Theta \frac{1}{\eta} \exp \left(-\eta \int_{\mathbf{R}^e_{T+1}} \int_{\mathbf{F}_{T+1}} \frac{\left[\left(1 - \psi \right) R_f + \psi \omega' \left(\tilde{\mathbf{R}}^e_{T+1} + R_f \right) \right]^{1-\gamma}}{1-\gamma} \right.$$
$$\left. d\mathbf{F}_{T+1} \left(\mathbf{E}_F, \mathbf{V}_F \right) d\mathbf{R}^e_{T+1} \left(\mathbf{B}, \mathbf{F}, \mathbf{\Sigma} \right) \right) d\pi_\theta d\mu \left(\theta \right) \tag{7}$$

where $\mu \left(\theta \right)$ is the joint posterior distribution of the parameters θ.

Because the optimization problem in (7) does not admit an analytical solution, we use numerical methods to solve the problem. We first simulate $\{\alpha, \beta, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F\}$ from their posterior densities and $\mathbf{F}$ from its predictive density. The inner expectation with respect to the joint normal densities that appear in (7) is evaluated by numerical integration. Finally, we use a numerical optimization routine to solve for optimal portfolio weights. Our numerical algorithm is discussed in detail in Appendix A. In the expected utility case, we rely on the standard Bayesian approach to solve the problem (see Pastor (2000)). In particular, we draw $\mathbf{F}_{T+1}$ and $\mathbf{R}^e_{T+1}$ using a Gibbs sampling procedure and the optimal portfolio weights can then be found by numerically maximizing

$$\frac{1}{M}\sum_{m=1}^{M}\frac{\left[(1-\psi)R_f+\psi\omega'\left(\tilde{\mathbf{R}}^e_{m,T+1}+R_f\right)\right]^{1-\gamma}}{1-\gamma}$$

where M is the number of Gibbs sampler draws and $\tilde{\mathbf{R}}^e_m$ are simulated returns on all investable assets.

4 International Asset Allocation

In this section, we study the application of our methodology to a typical international asset allocation problem, in the perspective of a US investor, similarly to Lewis (1999), among others. To gain intuition, we start by focussing on the simplest possible case of $K = N = 1$. $K = 1$ means that the only benchmark asset is the market portfolio.[6] Following Pastor (2000), the proxy for the market portfolio throughout this paper is the US value-weighted portfolio (VWUS). This choice is motivated by previous studies that have tested the global mean-variance efficiency of the US market portfolio (see e.g., Tesar and Werner (1995)). The monthly market returns are retrieved from the Center for Research in Security Prices (CRSP). We use the "World-except-US" (WXUS) portfolio as a proxy for all the remaining non-benchmark assets (equities), which implies that $N = 1$. The monthly returns on the WXUS portfolio are from Morgan Stanley Capital International (MSCI). The US 1-month Treasury bill rate is used as the risk-free rate. The data in this study span the sample period January 1988 - December 2009.

It has been well documented that US investors tend to hold domestic equities in a substantially larger proportion than what would be rationally implied by a standard mean-variance model or than the weight of US stocks in the world-wide equity market portfolio. This is often referred to as a "home equity bias puzzle", see French and Poterba (1991) and Tesar and Werner (1995). Previous studies (e.g., Pastor (2000) and Herold and Maurer (2003)) have investigated international asset allocation using a Bayesian approach and found that a strong belief of US investors in the global mean-variance efficiency of the US market portfolio (i.e., the *domestic* CAPM) would be required to justify the substantial home bias observed in the data on the portfolio choices of US investors.

Our main goal is then to examine the impact of ambiguity aversion on optimal weights in domestic and foreign stocks in the risky portfolio and explore the potential of our model in explaining the home bias of US investors. In other words, we attempt to understand (1) the conditions that need to be imposed on investors' preferences for a standard, partial equilibrium asset allocation model to generate a significantly under-diversified portfolio that would favor US

[6] Our approach can also be applied to study asset allocation with multiple benchmark portfolios (e.g., the Fama-French-Carhart factors).

stocks over non-US ones, and (2) assuming that ambiguity may be part of such a specification of preferences, whether the required degree of ambiguity aversion is plausible, given the experimental evidence in the literature (see e.g., Camerer (1999)).

4.1 Calibration of parameters

Based on the sample period January 1991 - December 2009, the regression of WXUS portfolio returns on VWUS returns yields the following results: $\hat{\alpha} = 1.8$ percent per month with a t-statistic of 0.96; $\hat{\beta} = 0.88$ with a t-statistic of 20.46; $\hat{\sigma}^2 = 8e - 4$, and $R^2 = 0.65$. The prior hyper-parameters are specified as follows. The parameters v and Ω are set such that the priors on Σ and β are uninformative. We use the first 36 months of our data (January 1988 - December 1990) to estimate the prior parameter s^2. This gives us a sample estimate of the residual variance of 0.002, and thus $s^2 = 0.002$. We vary the values of $\bar{\alpha}$ and σ_α to study the comparative static results of various beliefs over the domestic CAPM as well as the expected level of mispricing.

Table 1 reports the optimal weights ψ and the relative weights ω in the WXUS and VWUS portfolios, respectively, when the prior belief about the expected mispricing $\bar{\alpha}$ takes on the value of zero as implied by the domestic CAPM. Tables 3 and 4 present instead results for $\bar{\alpha} = \pm 0.02$, when the investor expects some plausible degree of mispricing. In order to study the comparative static effects of ambiguity and its interaction with mispricing uncertainty, we vary the degree of ambiguity aversion and the level of mispricing uncertainty. In particular, we set the coefficient of relative risk aversion, γ, at a typical value of 5, and consider two different selections for the ambiguity aversion parameter, $\eta = 15$ and 60. We assess a plausible range of the parameter η based on the magnitude of the ambiguity premium in Ellsberg-type experiments. Appendix C contains the details of these choices and explains how a parameterization of η ranging between 15 and 60 will produce plausible premia. Camerer (1999) suggests that an ambiguity premium of the order of 10-20% of the expected value of a bet in Ellsberg-type experiments is reasonable. For the power-exponential preferences specification adopted in this paper, a 10% ambiguity premium, expressed as a percentage of the expected value of a bet, is associated with $\eta = 60$ for relatively small bets.[7] Table C1 shows the ambiguity premia computed for different sizes of the bet and degrees of risk and ambiguity aversion. The uncertainty about the level of mispricing, σ_α, takes on values from low to high as: $0, 0.005, 0.01, 0.02, 0.05$ and 0.1, all of which are annualized values.[8] A value of σ_α in excess of 0.1 leads to insignificant changes in optimal portfolio weights, all else being equal. An extremely small σ_α indicates a strong belief in the expected level of mispricing.

[7] In the limit, as shown by Klibanoff et al. (2005), the ex ante welfare $V(W_{t+1})$ essentially exhibits a kinked maxmin expected utility functional à la Gilboa and Schmeidler (1989) when the degree $\eta \longrightarrow \infty$. Because a finance literature exists that has documented the effects of ambiguity under maxmin preferences (see e.g., Epstein and Wang (1994); Garlappi et al. (2007)), this means that any finite η may be taken as sufficiently realistic to deserve consideration.

[8] In our numerical simulations, we choose an extremely small value of σ_α as an approximation of $\sigma_\alpha = 0$.

Of course, the case $\sigma_\alpha = \bar{\alpha} = 0$ corresponds to complete confidence in the domestic CAPM.

4.2 Implications for international portfolio diversification

Previous papers such as Pastor (2000) find that (1) when the prior belief $\bar{\alpha}$ is zero, the relative weight in foreign, non-US stocks (WXUS) increases with σ_α, up to some level to which the weight goes asymptotically as $\sigma_\alpha \to \infty$, and (2) a strong belief in the global mean-variance efficiency of the US market portfolio, i.e., a remarkably low σ_α together with $\bar{\alpha} = 0$, is required to generate a substantially under-diversified international portfolio that is consistent with the actual allocation to foreign stocks observed in the equity holdings of US investors. These results are replicated in Panel A of Table 1 under expected utility, i.e., when there is no ambiguity aversion.[9] As σ_α rises, the investor has more doubt on the validity of the domestic CAPM. In this case, he will rely more on the sample estimate of α to make asset allocation decisions. This results in a larger allocation to foreign stocks, given a positive sample estimate of α as the one we have found in our data. Therefore, in a Bayesian asset allocation framework, a substantial international under-diversification would derive from the fact that investors—assumed to Bayesian portfolio decision makers who are neutral to ambiguity—would perceive very little doubts on the validity of the domestic CAPM and hold a neutral prior on the sign of any mispricings ($\bar{\alpha} = 0$). For instance, when $\sigma_\alpha = 0.005$, a Bayesian investor would optimally hold 68% of her portfolio in stocks, of which 85% would go to US stocks and only 15% to foreign ones.

What is the impact of ambiguity aversion on portfolio choice? Panels B and C of Table 1 present optimal and relative weights in WXUS and VWUS under ambiguity aversion. To also assess how risk and ambiguity aversion may interact, we also report portfolio weights computed for $\gamma = 30$ in Panel D. Two results are noteworthy. The first result is related to the effect of ambiguity aversion on the size of the optimal risky portfolio. Does ambiguity aversion reduce investment in stocks relative to the risk-free asset? Not necessarily in our case. Comparing Panel A and B, we find that ambiguity aversion *increases* the sum of optimal weights in WXUS and VWUS. For instance, when $\sigma_\alpha = 0.005$, the total investment in risky assets increases from 68 to 85 percent. This result is in line with Gollier (2011), who studies a static portfolio choice problem, and also with Chen et al. (2011), who investigate dynamic portfolio choice. In our model, an ambiguity-averse investor distorts her beliefs about the primitive parameters describing investment opportunities in a pessimistic way. Consistent with Gollier (2011) and Chen et al. (2011), a pessimistic distortion affecting the perceived distribution of stock returns does not

[9] The optimal weights computed under expected utility are slightly different from those in Pastor (2000) (Table I) because we assume a power form for the von Neumann-Morgenstern felicity function $u(\cdot)$, while Pastor (2000) uses a standard mean-variance approximation.

necessarily lead to a monotonically declining demand for stocks.[10]

Second and most importantly, we observe that an ambiguity-averse US investor optimally chooses an under-diversified portfolio that implies a substantial degree of home bias, regardless of her beliefs about the validity of the domestic CAPM. For example, an investor with $\gamma = 5$, $\eta = 60$ and a prior belief in the CAPM represented by $\sigma_\alpha = 0.1$ per year should invest 65 percent of his wealth in domestic stocks and 7 percent in foreign stocks, which implies a 90 percent US equity holdings as a fraction of her optimal risky portfolio. This is in contrast to the expected utility, Bayesian case in Panel A, where an expected utility investor with $\gamma = 5$ and $\sigma_\alpha = 0.1$ allocates 21 percent of wealth in domestic stocks and 51 percent in foreign stocks; thus the relative weight assigned to domestic stocks is a modest 30 percent which appears in line both with standard mean-variance calculations (see e.g., Lewis (1999)) and with a classical international CAPM estimate. For an ambiguity-averse investor, the optimal weight in WXUS still increases as σ_α rises, but with a "slope" that is substantially smaller than in the expected utility case. Moreover, the optimal weight in the US market portfolio remains almost unchanged across different values of σ_α. For instance, the weight assigned to US stocks goes from 8 percent in the case of $\sigma_\alpha = 0$ to 10 percent when $\sigma_\alpha = 0.1$. Taken together, both patterns give rise to strong home bias in favor of US stocks, no matter how a dispersed the beliefs of an investor are about the pricing model. Interestingly, at least in a qualitative perspective, the specific value taken by η does not seem to matter: when in panel B, $\eta = 15$ is assumed, for the case of $\sigma_\alpha = 0.1$ we obtain that an investor should invest 67 percent in US stocks, 22 percent in foreign stocks, and 11% in cash, which still implies a 75% weight assigned to domestic equities and a considerable home bias. Importantly, the relative weight of US stocks is even higher when σ_α is smaller, for instance 79% in the case of $\sigma_\alpha = 0.01$.

Another question that usually arises in the literature studying portfolio choice and ambiguity aversion is: does ambiguity aversion just replaces aversion under a different label? Our quantitative analysis clearly illustrates that this is not the case. Panel D of Table 1 shows expected utility results for the case $\gamma = 30$. As is standard in the literature, it shows that a more risk-averse investor will invest less in stocks and more in the risk-free asset. However, the relative weight in domestic stocks remains approximately the same as in panel A, regardless of the degree of risk aversion, i.e., just increasing risk aversion has no chance to create home bias in equity portfolio decisions, while ambiguity-averse preferences can.[11]

[10] In a way, this is a well-know result that holds under expected utility preferences that has been extended to the case of ambiguity aversion. For instance, Rothschild and Stiglitz (1971) had already analytically proven that an increase in the riskiness of an asset's payoffs may not decrease risk averse investors' demand for this asset.

[11] For instance, holding constant a relatively doubtful choice for $\sigma_\alpha = 0.1$, when $\gamma = 5$ and $\eta = 60$ we have that an investor should hold 72% of her portfolio in stocks, of which 90% in domestic stocks; when $\gamma = 30$, an investor should hold only 12% of her portfolio in stocks, of which 29% in domestic stocks. Higher (and hardly plausible)

4.3 Inspecting the mechanism

To understand why an ambiguity-averse investor tends to hold an under-diversified portfolio, it is important to investigate how ambiguity aversion alters the investor's belief about the distribution of asset returns. However, since it is impossible to derive analytical results, in this section we rely on numerical results to illustrate the mechanism through which ambiguity attitudes generate the observed equity home bias by US investors. The smooth ambiguity utility model introduced in Section 2 implies that an ambiguity-averse investor attaches more weight to states characterized by lower conditional expected utility and this reflects pessimism. In our model, the value of conditional expected utility depends on the parameters $\theta \equiv (\mathbf{B}, \mathbf{F}, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F)'$ and the investor's portfolio decisions. Based on this reasoning, we want to check the possible regions of values for the key parameters that are associated with relatively low conditional expected utility values.

Table 2 summarizes the posterior means, minima and maxima of the parameters in θ, and the same set of summary statistics computed with reference to parameter values that are associated with conditional expected utility values that fall in the lowest decile of the empirical realized distribution of expected utility values. Note that the empirical distribution of expected utility values also depends on the optimal portfolio weights obtained under smooth ambiguity preferences, and thus represent a sort of "fixed point". We first examine the case of perfect confidence in the domestic CAPM ($\bar{\alpha} = \sigma_\alpha = 0$), which is presented in Panel A of Table 2. The results show that in comparison with the posterior statistics computed under a Bayesian approach, low expected value states implied by the smooth ambiguity model are on average associated with low predicted benchmark returns (F), a low posterior mean of benchmark returns (E_F), a marginally higher posterior variance of the benchmark returns (V_F) and of the assets' betas. For example, the mean values of F and E_F are -5% and 0.3%, less than their posterior mean counterparts (0.6% and 0.6%) obtained using the Bayesian approach. Moreover, when we allow for uncertainty about the validity of the CAPM (e.g., $\sigma_\alpha = 10\%$), the posterior mean of α in low conditional expected value states declines to 0.16%, which is lower than its Bayesian counterpart 0.18% (Panel B). These results suggest that ambiguity aversion distorts the investor's belief about both domestic and foreign stocks returns in a pessimistic way, in which lower and more volatile benchmark returns are predicted and a lower positive mispricing of foreign stocks is expected when there is doubt on whether the domestic CAPM applies.

Since an ambiguity-averse investor puts more weight on states with low conditional expected utility values when decisions are taken, this feature of her preferences will crucially alter the relative weights optimally assigned to domestic and foreign stocks. Our quantitative analysis

levels of risk aversion do not affect this result.

reveals that the investor's pessimism over foreign stocks tends to dominate her pessimism over domestic stocks.[12] When the investor holds a belief in an asset pricing model, low benchmark returns together with a high systematic risk imply remarkably low conditional expected returns of the foreign, ex-US portfolio, all else being equal. In other words, pessimism makes the investor rely more on states where mean returns of domestic stocks are low while the conditional mean returns of foreign stocks are much lower. This intuition is confirmed in Table 2. Thus, the investor steers her portfolio composition away from foreign stocks and toward domestic stocks, even though the overall sample estimate of α is positive. When an investor holds a strong belief in the domestic CAPM and simply maximizes expected utility, her optimal portfolio is already under-diversified and therefore can naturally hedge against extremely low conditional mean returns of foreign stocks. In this case, ambiguity aversion does not affect the optimal risky portfolio much, as shown in Table 1.

When prior beliefs are centered around non-zero values of α, in which case the investor nurses strong doubts on the average level of mispricing of foreign stocks, ambiguity aversion has a similar effect on portfolio allocations as in the case $\bar{\alpha} = 0$. This can be seen in Table 3 and 4, where the prior mean of α takes on two alternative values, -2 and +2 percent per year, respectively. For $\bar{\alpha} = -0.02$ and $\sigma_\alpha = 0$, the optimal asset allocation under expected utility ($\gamma = 5$) involves a negative position in the foreign equity portfolio (-45 percent of wealth) and a large and positive position in the domestic US portfolio (107 percent of wealth), due to the maintained strong belief of a negative mispricing that would penalize foreign equity holdings by placing them above the domestic security market line. The aversion toward ambiguity makes this asset allocation decision even more conservative: an investor with $\gamma = 5$ and $\eta = 60$ would like to invest 65 percent of wealth in domestic stocks and less than 4 percent in foreign stocks. In this case, ambiguity aversion-induced pessimism tends to increase the investment in foreign stocks because the initial position in WXUS under expected expected utility is negative. By contrast, when the prior belief is centered around a positive α, an expected utility investor who is ambiguity-neutral would allocate a significant proportion of her wealth to foreign stocks even if σ_α is extremely small. Nevertheless, Panel C of Table 4 shows that the effect of ambiguity aversion clearly dominates that of a positive prior mean of α, and that the investor with $\eta = 60$ would still like to hold a significantly under-diversified portfolio. For instance, even assuming $\sigma_\alpha = 0.1$, an investor ought to hold only 10% of her risky portfolio in foreign equities.

[12] Because it is well known from expected utility theory that pessimistic deteriorations in beliefs do not always reduce the demand for the risky asset (see Gollier (2011), and references therein), this also sheds line on the finding commented earlier that an increase in ambiguity-aversion does not necessarily lead to monotonically less demand for stocks.

4.4 Real time results

The results above are based on the full sample from January 1988 to December 2009. Because the latter date did fall in the midst of the recent financial crisis that has also severely affected world equity markets between 2008 and 2009, a Reader may wonder whether any of our earlier results may be unduly driven by our use of full sample estimates. To deal with this concern, we have alternatively implemented a recursive, expanding window approach and compute the relative weights from every optimal monthly portfolio over our sample period.[13] By doing so, we can examine the time series evolution of composition of the optimal risk portfolio of an investor and obtain a dynamic view of the comparative static effects of ambiguity aversion. We specify the investors' priors as $\bar{\alpha} = 0$ and $\sigma_\alpha = 0.1$ per year, throughout the entire sample period.[14] As before, the first 36 months of the data are used to estimate the prior parameter s. In the beginning, the sample period spans 120 months from January 1991 to December 2000. Then the sample expands through December 2009. For every portfolio formation month, we compute the relative weights in the WXUS and VWUS portfolios.

Figure 1 plots the relative weights in WXUS under expected utility and smooth ambiguity preferences with reference to the period December 2000 - December 2009. In this plot, under expected utility, the relative weight attached to WXUS is unstable and varies from -30 percent to 80 percent. The foreign stock negative weights between late 2000 and early 2004 are mainly due to low sample estimates of α during the corresponding period, i.e., to the fact that an ambiguity-neutral investor would have simply perceived foreign stocks to lie above the domestic security market line. In more recent times, expected utility implies weights in foreign stocks that are persistently increasing and—especially after 2006—systematically in excess of 50% of the overall risky portfolio. Such a strong international diversification tilted towards foreign equities has of course no correspondence to observed portfolio weights by US investors (see e.g., Bekaert and Wang (2009)).

On the contrary, the smooth ambiguity KMM model generates consistently low and stable relative weights in foreign stocks, that appear to be consistent with the available empirical evidence on this phenomenon. In the periods when expected utility implies negative weights in WXUS, the optimal weights with ambiguity aversion are also small but positive, consistent with the results provided in Table 3. In recent periods when the sample estimate of α improves signalling that foreign stocks may be an attractive investment opportunity, the impact of ambiguity aversion

[13] Similar experiments based on a recursive, 60-month rolling window approach have given similar results that are available upon request.

[14] For $\sigma_\alpha = 0$ or modest levels of doubt on the size of the misspecifications affecting the domestic CAPM, the expected utility and the smooth ambiguity models generate similar relative portfolio weights, as seen in Table 1. Hence, we perform our discussion by focussing on the case of $\sigma_\alpha = 0.1$.

still dominates and the implied relative weights in the WXUS portfolio remain at low levels.

4.5 Extension to multi-asset problems

The results illustrated above are stark because derived within a simple two-asset toy model in which the risky portfolio choice involves only US, domestic stocks vs. foreign ones. However, the analysis above can be extended to multiple foreign stock index portfolios. We therefore consider an extension of the asset menu to four major foreign stock portfolios, Japan (JP) "Pacific-except-Japan" (PAXJP), United Kingdom (UK) and "Europe-except-UK" (EUXUK). As before, the first 36 monthly observations are used to estimate the prior parameters. In particular, the parameter s^2 in (5) is set to the average of the diagonal elements of the sample estimate of the residual covariance matrix, which is equal to 0.0031. We run the multivariate regression of the excess dollar returns on the PAXJP, JP, EUXUK and UK portfolios onto the excess returns on the VWUS portfolio, using standard MSCI total return data from January 1991 to December 2008.[15] The regression estimates for PAXJP, JP, EUXUK and UK, respectively, are

$$\hat{\alpha} = \begin{bmatrix} 2.16 & -2.64 & 2.64 & 0.60 \end{bmatrix}' \text{ (percent per year)},$$
$$\hat{\beta} = \begin{bmatrix} 0.87 & 0.56 & 0.85 & 0.71 \end{bmatrix}'.$$

Throughout the analysis, the prior mean for α is set to $\mathbf{0}$ for all foreign stock portfolios. To achieve additional realism, portfolio weights are constrained to be nonnegative in solving the optimization problem. Panel A of Table 5 reports the optimal and relative weights on all stock portfolios for an expected utility investor who is neutral to ambiguity. Similar to the two-asset case, the investor must possess a strong belief in the domestic CAPM in order to justify the low foreign equity holdings observed in the data. In particular, only when $\sigma_\alpha \leq 0.01$ we obtain that the relative weight of domestic, US stocks exceeds the 80-85% that is typically found in the data. For higher values of σ_α, such a relative weight declines towards 60 percent, that is more in line with the implications of a standard international CAPM (or even simply mean-variance) model. When the investor has low confidence in the CAPM (e.g., $\sigma_\alpha = 0.2$), the relative weight in US stocks drops to 60 percent, and the allocation to PAXJP and EUXUK together accounts for 40 percent, which is a rather strong foreign tilt when compared to the typical US portfolios. The optimal weights in JP and UK are equal to zero because the sample estimates of α are rather low for these country portfolios (-2.6 and 0.6 percent, respectively).

On the other hand, Panel B of Table 5 shows that the effect of ambiguity aversion ($\gamma = 5$

[15] Because stock returns are all expressed in US dollars, excess returns are computed with reference to the 1-month US T-bill as before. Using unhedged, US dollar returns seems to be typical of much literature on the home country bias, see e.g., Ahearne et al. (2004) and Tesar and Werner (1995).

and $\eta = 60$) are remarkable irrespective of the multi-asset nature of the problem solved. Even if an ambiguity-averse investor is willing to invest in foreign stocks such as PAXJP and EUXUK, their weights in the optimal risky portfolio are substantially smaller than under expected utility. Regardless of the choice of the "doubt parameter" σ_α, the relative weight in US stocks accounts for over 90 percent, consistent with our findings in the two-assets case. Interestingly however, ambiguity aversion fails to increase the overall amount of spread of the low holdings (7-8% at most) of foreign stocks among the available international portfolios.[16]

5 Portfolio Choice under Regime Switching

In this section, we extend our analysis to an alternative, more realistic statistical model of the dynamics of relationship among benchmark and asset returns that features regime shifts, i.e., discontinuities/time instability in either the matrix of loadings $\mathbf{B}$ or in the idiosyncratic risk matrix, $\mathbf{\Sigma}$. As we have discussed in the Introduction, previous studies (e.g., Guidolin and Timmermann (2007) and Guidolin and Timmermann (2008)) have shown that regime switching in investment opportunities is important in portfolio decisions when an investor has long horizons. Guidolin and Timmermann (2008) show that in fact regime switching in itself may generate priced co-skewness and co-kurtosis factors in an extended four-moment international CAPM that may go a long way towards explaining the observed home country bias in equity portfolios. Their intuition is that (negative) co-skewness and co-kurtosis would be spiking exactly when their diversification is most needed and this would drive a general under-diversification result that affects not only the bear regimes—when cross-asset correlations are high—but also the bull ones.[17] Tu (2010) considers regime switching in a Bayesian framework that accommodates asset pricing model uncertainty as well as parameter uncertainty. Given that investment opportunities are time-varying, both frameworks have the potential to generate not only a degree of international portfolio diversification comparable to the available evidence, but also rich time-variation in the extent of such under-diversification. Here, we embed smooth ambiguity preferences into the model proposed by Tu (2010) in order to study asset allocation under regime switching and ambiguity aversion. We have two goals in mind. First, because previous literature has argued that regimes may yield

[16] For instance, while in the by now standard case of $\sigma_\alpha = 0.1$, we have that an expected utility investor would invest approximately 116% of her wealth in stocks (i.e., she would borrow 16% of her initial wealth at the riskless rate), of which 60% of the total risky portfolio should be committed to US stocks, 30% to continental European stocks, and 10% to Pacific stocks. No wealth would be directed to investments in UK or Japanese stocks. On the contrary, an ambiguity-averse investor would invest approximately 114% of her wealth in stocks (i.e., she would borrow 14% at the riskless rate), of which 93% of her total risky portfolio should be committed to US stocks, 7% to continental European stocks, and a very tiny fraction to Pacific stocks.

[17] However, in Guidolin and Timmermann (2008) the extent of under-diversification remains considerably stronger in the bear than in the bull states and it is questionable whether there is sufficient evidence of strong time variation of the extent of home equity bias in the data.

substantial under-diversification in realistic rational asset pricing models with alternative Markov states, it is important to verify that the equally realistic assumption of ambiguity averse investors will not cause this result to unravel. Second, we conjecture that KMM-type smooth ambiguity has the potential to *persistently* raise the level of international under-diversification to the levels typically observed, *both in bad and good states.*

When regime switching dynamics is taken into account, we consider the multivariate regression of the non-benchmark returns on the benchmark returns to be

$$\mathbf{R}_t^e = \alpha^{z_t} + \mathbf{B}_2^{z_t}\mathbf{X}_t + \mathbf{u}_t \tag{8}$$

where α^{z_t} is an $N \times 1$ vector, $\mathbf{B}_2^{z_t}$ is an $N \times K$ matrix of beta exposures to the benchmark factors/portfolios, and $\mathbf{u}_t$ is an $N \times 1$ disturbance vector with zero mean and non-singular covariance matrix $\mathbf{\Sigma}^{z_t}$. Note that differently from the simple, single-state regression framework in (3), the mispricing vector α, the non-benchmark assets' beta matrix $\mathbf{B}_2$, and the covariance matrix $\mathbf{\Sigma}$ all depend on a unobservable state z_t. For simplicity, we assume the simplest possible state space, $Z = \{1, 2\}$, and $z_t \in S$. The transition probability matrix is given by

$$\Pi = \begin{pmatrix} P & 1-P \\ 1-Q & Q \end{pmatrix}$$

where $P = \Pr(z_t = 1|z_{t-1} = 1)$ and $Q = \Pr(z_t = 2|z_{t-1} = 2)$. It can be shown that the above specification is equivalent to a Markov switching model where excess returns have multivariate normal distributions conditional on each of the two regimes.[18]

To allow for mispricing uncertainty, we assume the following Normal-Inverted Wishart priors:

$$\begin{aligned} \alpha|\,\mathbf{\Sigma} &\sim N\left(\mathbf{0}, \frac{\sigma_\alpha^2}{(s^{z_t})^2}\mathbf{\Sigma}^{z_t}\right) \\ \mathbf{\Sigma} &\sim IW\left((s^{z_t})^2(v-N-1)\,\mathbf{I}_N, v\right), \quad v > N+1 \end{aligned}$$

The prior mean of $\mathbf{\Sigma}^{z_t}$ is therefore regime-dependent and is given by $(s^{z_t})^2\,\mathbf{I}_N$. Similarly to the case without regimes, the value of $(s^{z_t})^2$ is set to equal the average of the diagonal elements of the sample estimate of $\mathbf{\Sigma}^{z_t}$ in the quantitative analysis that follows. The parameter σ_α quantifies also in this case the level of mispricing uncertainty. For simplicity, we assume that the prior mean of α is equal to a $N \times 1$ vector of zeros. It follows that $\sigma_\alpha = 0$ corresponds to complete confidence in the domestic CAPM even under regime switching. As usual, we impose an uninformative prior on $\mathbf{B}_2^{z_t}$ conditional on each state. The benchmark returns are normally distributed within each regime:

$$\mathbf{F}^{z_t} \sim N\left(\mathbf{E}_F^{z_t}, \mathbf{V}_F^{z_t}\right).$$

[18] See Tu (2010) for the exact relationship of parameters between the two model representations.

The priors on $E_F^{z_t}$ and $V_F^{z_t}$ are also diffuse:

$$p\left(\mathbf{E}_F, \mathbf{V}_F\right) \propto |\mathbf{V}_F|^{-(K+1)/2}$$

Following Chib (1996), we let the priors on P and Q be a Dirichlet distribution. Further details on the numerical solution methods and the statistical frameworks are reported in Appendix B.

5.1 Calibration

We use a Gibbs sampling procedure developed by Tu (2010) to simulate the posterior distributions of the parameters $\{\alpha^{z_t}, \mathbf{B}_2^{z_t}, \mathbf{\Sigma}^{z_t}, \mathbf{E}_F^{z_t}, \mathbf{V}_F^{z_t}\}$. Notice that the predictive distribution of the benchmark returns $\mathbf{F}$ is also regime-dependent. Since the state $\{z_t\}$ is unobservable, we follow Chib (1996) and simulate states $\{z_t\}_{t=1}^{T}$ for each observation of the data. The full sample data are then grouped into two sets, according to the associated states. This procedure is repeated many times, and thus we can draw $\{\alpha^{z_t}, \mathbf{B}_2^{z_t}, \mathbf{\Sigma}^{z_t}, \mathbf{E}_F^{z_t}, \mathbf{V}_F^{z_t}\}$ from their posterior densities and $\mathbf{F}^{z_t}$ from its predictive density in each regime.

To start the Gibbs sampling procedure, we need prior values on the parameters. These values are set at the corresponding (maximum likelihood style) expectation maximization estimates of a multivariate regime-switching model of excess returns on both the WXUS and VWUS portfolios.[19] The estimation results suggest that the prior beliefs on P and Q are centered around 0.983 and 0.974 respectively, which—after labelling the regimes as bull (1) and bear (2)—realistically suggests that bull states are more persistent (their average duration is 59 months) than bear states are (approximately 39 months). In any event, the posterior distributions confirm an overwhelming evidence of regimes in our asset return data, as confirmed by standard posterior odds ratio comparing two- with single-state models. The monthly estimates of mean excess returns on the WXUS and US portfolios are $\{1.59, 1.01\}$ percent in the bull regime (regime 1) and $\{0.22, 0.44\}$ percent in the bear regime (regime 2, respectively). Moreover, the monthly standard deviation estimates of the WXUS and US portfolios are $\{3.16, 2.32\}$ percent in the bull regime and $\{5.61, 5.22\}$ percent in the bear regime, and the correlations between the two portfolios in the bull and bear regimes are 0.59 and 0.80 respectively. These estimates suggest that there are two distinctive regimes for the US and international stock returns, one with high mean returns, low volatilities and a moderate correlation whereas the other with low mean returns, high volatilities and a high correlation.

[19] Our results are robust to different values of the prior parameters.

5.2 Under-diversification results

As shown by previous studies (Ang and Bekaert (2002), Guidolin and Timmermann (2008) and Tu (2010)), optimal portfolio weights under regime switching differ from those implied by the single regime models in important ways. Since the primitive parameters describing returns processes depend on regimes, the implied optimal portfolio weights are regime-dependent. In addition, parameter uncertainty and mispricing uncertainty also affect the investor's portfolio choice in Bayesian context, see in particular Tu (2010). In this paper we extend this well-known results to an analysis of the effects of ambiguity aversion on the extent of portfolio under-diversification in each of the two regimes. We solve for the optimal portfolio weights numerically in each regime under expected utility and smooth ambiguity utility. The optimal weights are constrained to be nonnegative. The numerical procedure is explained in Appendix B.

Panel A of Table 6 shows the results for the benchmark expected utility case, where the risk aversion parameter γ is set at 10 to avoid implausibly large position in the optimal risky portfolio. We find that the optimal weights in the bull regime are significantly different from those in the bear regime. This is because the posterior distributions of the key parameters are drastically different in the two regimes. However, characterizing the two regimes by looking at their asset allocation implication remains non-trivial as optimal porfolio decisions under power utility remain a highly non-linear function of the posterior distributions of the parameters (via the implied predictive distributions). First, while in the bull regime a Bayesian investor would massively leverage her portfolio to invest in stocks in excess of 100%, in the bear state the same investor would adopt a very cautious stance. For instance, in the case of $\sigma_\alpha = 0.2$, in the bull state the investor borrows 98% of her initial wealth at the riskless rate (an almost perfect 1:1 leverage) to invest 116% in domestic US stocks and 82% in foreign stocks. These absolute weights yield relative weights of 59 and 41 percent in domestic and foreign stocks, respectively. In bear regime, the investor invests 14% of her initial wealth in stocks, and leaves 86% in cash, which seems appropriate when investment opportunities are on average poor, see e.g., Table 7. These absolute weights yield a striking relative weight of 100% in domestic stocks.[20] Therefore, regime switching per se (i.e., under expected utility) yields very strong under-diversification in the bear regime and a wearker bias to domestic stocks in the bull state.[21] Implicitly then, an expected utility short-horizon asset allocation model under regime switching would imply considerable time

[20] Interestingly, these results closely mimic those in Guidolin and Timmermann (2008) for the two-state case and mean-variance preferences, when the investment horizon does not matter and as such can be taken to correspond to 1-month, as in our paper. This is not surprising because over short-investment horizons, expected power utility preferences are well-known to be locally mean-variance.

[21] Yet, the increase in under-diversification is considerable, for instance, in the single-state case with $\sigma_\alpha = 0.1$ a Bayesian investor would only hold 30 of her risky portfolio or less in foreign stocks when the prior on α is diffuse.

variation in the extent of the home country bias. Also in this case, it takes a considerable faith (one may argue, unreasonable) in the domestic CAPM (i.e., $\sigma_\alpha \leq 0.005$) for the extent of home bias in the bull state to grow to levels comparable to those commonly found in the data.

To better understand these results, we present in Table 7 summary statistics (mean, minimum and maximum) of the key parameters computed from the corresponding posterior and predictive densities. Panel A of Table 7 shows the results for the case with complete confidence in the CAPM ($\sigma_\alpha = 0$) and Panel B for the case with high mispricing uncertainty ($\sigma_\alpha = 0.5$). These statistics suggest that returns on the WXUS and VWUS portfolios perform worse in the bear regime than in the bull regime in the perspective of their posterior distributions/properties. Specifically, in the bear regime posterior variances tend to be high while posterior mean returns are low. Thus, in the bear regime the investor will optimally invest less in the stock portfolios but more in the risk-free rate, as we have commented above. When we allow for high mispricing uncertainty, there arises a notable difference in the posterior means of α between the two regimes. In particular, the posterior mean of α is 0.7 percent at a monthly frequency in the bull regime while -0.2 percent in the bear regime. Thus, the poor performance of the foreign stock portfolio in the bear regime leads to a remarkably under-diversified portfolio held by a typical US investor, as observed in Table 6.

Turning to the effects of ambiguity aversion, Panel B of Table 6 presents optimal and relative weights for an investor with smooth ambiguity preferences. The results reveal that the effect of ambiguity aversion can be understood in a similar way as in the single regime model. In the bull regime, the optimal risky portfolio is significantly under-diversified, even though the mispricing uncertainty is high. For instance, when $\sigma_\alpha = 0.1$ percent per year, in the bull state the investor still invests 198% in stocks, but in this case this goes now for only 38% to domestic US stocks and with a weight of 160% in foreign stocks. These absolute weights imply relative weights of 81% in foreign equities and 19% in domestic US stocks, which are highly realistic outcomes that substantially differ from the expected utility ones. In the bear regime, the investor becomes slightly more prudent and allocates only 13% to stocks, of which 95% to domestic, US stocks and only 5% to foreign stocks. Interestingly, the outcome of complete home bias preference observed in the bear regime under expected utility does not obtain in this case: the ambiguity-averse investor still invests in the WXUS portfolio, albeit in a small amount. This is because under expected utility the optimal weight in WXUS would be negative without the nonnegative constraint. In such a case, the comparative statics analysis for the single regime model in Section 4 suggests that adding ambiguity aversion into the problem tends to increase the investment in the foreign stock portfolio to a small and positive level. The mechanism driving the observed under-diversification in the regime switching model is similar to that in the single regime model, as revealed in Panel B of Table 7 for a relatively large σ_α. In the bull regime, low conditional expected utility states are associated with pessimistic beliefs over the distributions of both benchmark and non-benchmark returns. However, when we allow for a relatively high prior degree of belief in the domestic CAPM, pessimism over the foreign stock portfolio dominates and results in an under-diversified portfolio.

Finally, we re-examine the case with multiple non-benchmark assets. We compute the optimal weights in PAXJP, JP, EUXUK, UK and VWUS in each regime. This asset menu is similar to the one investigated by Guidolin and Timmermann (2008). The optimal weights are constrained to be nonnegative. Table 8 reports results for an expected utility investor. We see that our findings regarding the impact of regime switching on under-diversification are similar to the case of only one non-benchmark asset. The bull and bear regimes feature significantly different international diversification patterns in the optimal risky portfolio. However, under multiple non-benchmark assets, we can further obtain the composition of the optimal foreign stock portfolio. Panel A of Table 8 shows that in the bull regime with high mispricing uncertainty ($\sigma_\alpha = 0.05$ or higher), the investor prefers to hold more in PAXJP than other foreign stocks. This is because the posterior mean of PAXJP's α is the highest among all the non-benchmark assets. The relative weights in JP and EUXUK are comparable in magnitude, followed by the weight attached to continental European stocks. However, also under a multiple asset menu, it remains the case that only a Bayesian investor with a very high (probably implausible) faith in the domestic CAPM (say, $\sigma_\alpha = 0.01$ or lower) would exhbit a realist degree of home bias in the bull regime. However, for the same parameters, the investor would also leverage her portfolio to a very high extent, always in excess of 142% of her initial wealth, which may be considered not entirely realistic. In addition, Panel B reveals that the effect of the bear regime on international diversification is robust to multiple non-benchmark assets: in this case, even under a very low degree of doubt on the domestic CAPM, the extent of home bias is almost perfect, i.e., close or equal to 100%.

Table 9 shows instead that the impact of ambiguity aversion is also considerable in the case of a richer and more realistic asset menu, especially in the bull regime. For instance, when $\sigma_\alpha = 0.1$ and in the bull regime, an ambiguity-averse investor will leverage her portfolio to invest 86% in US stocks, 9% in Pacific stocks, and 5% in Japanese stocks. The resulting under-diversification is therefore large. However, also in this case in the bear regime under-diversification is considerable. Implicitly, a smooth ambiguity model implies that not only the home bias in optimal equity allocation is large, but also that this will hardly fluctuate over bull and bear market phases, contrary to what a Bayesian asset allocation model under regime switching will imply.

6 Conclusion

In this paper, we have proposed a new approach to explain the substantial amount of under-diversification in international portfolio decisions that has been reported by the empirical literature. We have done that by incorporating a simple and yet powerful smooth aversion to ambiguity in a standard framework traditionally employed to examine asset allocation with parameter uncertainty and an uncertain belief in an asset pricing model. Our approach is developed by embedding the smooth ambiguity utility model of Klibanoff et al. (2005) into the Bayesian framework. The key feature of the resulting model is that unlike a Bayesian portfolio methodology, the ambiguity-averse investor in our model not only will rely on the conditional distribution of returns and the posterior over the parameters in making asset allocation decisions, but she will also place more weight on states with low conditional expected utility values when decisions are taken. In our paper, we show that this feature of her preferences will crucially alter the relative weights optimally assigned to different stocks.

We use this approach to study international asset allocation from the perspective of a US investor, who has a prior degree of belief in the global mean-variance efficiency of the US market portfolio. We find that ambiguity aversion can lead to strong home bias in equity holdings, regardless of an investor's belief in the domestic CAPM. Moreover, we extend our analysis to regime-switching investment opportunities following Ang and Bekaert (2002), Guidolin and Timmermann (2008) and Tu (2010). We find that, consistently with earlier literature, regime switching generates substantial under-diversification in bear regimes, when non-benchmark assets tend to be under-priced, also the benchmark ones are expected to yield relatively low returns, and both volatilities and correlations are higher than on the average. However, under-diversification is milder in the bull regime and tends to obtain only when a Bayesian investor places a lot of faith in the conditional CAPM, that is almost the same as assuming home bias on an ex-ante basis (notice that if the domestic CAPM holds, then the US equity portfolio is mean-variance efficient). More importantly, when smooth ambiguity is introduced in the model, it persistently raises the level of international under-diversification to the levels typically observed, both in bad and good states.

Appendix

In Appendix A and B, we describe the numerical procedures that we use to solve for optimal portfolio weights in the single regime model and the regime switching model, respectively. Appendix C describes the calibration of the ambiguity aversion parameter.

A: Numerical algorithm for the single regime model

This numerical algorithm consists of three steps. First, we employ a sampling procedure to simulate the parameters describing investment opportunities. Second, we compute the value function given the simulated parameters and a portfolio choice, using numerical integration and Monte Carlo simulation. Third, we use a numerical optimizer to solve the optimization problem with multiple control variables.

We use the sampling procedure proposed by Pastor (2000) to simulate $\{\alpha, \mathbf{B}_2, \mathbf{\Sigma}, \mathbf{E}_F, \mathbf{V}_F\}$ from their posterior densities and the vector of factors $\mathbf{F}$ from its predictive density. The predictive density of the benchmark returns $\mathbf{F}$ is a multivariate Student-t with $T-K$ degrees of freedom (see Zellner (1971)). Given the data on the benchmark returns, we compute the statistics $\hat{\mathbf{E}}_F$ and $\hat{\mathbf{V}}_F$ as follows:

$$\begin{aligned}\hat{\mathbf{E}}_F &= \frac{1}{T}\sum_{t=1}^{T}\mathbf{F}_t' \\ \hat{\mathbf{V}}_F &= \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{F}_t - \hat{\mathbf{E}}_F\right)'\left(\mathbf{F}_t - \hat{\mathbf{E}}_F\right)\end{aligned}$$

The posterior of $\mathbf{V}_F^{-1}$ is a Wishart distribution with parameter matrix $\left(T\hat{\mathbf{V}}_F\right)^{-1}$ and $T-1$ degrees of freedom. The posterior of $\mathbf{E}_F$ given $\mathbf{V}_F$ is normal with mean $\hat{E}_F$ and covariance matrix $\mathbf{V}_F/T$. Thus, we first draw $\mathbf{V}_F$ and $\mathbf{E}_F$ from their posteriors and then draw $\mathbf{F}_{T+1}$ from its normal density with mean $\mathbf{E}_F$ and covariance matrix $\mathbf{V}_F$.

The joint posterior density of $\mathbf{B}$ and $\mathbf{\Sigma}$ is

$$p\left(\mathbf{B}, \mathbf{\Sigma}|\Phi\right) \propto |\mathbf{\Sigma}|^{-(T+v+N+2)/2}\exp\left\{-\frac{1}{2}\left(\beta-\bar{\beta}\right)'\mathbf{\Omega}^{-1}\left(\beta-\bar{\beta}\right)\right\}\exp\left\{-\frac{1}{2}tr(\mathbf{\Sigma}^{-1}\mathbf{C})\right\}$$

where the parameter matrix $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{S} + \mathbf{H} + \left(\alpha-\bar{\alpha}\right)\left(\alpha-\bar{\alpha}\right)' s^2/\sigma_\alpha^2 + \left(\mathbf{B}-\hat{\mathbf{B}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{B}-\hat{\mathbf{B}}\right)$$

in which $\mathbf{S} \equiv (\mathbf{R}-\mathbf{X}\hat{\mathbf{B}})'(\mathbf{R}-\mathbf{X}\hat{\mathbf{B}})$, $\hat{\mathbf{B}}' = \begin{bmatrix} \hat{\alpha} & \hat{\mathbf{B}}_2 \end{bmatrix} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{R}$ and $\mathbf{H} \equiv s^2\left(v-N-1\right)\mathbf{I}_N$.

Since the prior covariance matrix of β is set such that there is no prior information about the assets' betas, the marginal posterior density of $\mathbf{B} = \left[\begin{array}{cc} \alpha & \mathbf{B}_2 \end{array} \right]'$ is

$$p\left(\mathbf{B}|\Phi\right) \propto \left|\mathbf{S} + \mathbf{H} + \left(\alpha - \bar{\alpha}\right)\left(\alpha - \bar{\alpha}\right)' \frac{s^2}{\sigma_\alpha^2} + (\mathbf{B} - \hat{\mathbf{B}})'\mathbf{X}'\mathbf{X}(\mathbf{B} - \hat{\mathbf{B}})\right|^{-\frac{T+v+1}{2}}$$

The conditional posterior of $\mathbf{\Sigma}$ given $\mathbf{B}$ is Inverted Wishart with parameter matrix $\mathbf{C}$ and $T+v+1$ degrees of freedom:

$$p\left(\mathbf{\Sigma}|\mathbf{B}, \Phi\right) \propto |\mathbf{\Sigma}|^{-\frac{T+v+N+2}{2}} \exp\left\{-\frac{1}{2} tr(\mathbf{\Sigma}^{-1}\mathbf{C})\right\}$$

The components of $\mathbf{B} \equiv \left[\begin{array}{cc} \alpha & \mathbf{B}_2 \end{array} \right]'$ can be drawn using Gibbs sampler. To do so, we draw α and $\mathbf{B}_2$, respectively, from the conditional posterior density $p\left(\alpha|\mathbf{B}_2, \Phi\right)$ and $p\left(\mathbf{B}_2|\alpha, \Phi\right)$. The conditional posterior $p\left(\alpha|\mathbf{B}_2, \Phi\right)$ takes the form

$$\begin{aligned}
p\left(\alpha|\mathbf{B}_2, \Phi\right) &\propto \left|1 + c^*\left(\alpha - \alpha^*\right)' \mathbf{D}^{-1}\left(\alpha - \alpha^*\right)\right|^{-\frac{T+v+1}{2}} \\
c^* &\equiv T + \frac{s^2}{\sigma_\alpha^2} \\
\alpha^* &\equiv \frac{T\hat{\alpha} + s^2\bar{\alpha}/\sigma_\alpha^2 - (\mathbf{B}_2 - \hat{\mathbf{B}}_2)'\left(\mathbf{F}^T\right)'\iota_T}{T + s^2/\sigma_\alpha^2} \\
\mathbf{D} &\equiv \mathbf{S} + \mathbf{H} + T\hat{\alpha}\hat{\alpha}' + \bar{\alpha}\bar{\alpha}' s^2/\sigma_\alpha^2 - \hat{\alpha}\iota_T'\mathbf{F}^T(\mathbf{B}_2 - \hat{\mathbf{B}}_2) - (\mathbf{B}_2 - \hat{\mathbf{B}}_2)'\left(\mathbf{F}^T\right)'\iota_T\hat{\alpha}' + \\
&\quad + (\mathbf{B}_2 - \hat{\mathbf{B}}_2)'\left(\mathbf{F}^T\right)'\left(\mathbf{F}^T\right)(\mathbf{B}_2 - \hat{\mathbf{B}}_2) - c^*\alpha^*\left(\alpha^*\right)'
\end{aligned}$$

which is a multivariate Student-t distribution with mean α^*, covariance matrix $\mathbf{D}/\left(c^*\left(T + v - N - 1\right)\right)$, and $(T + v - N - 1)$ degrees of freedom.

The conditional posterior $p\left(\mathbf{B}_2|\alpha, \Phi\right)$ takes the form

$$\begin{aligned}
p\left(\mathbf{B}_2|\alpha, \Phi\right) &\propto \left|\mathbf{A} + \left(\mathbf{B}_2 - \mathbf{B}_2^*\right)' \mathbf{M}^*\left(\mathbf{B}_2 - \mathbf{B}_2^*\right)\right|^{-\frac{T+v+1}{2}} \\
\mathbf{M}^* &\equiv \left(\mathbf{F}^T\right)'\left(\mathbf{F}^T\right) \\
\mathbf{B}_2^* &\equiv \hat{\mathbf{B}}_2 + \left(\mathbf{M}^*\right)^{-1}\left(\mathbf{F}^T\right)'\iota_T\left(\hat{\alpha} - \alpha\right)' \\
\mathbf{A} &\equiv \mathbf{S} + \mathbf{H} + \left(\alpha - \bar{\alpha}\right)\left(\alpha - \bar{\alpha}\right)' s^2/\sigma_\alpha^2 + T\left(\alpha - \hat{\alpha}\right)\left(\alpha - \hat{\alpha}\right)' - \hat{\mathbf{B}}_2'\left(\mathbf{F}^T\right)'\iota_T\left(\alpha - \hat{\alpha}\right)' + \\
&\quad - \left(\alpha - \hat{\alpha}\right)\iota_T'\mathbf{F}^T\hat{\mathbf{B}}_2 + \hat{\mathbf{B}}_2'\mathbf{M}^*\hat{\mathbf{B}}_2 - \left(\mathbf{B}_2^*\right)'\mathbf{M}^*\mathbf{B}_2^*
\end{aligned}$$

which is a generalized multivariate Student-t distribution. To draw $\mathbf{B}_2$ from its density, we first draw a matrix, denoted by $\mathbf{G}$, from an Inverted Wishart distribution with parameter matrix $\mathbf{A}$ and $T + v + 1 - K$ degrees of freedom. Then we sample $\mathbf{B}_2$ from a normal density with mean $\mathbf{B}_2^*$ and covariance matrix $\mathbf{G}\left(\mathbf{M}^*\right)^{-1}$.

We initialize the Gibbs chain at $\mathbf{B}_2 = \hat{\mathbf{B}}_2$ and repeat draws of α and $\mathbf{B}_2$ from their conditional posterior densities. Given each draw of $\mathbf{B}$, $\mathbf{\Sigma}$ is drawn from its conditional posterior density. The number of random draws is 50,000, after an initial burn-in stage with 2,000 draws.

In this paper, we consider a single benchmark asset ($N = 1$) and the vector β contains the non-benchmark assets' betas. For each set of simulated parameters $\theta \equiv (\alpha, \beta, \mathbf{F}, \boldsymbol{\Sigma}, \mathbf{E}_F, \mathbf{V}_F)$, we compute the inner expectation in the value function (1). The conditional expectation $\mathbb{E}_\pi \left[u \left(W_{T+1} \right) | \theta \right]$ can be approximated by numerical integration. Note that $\mathbf{F}_{T+1}$ is normal conditional on $\mathbf{E}_F$ and $\mathbf{V}_F$ and that $\mathbf{R}_{T+1}$ is normal conditional on α, β, $\mathbf{F}_{T+1}$ and $\boldsymbol{\Sigma}$. In addition, the benchmark returns are assumed to be independent of the disturbance matrix $\mathbf{U}$ in (3). For $N = 1$, the expectation $\mathbb{E}_\pi \left[u \left(W_{T+1} \right) | \theta \right]$, which is given by

$$\mathbb{E}_\pi \left[u \left(W_{T+1} \right) | \theta \right] = \int_{\mathbf{R}^e_{T+1}} \int_{\mathbf{F}_{T+1}} \frac{\left[\left(1 - \psi \right) R_f + \psi \omega' \left(\tilde{\mathbf{R}}^e_{T+1} + R_f \right) \right]^{1-\gamma}}{1 - \gamma} d\mathbf{F}_{T+1} d\mathbf{R}^e_{T+1},$$

can be approximated by two-dimensional Gaussian quadrature with 5 nodes, where $\tilde{\mathbf{R}}^e_{T+1} \equiv \left[\begin{array}{cc} \mathbf{R}^e_{T+1} & \mathbf{F}_{T+1} \end{array} \right]$ denotes the returns on all investable assets, and the initial wealth W_T is assumed to be 1.[22] For $N > 1$, the dimension of numerical integration is large, and thus the curse of dimensionality makes Gaussian quadrature method unappealing in practical applications. In this case, we resort to non-product monomial rules to approximate high dimensional integrals (see Judd (1998)). A degree 5 monomial rule is suitable for integration with relatively large dimension and a degree 7 monomial rule for integration with extremely large dimension (for instance, $n = 25$). The degree 5 and 7 monomial rules use, respectively, $2n^2 + 1$ and $2n$ grid points, where n is the dimension of integration.

Finally, we compute the value function using Monte Carlo methods and employ a numerical optimizer to solve for optimal portfolio weights. In the expected utility case, we follow the standard approach and simulate returns on investable assets from their respective predictive densities. The optimal portfolio weights are obtained by solving the optimization

$$\max_{\psi, \omega} \frac{1}{M} \sum_{m=1}^{M} \frac{\left[\left(1 - \psi \right) R_f + \psi \omega' \left(\tilde{\mathbf{R}}^e_{T+1} + R_f \right) \right]^{1-\gamma}}{1 - \gamma}$$

where M is the number of draws in Gibbs sampling.

B: Numerical algorithm for the regime switching model

We briefly describe a Gibbs sampling procedure used to solve Bayesian portfolio problems under regime switching, when there are two distinct regimes. Interested readers may consult Tu (2010) for additional details. The numerical algorithm consists of three stages. First, we simulate the states $\{z_t\}_{t=1}^T$ using the approach proposed by Chib (1996), and group the full sample data into two sets according to the associated states. Specifically, we obtain two groups of data $\tilde{\mathbf{R}}^e|_i = \left[\begin{array}{cc} \mathbf{R}^e|_i & \mathbf{F}|_i \end{array} \right]$ $i = 1, 2$ where $\mathbf{R}^e|_i = \{\mathbf{R}^e_t | z_t = i\}$ is a $T^i \times N$ matrix, $\mathbf{F}|_i = \{\mathbf{F}_t | z_t = i\}$

[22] We find that increasing the number of quadrature nodes further does not significantly change our quantitative results.

a $T^i \times K$ matrix (in this paper $K = 1$), and T^i denotes the number of observations in regime i. Second, we draw parameters $\left(\alpha^i, \mathbf{B}_2^i, \mathbf{F}^i, \boldsymbol{\Sigma}^i, \mathbf{E}_F^i, \mathbf{V}_F^i\right)$ in regime i based on a Gibbs sampling procedure. Third, we solve the optmization problem in each regime, using numerical integration and Monte Carlo method.

We assume that the prior distributions of $(P, 1-P)$, and $(1-Q, Q)$ are two independent Dirichlet distributions on the two-dimensional simplex as in Chib (1996) and Tu (2010):

$$(P, 1-P) \sim D\left(d_{p1}, d_{p2}\right), \quad (1-Q, Q) \sim D\left(d_{q1}, d_{q2}\right).$$

where the constants d_{p1}, d_{p2}, d_{q1} and d_{q2} are set to deliver the required prior belief on regimes.

We let $\mathbf{b}^i \equiv [\alpha^i \; \beta^i]'$. Combining the likelihood of the returns data (not shown here) with the prior on $\left(\mathbf{b}^i, \boldsymbol{\Sigma}^i\right)$ and $\left(\mathbf{E}_F^i, \mathbf{V}_F^i\right)$, it can be shown that the joint posterior of the regression parameters $\left(\mathbf{b}^i, \boldsymbol{\Sigma}^i\right)$ is (see Pastor and Stambaugh (2000))

$$\begin{aligned} p\left(\mathbf{b}^i, \boldsymbol{\Sigma}^i | \Phi\right) \;\propto\; & \left|\boldsymbol{\Sigma}^i\right|^{-(K+1)/2} \exp\left\{-\frac{1}{2}\left[(\mathbf{b}^i - \tilde{\mathbf{b}}^i)'\left(\left(\boldsymbol{\Sigma}^i\right)^{-1} \otimes \mathbf{F}^i\right)(\mathbf{b}^i - \tilde{\mathbf{b}}^i)\right]\right\} \\ & \times \left|\boldsymbol{\Sigma}^i\right|^{-\left(T^i+v+N-K+1\right)/2} \exp\left\{-\frac{1}{2} tr\left[\left(\mathbf{H}^i + T^i \hat{\boldsymbol{\Sigma}}^i + \left(\hat{\mathbf{B}}^i\right)' \mathbf{Q}^i \hat{\mathbf{B}}^i\right)\left(\boldsymbol{\Sigma}^i\right)^{-1}\right]\right\} \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathbf{F}}^i &\equiv \tilde{\mathbf{D}}^i + \left(\mathbf{X}^i\right)' \mathbf{X}^i \\ \tilde{\mathbf{b}}^i &\equiv \left(\mathbf{I}_N \otimes \left(\tilde{\mathbf{F}}^i\right)^{-1} \left(\mathbf{X}^i\right)' \mathbf{X}^i\right) \hat{\mathbf{b}}^i \\ \mathbf{Q}^i &\equiv \left(\mathbf{X}^i\right)' \mathbf{X}^i - \left(\mathbf{X}^i\right)' \mathbf{X}^i \left(\tilde{\mathbf{F}}^i\right)^{-1} \left(\mathbf{X}^i\right)' \mathbf{X}^i \\ \mathbf{H}^i &\equiv \left(s^i\right)^2 (v - N - 1)\, \mathbf{I}_N \end{aligned}$$

and $\tilde{\mathbf{D}}^i$ is a $(K+1) \times (K+1)$ matrix with $(1,1)$ element being $\left(s^i\right)^2 / \sigma_\alpha^2$ and other elements being zeros. Thus, the posterior of $\left(\boldsymbol{\Sigma}^i\right)^{-1}$ is Wishart:

$$\left(\boldsymbol{\Sigma}^i\right)^{-1} | \Phi \sim W\left(T^i + v - K, \left(\mathbf{H}^i + T^i \hat{\boldsymbol{\Sigma}}^i + \left(\hat{\mathbf{B}}^i\right)' \mathbf{Q}^i \hat{\mathbf{B}}^i\right)^{-1}\right)$$

and the conditional posterior of $\mathbf{b}^i$ is normal:

$$\mathbf{b}^i \left| \left\{ \left(\boldsymbol{\Sigma}^i\right)^{-1}, \Phi \right\} \right. \sim N\left(\tilde{\mathbf{b}}^i, \boldsymbol{\Sigma}^i \otimes \left(\tilde{\mathbf{F}}^i\right)^{-1}\right)$$

In addition, we have

$$
\begin{aligned}
\left(\mathbf{V}_F^i\right)^{-1}\Big|\,\Phi &\sim W\left(T^i-1,\left(T^i\hat{\mathbf{V}}_F^i\right)^{-1}\right)\\
\mathbf{E}_F^i\big|\left\{\mathbf{V}_F^i,\Phi\right\} &\sim N\left(\hat{\mathbf{E}}_F^i,\frac{\mathbf{V}_F^i}{T^i}\right)\\
\mathbf{F}^i\big|\left\{\mathbf{E}_F^i,\mathbf{V}_F^i\right\} &\sim N\left(\mathbf{E}_F^i,\mathbf{V}_F^i\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\mathbf{E}}_F^i &= \frac{1}{T^i}\sum_{t=1}^{T^i}\left(\mathbf{F}_t^i\right)'\\
\hat{\mathbf{V}}_F^i &= \frac{1}{T^i}\sum_{t=1}^{T^i}\left(\mathbf{F}_t^i-\hat{\mathbf{E}}_F^i\right)'\left(\mathbf{F}_t^i-\hat{\mathbf{E}}_F^i\right)
\end{aligned}
$$

The posterior distributions of P and Q are (see Chib (1996))

$$
(P,1-P)\sim D\left(S_{11}+d_{p1},S_{12}+d_{p2}\right),\quad (1-Q,Q)\sim D\left(S_{21}+d_{q1},S_{22}+d_{q2}\right)
$$

where S_{ij}, $i,j=1,2$, is the total number of one-step transitions from regime i to regime j.

The Gibbs sampling procedure can be summarized as follows:

1) simulate states $\{z_t\}_{t=1}^T$ for each observation and group the full sample data into two sets according to the associated states;

2) in regime i, draw the parameters $\mathbf{\Sigma}^i$, $\mathbf{b}^i$, $\mathbf{V}_F^i$ and $\mathbf{E}_F^i$ from their posterior densities and $\mathbf{F}^i$ from a normal density with mean $\mathbf{E}_F^i$ and covariance matrix $\mathbf{V}_F^i$;

3) simulate P and Q;

4) repeat the steps above.

The number of draws is 50,000, after an initial burn-in stage with 2,000 draws. Afer the Gibbs sampling algorithm is terminated, we solve the optimization problem conditional on each regime, using numerical integration and Monte Carlo methods. Note that in regime i, $i=1,2$, $\mathbf{F}_{T+1}^i$ is normal conditional on $\mathbf{E}_F^i$ and $\mathbf{V}_F^i$ and that $\mathbf{R}_{T+1}^e|_i$ is normal conditional on $\mathbf{b}^i$, $\mathbf{F}_{T+1}^i$, and $\mathbf{\Sigma}^i$.

C: Calibrating the ambiguity avesion parameter

We use thought experiments similar to those used in the classical statement of the Ellsberg paradox to calibrate the ambiguity aversion parameter. Suppose there are two urns filled with black and white balls. Subjects are told that one urn contains 50 white and 50 black balls. The second urn contains 100 balls, which are either white or black. The exact composition of the second urn is unknown to the subjects. Subjects are asked to place a bet on the color of the ball drawn from each urn. The bet could be on either black or white. Subjects win a prize worth d

dollars if a bet on a specific urn is correct; otherwise they do not win or lose anything. Halevy (2007) reports that the majority of subjects prefer a bet on the first urn over the second urn.

The standard expected utility framework fails to explain this behavior, regardless of the degree of risk aversion or beliefs held by the subjects because the certainty equivalents of the two bets are identical for the two urns when subjects simply compute expected utility. Ambiguity aversion, however, can create a difference between the certainty equivalents over the two urns, which gives rise to the ambiguity premium. The ambiguity premium is formally defined as (see Chen et al. (2011))

$$u^{-1}\left(\int_{\Theta}\int_{S} u(c)d\pi_{\theta}d\mu\left(\theta\right)\right)-\phi^{-1}\left(\int_{\Theta}\phi\left(u^{-1}\left(\int_{S}u\left(c\right)d\pi_{\theta}\right)\right)d\mu\left(\theta\right)\right)$$

In our model, the functional forms of u and ϕ are given by

$$\begin{aligned} u\left(W\right) &= \frac{W^{1-\gamma}}{1-\gamma}, \quad \gamma>0,\neq 1 \\ \phi\left(x\right) &= -\frac{1}{\eta}e^{-\eta x}, \quad \eta>0 \end{aligned}$$

We specify the set of probability distributions for the bet on the second urn as (0,1) and (1,0) ($\Theta=\{(0,1),(1,0)\}$), and the subjective prior as $\mu=(0.5,0.5)$. Let w be the subject's wealth level. The ambiguity premium is defined as

$$\left(0.5(d+w)^{1-\gamma}+0.5w^{1-\gamma}\right)^{\frac{1}{1-\gamma}}-\left(-\frac{1}{\eta}\ln\left(-\left[0.5\left(-e^{-\eta(d+w)}\right)+0.5\left(-e^{-\eta w}\right)\right]\right)\right)$$

Obviously, the size of this premium naturally depends on the prize-wealth ratio d/w and the degrees of risk aversion and ambiguity aversion. Table 0 presents the ambiguity premium, expressed as a percentage of the expected value of the bet $d/2$, for various values of η and γ. We consider three different prize-wealth ratios, $d/w=1\%$ (Panel A), $d/w=0.75\%$ (Panel B) and $d/w=0.5\%$ (Panel C). All else being equal, a smaller bet or a higher degree of risk aversion is associated with a lower ambiguity premium. In the light of previous experimental evidence (see Camerer (1999) and Halevy (2007)), the ambiguity premium is on the order of 10-20 percent of the expected value of a bet. Table 0 shows that the implied ambiguity premium for various values of the ambiguity aversion parameter η is consistent with the experimental results documented in the literature. Thus, the values of η used in the calibration of this paper are plausible.

Table C1: **Ambiguity premia for alternative choices of γ and η**

$\gamma \diagdown \eta$	20	30	40	50	60	80	100
Panel A: Prize-wealth ratio=1%							
2	0.045	0.070	0.094	0.119	0.143	0.190	0.235
5	0.037	0.062	0.087	0.111	0.135	0.182	0.228
10	0.025	0.050	0.074	0.099	0.123	0.170	0.215
Panel B: Prize-wealth ratio=0.75%							
2	0.034	0.052	0.071	0.089	0.108	0.144	0.180
5	0.028	0.047	0.065	0.084	0.102	0.138	0.174
10	0.019	0.037	0.056	0.075	0.093	0.129	0.165
Panel C: Prize-wealth ratio=0.50%							
2	0.022	0.035	0.047	0.060	0.072	0.097	0.121
5	0.019	0.031	0.044	0.056	0.068	0.093	0.117
10	0.013	0.025	0.037	0.050	0.062	0.087	0.111

This table reports the ambiguity premium, expressed as a percentage of the expected value of a bet, $d/2$, for various values of the ambiguity aversion parameter η, the risk aversion parameter γ, and the prize-wealth ratio d/w.

Table 1: **Optimal and relative portfolio weights:** $\bar{\alpha} = 0$

σ_α	0	0.005	0.01	0.02	0.05	0.1
Panel A: Expected utility $\gamma = 5$						
			Optimal weights			
WXUS	1.03	10.55	25.25	40.88	49.74	51.31
VWUS	66.14	57.68	44.66	30.77	22.92	21.54
			Relative weights			
WXUS	1.53	15.46	36.12	57.06	68.45	70.44
VWUS	98.47	84.54	63.88	42.94	31.55	29.56
Panel B: KMM preferences $\gamma = 5, \eta = 15$						
			Optimal weights			
WXUS	16.86	17.86	19.44	21.12	22.07	22.24
VWUS	67.21	67.19	67.16	67.17	67.14	67.14
			Relative weights			
WXUS	20.06	21.00	22.45	23.92	24.73	24.88
VWUS	79.94	79.00	77.55	76.08	75.27	75.12
Panel C: KMM preferences $\gamma = 5, \eta = 60$						
			Optimal weights			
WXUS	5.44	5.78	6.30	6.86	7.18	7.24
VWUS	64.71	64.70	64.67	64.71	64.69	64.69
			Relative weights			
WXUS	7.76	8.20	8.88	9.59	9.99	10.06
VWUS	92.24	91.80	91.12	90.41	90.01	89.94
Panel D: Expected utility $\gamma = 30$						
			Optimal weights			
WXUS	0.19	1.74	4.21	6.84	8.32	8.58
VWUS	11.02	9.63	7.44	5.11	3.80	3.57
			Relative weights			
WXUS	1.66	15.32	36.15	57.24	68.63	70.63
VWUS	98.34	84.68	63.85	42.76	31.37	29.37

This table reports optimal and relative weights in the WXUS and VWUS portfolios when the prior mean of the mispricing α is zero ($\bar{\alpha} = 0$). The WXUS portfolio is the "World-Except-US" portfolio provided by MSCI. The "VWUS" portfolio is the value-weighted market portfolio provided by CRSP. The optimal weights are computed for different annualized values of σ_α, using the numerical methods in Appendix A. The relative weight is defined as the weight of a portfolio in the optimal risky portfolio.

Table 2: **Posterior statistics of parameters**

Statistics	α	σ^2	β	F	E_F	V_F
Panel A: $\bar{\alpha} = 0, \sigma_\alpha = 0$						
	Posterior statistics					
Mean	0.00	0.07	0.88	0.64	0.65	0.19
Min	-0.01	0.05	0.71	-16.77	-0.54	0.13
Max	0.01	0.11	1.06	18.92	1.84	0.29
	Posterior statistics for low conditional expected utility states under KMM					
Mean	0.00	0.07	0.89	-5.23	0.30	0.19
Min	-0.01	0.05	0.75	-16.77	-0.54	0.14
Max	0.01	0.11	1.04	5.06	1.11	0.27
Panel B: $\bar{\alpha} = 0, \sigma_\alpha = 0.10$						
	Posterior statistics					
Mean	0.18	0.07	0.88	0.64	0.65	0.19
Min	-0.55	0.05	0.71	-16.77	-0.54	0.13
Max	0.97	0.11	1.05	18.92	1.84	0.29
	Posterior statistics for low conditional expected utility states under KMM					
Mean	0.16	0.07	0.89	-5.81	0.35	0.19
Min	-0.51	0.05	0.74	-16.77	-0.54	0.14
Max	0.75	0.11	1.04	3.78	1.37	0.27

Panel A reports the posterior statistics of the parameters describing return distributions, where α is the mispricing of the non-benchmark asset (WXUS), β is the asset beta, σ^2 is the variance of the disturbance term in the CAPM regression, F is the benchmark asset return, and E_F and V_F are the mean and variance of the benchmark returns. Panel A also shows the posterior statistics of the parameters values that are associated with conditional expected utility values in the lowest 10 decile of the empirical distribution of expected utility values. The smooth ambiguity model is used to compute optimal portfolio weights, which give us expected utility conditional on the parameters. Panel B reports the same set of results for $\bar{\alpha} = 0$ and $\sigma_\alpha = 0.10$. All statistics are expressed in monthly and percentage terms except for β.

Table 3: **Optimal and relative portfolio weights:** $\bar{\alpha} = -0.02$

σ_α	0	0.005	0.01	0.02	0.05	0.1
Panel A: Expected utility $\gamma = 5$						
			Optimal weights			
WXUS	-45.07	-27.33	0.53	30.73	47.72	50.79
VWUS	106.99	91.24	66.59	39.80	24.71	21.99
			Relative weights			
WXUS	-72.80	-42.75	0.78	43.57	65.89	69.79
VWUS	172.80	142.75	99.22	56.43	34.11	30.21
Panel B: KMM preferences $\gamma = 5, \eta = 60$						
			Optimal weights			
WXUS	3.75	4.40	5.42	6.50	7.11	7.22
VWUS	64.78	64.76	64.71	64.66	64.69	64.69
			Relative weights			
WXUS	5.47	6.36	7.72	9.13	9.90	10.04
VWUS	94.53	93.64	92.28	90.87	90.10	89.96

This table reports optimal and relative weights in the WXUS and VWUS portfolios when the prior mean of the mispricing α is equal to -0.02 (in annualized terms).

Table 4: **Optimal and relative portfolio weights:** $\bar{\alpha} = 0.02$

σ_α	0	0.005	0.01	0.02	0.05	0.1
Panel A: Expected utility $\gamma = 5$						
			Optimal weights			
WXUS	48.26	48.90	49.95	51.08	51.66	51.84
VWUS	24.23	23.67	22.74	21.72	21.19	21.08
			Relative weights			
WXUS	66.57	67.38	68.71	70.16	70.91	71.09
VWUS	33.43	32.62	31.29	29.84	29.09	28.91
Panel B: KMM preferences $\gamma = 5, \eta = 60$						
			Optimal weights			
WXUS	7.13	7.16	7.19	7.23	7.25	7.25
VWUS	64.69	64.69	64.69	64.69	64.69	64.69
			Relative weights			
WXUS	9.93	9.96	10.00	10.05	10.08	10.08
VWUS	90.07	90.04	90.00	89.95	89.92	89.92

This table reports optimal and relative weights in the WXUS and VWUS portfolios when the prior mean of the mispricing α is equal to 0.02 (in annualized terms).

Table 5: **Optimal and relative portfolio weights: multiple non-benchmark assets**

σ_α	0	0.01	0.02	0.05	0.1	0.2
Panel A: Expected utility $\gamma = 5$						
			Optimal weights			
PAXJP	0.00	4.07	10.84	11.46	11.67	11.68
JP	0.75	0.00	0.00	0.00	0.00	0.00
EUXUK	1.67	13.79	32.51	34.21	34.79	34.82
UK	2.57	0.00	0.00	0.00	0.00	0.00
VWUS	106.00	94.26	72.56	70.61	69.94	69.91
			Relative weights			
PAXJP	0.00	3.63	9.36	9.85	10.02	10.03
JP	0.68	0.00	0.00	0.00	0.00	0.00
EUXUK	1.50	12.30	28.05	29.42	29.89	29.91
UK	2.31	0.00	0.00	0.00	0.00	0.00
VWUS	95.50	84.08	62.60	60.73	60.09	60.06
Panel B: KMM preferences $\gamma = 5, \eta = 60$						
			Optimal weights			
PAXJP	1.58	1.41	0.28	0.18	0.15	0.15
JP	0.19	0.00	0.00	0.00	0.00	0.00
EUXUK	2.63	5.42	7.78	7.99	8.06	8.06
UK	2.26	0.00	0.00	0.00	0.00	0.00
VWUS	105.60	105.54	105.44	105.43	105.43	105.43
			Relative weights			
PAXJP	1.41	1.26	0.25	0.16	0.13	0.13
JP	0.17	0.00	0.00	0.00	0.00	0.00
EUXUK	2.35	4.82	6.86	7.03	7.09	7.10
UK	2.01	0.00	0.00	0.00	0.00	0.00
VWUS	94.07	93.92	92.90	92.81	92.77	92.77

This table reports optimal and relative weights in international stock portfolios when the prior mean of the mispricing α is zero ($\bar{\alpha} = 0$). The PAXJP, JP, EUXUK and UK portfolios are, respectively, the "Pacific-Except-Japan", Japan, "Europe-Except-UK" and UK portfolios, for which data are provided by MSCI. The optimal weights are computed for different annualized values of σ_α, using the numerical method described in Appendix A. The relative weight is defined as the weight of a portfolio on the optimal risky portfolio.

Table 6: **Optimal and relative portfolio weights under regime switching: single non-benchmark asset**

σ_α	0	0.005	0.01	0.05	0.1	0.5
Panel A: Expected utility $\gamma = 10$						
	Bull regime					
	Optimal weights					
WXUS	3.04	28.01	62.50	113.08	116.03	116.93
VWUS	177.26	149.30	122.60	84.17	82.09	81.55
	Relative weights					
WXUS	1.69	15.80	33.76	57.33	58.57	58.91
VWUS	98.31	84.20	66.24	42.67	41.43	41.09
	Bear regime					
	Optimal weights					
WXUS	0.00	0.00	0.00	0.00	0.00	0.00
VWUS	14.54	13.28	14.14	13.46	13.54	13.47
	Relative weights					
WXUS	0.00	0.00	0.00	0.00	0.00	0.00
VWUS	100.00	100.00	100.00	100.00	100.00	100.00
Panel B: KMM preferences $\gamma = 10, \eta = 80$						
	Bull regime					
	Optimal weights					
WXUS	18.02	21.28	27.39	37.08	37.57	37.76
VWUS	164.28	162.06	160.67	159.93	159.80	159.77
	Relative weights					
WXUS	9.88	11.61	14.57	18.82	19.04	19.11
VWUS	90.12	88.39	85.43	81.18	80.96	80.89
	Bear regime					
	Optimal weights					
WXUS	1.70	1.41	1.29	0.78	0.76	0.75
VWUS	14.01	13.56	13.03	13.01	13.03	13.04
	Relative weights					
WXUS	10.84	9.40	9.01	5.68	5.54	5.41
VWUS	89.16	90.60	90.99	94.32	94.46	94.59

This table reports optimal and relative weights in the WXUS and VWUS portfolios with regime switching, where the prior mean of the mispricing α is assumed to be zero in both regimes. The optimal weights are computed for different annualized values of σ_α, using the numerical method described in Appendix B.

Table 7: **Posterior statistics of parameters: regime switching model**

Parameters	α	σ^2	β	F	E_F	V_F
Panel A: $\sigma_\alpha = 0$						
	Posterior statistics					
			Bull regime			
Mean	0.00	0.07	0.94	1.04	1.04	0.06
Min	0.00	0.03	0.43	-10.94	-0.06	0.03
Max	0.00	0.13	1.42	12.23	2.43	0.14
			Bear regime			
Mean	0.00	0.11	0.86	0.41	0.41	0.28
Min	0.00	0.06	0.63	-28.37	-4.74	0.16
Max	0.00	0.20	1.11	23.97	2.31	0.87
	Posterior statistics of low conditional expected value states (lower 10% percentile) under KMM					
			Bull regime			
Mean	0.00	0.07	0.94	-2.01	0.73	0.06
Min	0.00	0.03	0.48	-10.94	-0.06	0.03
Max	0.00	0.12	1.42	4.65	1.52	0.14
			Bear regime			
Mean	0.00	0.12	0.87	-6.92	-0.13	0.29
Min	0.00	0.06	0.68	-28.37	-4.74	0.18
Max	0.00	0.18	1.10	9.30	1.43	0.87
Panel B: $\sigma_\alpha = 0.50$						
	Posterior statistics					
			Bull regime			
Mean	0.78	0.07	0.80	1.03	1.05	0.06
Min	-0.36	0.02	0.24	-10.76	-0.06	0.03
Max	1.93	0.12	1.34	12.11	2.15	0.13
			Bear regime			
Mean	-0.16	0.11	0.86	0.38	0.39	0.28
Min	-1.78	0.06	0.63	-28.37	-2.35	0.16
Max	1.12	0.19	1.09	24.52	2.47	0.56
	Posterior statistics of low conditional expected value states (lower 10% percentile) under KMM					
			Bull regime			
Mean	0.70	0.07	0.83	-2.66	0.82	0.06
Min	-0.28	0.02	0.40	-10.76	-0.06	0.03
Max	1.83	0.11	1.34	2.89	1.67	0.12
			Bear regime			
Mean	-0.20	0.11	0.87	-4.73	-0.34	0.30
Min	-1.36	0.08	0.67	-28.37	-2.35	0.20
Max	1.05	0.18	1.09	13.16	0.58	0.49

Panel A reports the posterior statistics of the parameters describing return distributions with regime switching, where the notation is the same as in Table 2. Panel A also shows the posterior statistics of the parameters that are associated with expected utility values in the lowest 10 decile of the empirical distribution of expected utility, conditioning on regimes. Panel B reports the same set of results for $\bar{\alpha} = 0$ and $\sigma_\alpha = 0.50$. All statistics are expressed in monthly and percentage terms except for β.

Table 8: **Optimal and relative weights under regime switching and expected utility: multiple non-benchmark assets**

	Expected utility, $\gamma = 10$					
σ_α	0	0.005	0.01	0.05	0.1	0.5
Panel A: Bull regime						
	Optimal weights					
PAXJP	0.00	5.95	24.54	69.46	73.24	74.53
JP	0.78	4.54	9.90	22.74	23.83	24.21
EUXUK	0.00	2.10	8.37	23.08	24.33	24.76
UK	0.00	2.34	6.62	16.66	17.50	17.79
VWUS	241.39	228.35	196.07	126.75	121.40	119.58
	Relative weights					
PAXJP	0.00	2.44	10.00	26.85	28.14	28.57
JP	0.32	1.87	4.03	8.79	9.16	9.28
EUXUK	0.00	0.86	3.41	8.92	9.35	9.49
UK	0.00	0.96	2.70	6.44	6.72	6.82
VWUS	99.68	93.86	79.86	49.00	46.64	45.84
Panel B: Bear regime						
	Optimal weights					
PAXJP	0.31	0.00	0.00	0.00	0.00	0.00
JP	0.00	0.00	0.00	0.00	0.00	0.00
EUXUK	0.92	0.59	0.00	0.00	0.00	0.00
UK	0.00	0.00	0.00	0.00	0.00	0.00
VWUS	18.67	19.27	19.80	19.80	19.80	19.80
	Relative weights					
PAXJP	1.56	0.00	0.00	0.00	0.00	0.00
JP	0.00	0.00	0.00	0.00	0.00	0.00
EUXUK	4.63	2.97	0.00	0.00	0.00	0.00
UK	0.00	0.00	0.00	0.00	0.00	0.00
VWUS	93.80	97.03	100.00	100.00	100.00	100.00

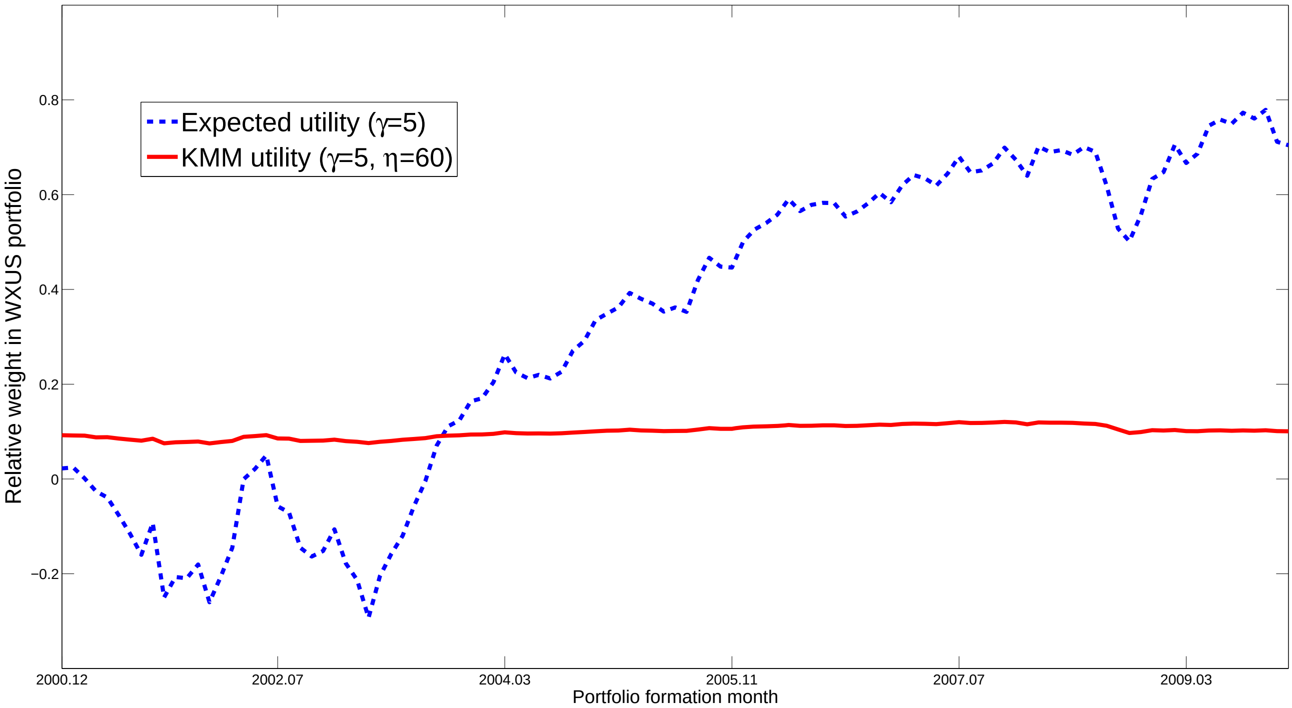
This table reports optimal and relative weights in international stock portfolios under the assumption of regime switching and expected utility maximization. The prior means of the mispricing α are zeros in both regimes. The optimal weights are computed conditional on each regime, for different annualized values of σ_α.

Table 9: **Optimal and relative weights under regime switching and smooth ambiguity preferences: multiple non-benchmark assets**

KMM preferences $\gamma = 10, \eta = 80$						
σ_α	0	0.005	0.01	0.05	0.1	0.5
Panel A: Bull regime						
	Optimal weights					
PAXJP	4.39	10.20	16.18	20.91	21.42	21.61
JP	0.00	2.53	6.24	12.91	13.40	13.56
EUXUK	9.97	3.85	0.00	0.00	0.00	0.00
UK	0.83	1.46	0.00	0.00	0.00	0.00
VWUS	212.15	211.94	211.47	210.05	209.93	209.89
	Relative weights					
PAXJP	1.93	4.43	6.92	8.58	8.75	8.82
JP	0.00	1.10	2.67	5.29	5.48	5.53
EUXUK	4.38	1.67	0.00	0.00	0.00	0.00
UK	0.37	0.64	0.00	0.00	0.00	0.00
VWUS	93.32	92.16	90.41	86.13	85.77	85.65
Panel B: Bear regime						
	Optimal weights					
PAXJP	0.40	0.14	0.00	0.00	0.00	0.00
JP	0.00	0.00	0.00	0.00	0.00	0.00
EUXUK	0.69	1.39	1.48	1.38	1.37	1.37
UK	0.54	0.00	0.00	0.00	0.00	0.00
VWUS	20.52	20.54	20.54	20.55	20.55	20.55
	Relative weights					
PAXJP	1.81	0.63	0.00	0.00	0.00	0.00
JP	0.00	0.00	0.00	0.00	0.00	0.00
EUXUK	3.13	6.28	6.70	6.28	6.25	6.24
UK	2.43	0.00	0.00	0.00	0.00	0.00
VWUS	92.63	93.09	93.30	93.72	93.75	93.76

This table reports optimal and relative weights in international stock portfolios under the assumption of regime switching and smooth ambiguity preferences. The prior means of the mispricing α are zeros in both regimes. The optimal weights are computed conditional on each regime, for different annualized values of σ_α.

Figure 1: Optimal relative percentage weights in foreign stocks: expanding sample



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