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Mixed Extensions of Decision Problems under Uncertainty*

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Abstract

In a decision problem under uncertainty, a decision maker considers a set of alternative actions whose consequences depend on uncertain factors outside his control. Following Luce and Raiffa (1957), we adopt a natural representation of such situation that takes as primitives a set of conceivable actions A, a set of states S and a consequence function $\rho : A \times S \to C$. With this, each action induces a map from states to consequences, or Savage act, and each mixed action induces a map from states to probability distributions over concequences, or Anscombe-Aumann act. Under a consequentialist axiom, preferences over pure or mixed actions yield corresponding preferences over the induced acts. The most common approach to the theory of choice under uncertainty takes instead as primitive a preference relation over the set $\Delta(C)^S$ of all Anscombe-Aumann acts. This allows to apply powerful convex analysis techniques, as in the seminal work of Schmeidler (1989) and the vast descending literature. This paper shows that we can maintain the mathematical convenience of the Anscombe-Aumann framework within a description of decision problems which is closer to applications and experiments. We argue that our framework is more expressive, it allows to be explicit and parsimonious about the assumed richness of the set of conceivable actions, and to directly capture preference for randomization as an expression of uncertainty aversion.

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1 Introduction

In the modern development of economic theory the role of uncertainty has been ubiquitous and the past two decades witnessed a number of studies that extended classical risk theory results to cope with new dimensions of uncertainty, in particular with Knightian uncertainty (often called ambiguity). These theoretical studies have found applications in a variety of fields, from asset pricing to market participation, from contract theory to risk management.[1] The framework which these studies relied upon remained the one adopted by the seminal paper of Schmeidler (1989), the so called *Anscombe-Aumann framework*. In the literature this term usually does not refer to the original framework of Anscombe and Aumann (1963), but to its simplified version proposed by Fishburn (1970). In such version, the objects of choice are functions $f : S \to \Delta(C)$,[2] where $S = \{s_1, ..., s_n\}$ is a set of *states* of the world, C is a set of deterministic *consequences*, and $\Delta(C)$ is the set of *random consequences*.[3] In particular, as Fishburn (1970) p. 176 writes

... We adopt the following pseudo-operational interpretation for $f \in \Delta(C)^S$. If f is 'selected' and $s \in S$ obtains then $f(s) \in \Delta(C)$ is used to determine a resultant consequence in C...

This framework gives the set of conceivable alternatives a tractable and mathematically familiar convex space structure that has been fundamental for the development of axiomatic models of choice under uncertainty. The translation of economic choice situations in this framework, however, is not always intuitive. For this reason in this paper we study the connection between the mathematically convenient framework of Anscombe and Aumann (1963) and the one of Luce and Raiffa (1957), which is more natural in terms of decision making. For Luce and Raiffa, a decision problem under uncertainty is described by a table

$$\begin{array}{c|c|c|c|c|} & s_1 & s_2 & ... & s_n \\ \hline a_1 & c_{11} & c_{12} & ... & c_{1n} \\ \hline a_2 & c_{21} & c_{22} & ... & c_{2n} \\ \hline ... & ... & ... & ... & ... \end{array} \qquad (1)$$

in which $c_{ij} \in C$ is the consequence of action $a_i \in A$ in state $s_j \in S$ and the decision maker can choose a *pure action* $a \in A$ or a *mixed action* $\alpha \in \Delta(A)$.[4] Mixed actions are interpreted in the usual game theoretic way as mixed strategies of a player choosing rows in the table above. The objects of choice of the Luce-Raiffa framework (the mixed actions $\alpha \in \Delta(A)$) are thus common in economics, statistics, and operations research, unlike those of the Anscombe-Aumann framework (the acts $f \in \Delta(C)^S$), which are peculiar to decision theory.

In this paper we show that the Luce-Raiffa framework is actually as tractable as the Anscombe-Aumann one, and can be immersed into it provided decision makers are *consequentialist*, that is, indifferent between actions that generate the same distribution of consequences in every state. Our analysis thus presents a vocabulary that allows to translate all decision theoretic results that have been expressed in the language of Anscombe-Aumann acts into the language of mixed actions. Such translation facilitates the access to uncertainty models to anybody familiar with mixed strategies, and allows to use these models directly 'off the shelves' in any application framed in the language of game theory (such as auction theory, matching, mechanism design, moral hazard, etc.). Moreover, mixed actions are easier to implement in the controlled setup of an experiment than Anscombe-Aumann acts. Therefore, the framework we consider here facilitates the exchange between theory and experiments, and in particular the experimental analysis of choice models derived within the Anscombe-Aumann framework.

The paper is organized as follows. After a preliminary analysis of decision problems and frameworks (Section 2), we first study (Section 3) the properties of randomization of pure actions and relate it to randomization of

[1] See Gilboa and Marinacci (2013) for a recent survey.

[2] Called horse lotteries by Anscombe and Aumann (1963) and *Anscombe-Aumann acts* in the subsequent literature.

[3] Formally, the set of all finitely supported probability distributions on C.

[4] This is the framework they describe on p. 276. Later in their book they replace consequences c_{ij} with their utilities u_{ij}.

consequences in the Anscombe-Aumann framework. The novelty here is not the introduction of mixed actions, already present in Luce and Raiffa (1957), but the relation between the two kinds of randomization.[5] Subsequently, we study the basic properties of preferences and choice correspondences in the Luce-Raiffa framework (Section 4), and we illustrate them by establishing behavioral characterizations of two classical choice criteria (Section 5). Finally, we relate uncertainty aversion with preference for randomization, and we show how it provides a natural explanation for commonly observed random choice behavior (Section 6). Two important remarks on independence and timing close the main text of the paper (Section 7).

All proofs are in Appendix B.

2 Decision problems under uncertainty

2.1 Setup

In a decision problem under uncertainty, a decision maker considers a set of alternative actions whose consequences depend on uncertain factors outside his control. Formally, there is a set A of conceivable pure actions a that can result in different material consequences c, within a set C, depending on which state s of the world (or of the environment) in a space $S = \{s_1, ..., s_n\}$ obtains. The dependence of consequences on actions and states is described by a *consequence function*

$$\begin{array}{rccc} \rho: & A \times S & \to & C \\ & (a, s) & \mapsto & \rho(a, s) \end{array}$$

that details the consequence $c = \rho(a, s)$ of each action a in each state s.[6] A *decision problem* (*under uncertainty* in *normal form*) is a quartet $(\mathcal{A}, S, C, \rho)$ where $\mathcal{A}$ is a nonempty subset of conceivable pure or mixed actions that are also feasible for the decision maker.[7] We call *decision framework* the quartet (A, S, C, ρ).

In such a framework, a *bet* on an *event* $E \subseteq S$ is an action $cEd \in A$ that delivers consequence c if E is true and d otherwise, that is,

$$\rho(cEd, s) = \begin{cases} c & s \in E \\ d & s \notin E. \end{cases} \tag{2}$$

In particular, if $E = S$, cSd delivers c in every state. For this reason such bet is called a *sure action* and denoted cS. We say that *all bets are conceivable* if and only if for every $c, d \in C$ and $E \subseteq S$ there exists a (possibly nonunique) action $cEd \in A$ such that (2) holds. Analogously, *all sure actions are conceivable* if and only if for every $c \in C$ there exists a (possibly nonunique) action $cS \in A$ for which $\rho(cS, \cdot) \equiv c$.

2.2 Examples

Example 1 (portfolio selection) *An investor in a frictionless financial market with J primary assets chooses at time 0 a portfolio $h \in \mathbb{R}^J$ being uncertain about the vector $r \in \{r_1, r_2, ..., r_n\} \subseteq \mathbb{R}^J$ of gross returns that will prevail at time 1. Denoting by S the set $\{1, 2, ..., n\}$, the investor's monetary payoff at time 1 is*

$$\nu(h, s) = r_s \cdot h \quad \forall (h, s) \in \mathbb{R}^J \times S$$

The decision framework is thus $\left(\mathbb{R}^J, S, \mathbb{R}, \nu\right)$. Depending on the investor's wealth w, his budget set is $B_w = \left\{h \in \mathbb{R}^J : \sum_{j \in J} h_j = w\right\}$ and the corresponding decision problem is $(B_w, S, \mathbb{R}, \nu)$.

[5] This relation was hinted at by Kreps (1988) in Chapter 7, where Savage acts (functions $f : S \to C$) are randomized. See the concluding Section 7 for details.

[6] In table (1), $\rho(a_i, s_j) = c_{ij}$ for each action $a_i \in A$ and each state $s_j \in S$.

[7] We use interchangeably the words feasible and available. We denote by capital script letters sets of (possibly degenerate) distributions or sets of Anscombe-Aumann acts.

Note that here all bets are conceivable if and only if the market is complete, while all sure actions are conceivable if and only if there exists a risk-free portfolio. ▲

In the previous example the distinction between the decision framework $\left(\mathbb{R}^J, S, \mathbb{R}, \nu\right)$ and the family of relevant decision problems $\{(B_w, S, \mathbb{R}, \nu) : w \in \mathbb{R}\}$ is neat. The next example shows that in other cases the focus is on a single decision problem that essentially coincides with the decision framework.[8]

Example 2 (normal game-forms) *An agent interacting with other agents chooses his action being uncertain about the choices of the others. The consequences of the interaction are determined by the profile of actions chosen by all agents. In this vein,* normal game-forms $\left\langle I, (S_i)_{i\in I}, C, g\right\rangle$ *are used to model players' strategic interaction in a game (see, e.g., Glazer and Rubinstein, 1996). They consist of a finite set* I *of players, a set* S_i *of available strategies for each player* $i \in I$*, a set* C *of consequences, and a function* $g : \Pi_{i\in I} S_i \to C$ *that associates consequences with strategy profiles.*[9] *The decision framework of each player* j *is* (S_j, S_{-j}, C, g)*, his actions are the strategies in* S_j *and the corresponding state space is* $S_{-j} = \Pi_{i\neq j} S_i$*.*

Depending on whether the player can commit his actions to a random device or not, that is, on whether mixed strategies are available or not, here the decision problem is either $(\Delta(S_j), S_{-j}, C, g)$ *or* (S_j, S_{-j}, C, g) *itself (unless there is an explicit description of the available randomizations).* ▲

Our final example is the famous omelet of Savage (1954), which we report verbatim because at the beginning of the next section it will be used to clarify the relations between pure actions and Savage acts.

Example 3 (the omelet) *... Your wife has just broken five good eggs into a bowl when you come in and volunteer to finish making the omelet. A sixth egg, which for some reason must either be used for the omelet or wasted altogether, lies unbroken beside the bowl. You must decide what to do with this unbroken egg. Perhaps it is not too great an oversimplification to say that you must decide among three acts only, namely, to break it into the bowl containing the other five, to break it into a saucer for inspection, or to throw it away without inspection. Depending on the state of the egg, each of these three acts will have some consequences of concern to you, say that indicated by...*

TABLE 1	*Good*	*Rotten*
break into bowl	*six-egg omelet*	*no omelet, and five good eggs destroyed*
break into saucer	*six-egg omelet, and a saucer to wash*	*five-egg omelet, and a saucer to wash*
throw away	*five-egg omelet, and one good egg destroyed*	*five-egg omelet*

... If two different acts had the same consequences in every state of the world, there would from the present point of view be no point in considering them two different acts at all... Or, more formally, an act is a function attaching a consequence to each state... ▲

2.3 Reduced decision problems and consequentialism

The final sentence of Savage in Example 3 refers to the identification of each pure action $a \in A$ with the section $\rho_a \in C^S$, which is called the *Savage act* induced by a

$$\begin{array}{rccc} \rho_a : & S & \to & C \\ & s & \mapsto & \rho(a,s) \end{array}$$

[8] That is, the decision framework is (A, S, C, ρ) and the decision problem is either $(\Delta(A), S, C, \rho)$ or (A, S, C, ρ) itself.

[9] By contrast, *normal-form games* also include a profile $(u_i)_{i\in I}$ of von Neumann-Morgenstern utility functions on C.

For example, the action 'break into bowl' is identified by Savage with the act $f = \rho_{\text{break into bowl}}$ given by

$$f\,(\text{Good}) = \text{six-egg omelet} \quad ; \quad f\,(\text{Rotten}) = \text{no omelet, and five good eggs destroyed}$$

Two actions a and b are thus identified if

$$\rho\,(a, s) = \rho\,(b, s) \quad \forall s \in S$$

that is, if $\rho_a = \rho_b$. Two such actions generate the same consequences in every state of the world and are called *realization equivalent*,[10] denoted $a \approx b$.

The identification of realization equivalent actions is pervasive in the modelling of individual and interactive decisions. For example, when $\left\langle I, (S_i)_{i\in I}, Z, g\right\rangle$ is the normal game-form of an extensive game-form (Example 2), then the set of consequences Z is the set of terminal nodes, the consequence function g is the outcome function, and our definition of realization equivalence corresponds to the one of Kuhn (1953). His Theorem 1 shows that two strategies are realization equivalent if and only if they induce the same decision plan (see Rubinstein, 1991, p. 911). In this case, the identification of realization equivalent strategies leads to the quasi-reduced normal game-form.[11]

Definition 1 *A decision framework* (A, S, C, ρ) *is* reduced *if and only if* $a \approx b$ *implies* $a = b$.

Non-reduced decision frameworks can always be reduced by identifying all elements of each realization equivalence class. Sometimes, the equivalence classes obtained in the reduction maintain a direct interpretation in terms of the original problem. For example, when the normal game-form $\left\langle I, (S_i)_{i\in I}, Z, g\right\rangle$ is considered, realization equivalence classes of strategies correspond to decision plans. In other cases, their interpretation is less immediate. In Example 1, the $S \times J$ matrix $R = [r_s]_{s=1}^n$, called *Arrow-Debreu tableau*, groups the state-contingent returns of the marketed assets. Therefore, the section $\nu_h = Rh$ of portfolio h, called the contingent claim replicated by h, describes the state-contingent payoffs generated by h.[12] The usual interpretation of a realization equivalence class $\left\{h \in \mathbb{R}^J : Rh = x\right\}$, geometrically an hyperplane, as an object of choice consists in identifying it with contingent claim x itself. But, while a portfolio is a collection of primary assets, a contingent claim $x \in \mathbb{R}^S$ is a contract that pays $x\,(s)$ if r_s is the true vector of gross returns, that is, it is a derivative asset.[13]

Be that as it may, from the decision maker's perspective the reduction of a nonreduced decision framework is an innocuous simplification as long as he is indifferent between pure actions that have the same consequences in every state. Denoting by $\succsim = \succsim_A$ the decision maker's preferences on A and by $\sim$ (resp. $\succ$) its symmetric (resp. asymmetric) part, this amounts to:

Consequentialism *If* $a, b \in A$*, then* $a \approx b$ *implies* $a \sim b$.

This assumption is reasonable when the decision maker only cares about consequences, and actions are just means to an end (to obtain 'desirable' consequences). Paraphrasing Marschak and Radner (1972, p. 12), if all consequences were directly available the 'most desirable' would be chosen. But, it is the actions, not the consequences, that are available for choice.

Given a decision framework (A, S, C, ρ), it is consequentialism that allows to elicit both the 'desirability' of consequences and the 'plausibility' of events, provided $\succsim$ is transitive.[14]

[10] Marschak and Radner (1972, Ch. 1) call them *essentially equivalent*.

[11] See Swinkels (1989) for a careful distinction among the various reduced forms of a game.

[12] In particular, the decision framework is reduced if and only if there are no redundant assets, that is, the rank of the Arrow-Debreu tableau is equal to the number of marketed assets. See, e.g., LeRoy and Werner (2000, Ch. 1) or Cerny (2009, Ch. 1).

[13] According to the Office of the Comptroller of the Currency (1999) a *derivative* is 'a financial contract whose value is derived from the performance of assets, interest rates, currency exchange rates, or indexes.'

[14] A transitive binary relation which is also reflexive is called *preorder*.

Proposition 1 *Let (A, S, C, ρ) be a decision framework and $\succsim$ a transitive binary relation on A.*

1. *If $\succsim$ satisfies consequentialism, then $\succsim$ is a preorder; the converse is true provided (A, S, C, ρ) is reduced.*

2. *If $\succsim$ satisfies consequentialism and all sure actions are conceivable, then*

$$c \succsim_C d \overset{\text{def}}{\iff} cS \succsim dS \tag{3}$$

is a well defined preorder on C.

3. *If $\succsim$ satisfies consequentialism and all bets are conceivable, then*

$$E \succsim^* F \overset{\text{def}}{\iff} cEd \succsim cFd \quad \forall c \succ_C d$$

is a well defined preorder on the power set of S.

The simple proof of this proposition shows that consequentialism is crucial for the well posedness of the preorders $\succsim_C$ and $\succsim^*$. These preorders are, respectively, interpreted as qualitative descriptions of the decision maker's tastes and beliefs on C and S. For later reference, we remark that to elicit tastes and beliefs all bets must be conceivable (even if not necessarily feasible). This assumption, although strong, is weaker than the Savage one that requires all acts to be conceivable.[15] In Savage's framework the role of bets cEd is played by binary acts $c_E d$ taking value c on E and d on E^c, and sure actions cS correspond to constants acts c_S.

3 Mixed actions and immersion

Following the common practice of game theory and statistics, we consider a decision maker that conceives the possibility of committing his actions to some random device. As a result, the set of conceivable actions is the set $\Delta(A)$ of all mixed actions. Let us reiterate that this should not be interpreted as an assumption about the feasibility of all mixed actions, but rather as an assumption about the ability of the decision maker to consider hypothetical alternatives. We will not discuss the difference between feasibility and conceivability anymore; but referring to Savage's omelet one last time, the toss of a coin before deciding where to break the sixth egg seems easier to conceive than an act that delivers a 'six-egg omelet' if the sixth egg is rotten and 'no omelet, and five good eggs destroyed' if the sixth egg is good.

Formally,

$$\Delta(A) = \left\{ \alpha : A \to [0,1] \;\middle|\; \alpha(a) > 0 \text{ for finitely many } a\text{'s in } A \text{ and } \sum_{a \in A} \alpha(a) = 1 \right\}$$

is the set of all probability distributions on A with finite support.[16] In particular, pure actions can be viewed as special mixed actions through the embedding $a \hookrightarrow \delta_a$ of points into point-masses. Conceptually, the elements $\alpha \in \Delta(A)$ should be interpreted as chance distributions, that is, objective probabilities such as those featured by random devices, and not as epistemic distributions, that is, subjective probabilities describing beliefs.[17] Mixed actions correspond to mixed strategies in game theory and to randomized decision rules in statistics.[18]

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[15] That is, for every $f : S \to C$ there exists $a \in A$ such that $\rho_a = f$. Formally, Savage assumes $(A, S, C, \rho) = (C^S, S, C, \varrho)$ where $\varrho(f, s) = f(s)$ is the evaluation pairing. In this decision framework, the number of bets is $|C| + |C|(|C| - 1)(2^{|S|-1} - 1)$ while the number of acts is $|C|^{|S|}$. With 10 states and 10 consequences there are 46 thousands of bets and 10 billions of acts.

[16] With the usual abuse we denote by the same greek letter a probability distribution on A and the probability measure it induces on the set of all parts of A.

[17] We refer to Luce and Raiffa (1957) and Anscombe and Aumann (1963) for a discussion of this distinction.

[18] One should distinguish between mixed decision rules and behavioral decision rules. But, since we are considering finite state spaces, the distinction is irrelevant.

The relevance of mixed actions is well illustrated by the decision problem:

$$\begin{array}{c|c|c|} & s_1 & s_2 \\ \hline a_1 & 0 & 1 \\ \hline a_2 & 1 & 0 \\ \hline \end{array} \tag{4}$$

with action set $A = \{a_1, a_2\}$, state space $S = \{s_1, s_2\}$, consequence space $C = \{0, 1\}$, and consequence function $\rho(a_1, s_1) = \rho(a_2, s_2) = 0$ and $\rho(a_1, s_2) = \rho(a_2, s_1) = 1$. As Luce and Raiffa (1957, p. 279) observe, the mixed action

$$\alpha = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$$

guarantees an expected value of 1/2 regardless of which state obtains, while the minimum guaranteed by both pure actions is 0. Randomization may thus hedge uncertainty, an obviously important feature in analyzing these decision problems (see also Debreu, 1959, p. 101).

Recall that $\Delta(C)$ is the collection of random consequences, that is, the set of all chance distributions on C with finite support. Each mixed action (a chance distribution of pure actions) induces a random consequence (a chance distributions of deterministic consequences) in every state: if the decision maker takes mixed action α, the chance of obtaining consequence c in state s is

$$\alpha(\{a \in A : \rho(a, s) = c\})$$

which we denoted by $\rho_\alpha(c \mid s)$. Note that

$$\rho_\alpha(\cdot \mid s) = \left(\alpha \circ \rho_s^{-1}\right)(\cdot)$$

is an element of $\Delta(C)$ for all $s \in S$, that is, each mixed action $\alpha \in \Delta(A)$ determines an Anscombe-Aumann act

$$\begin{array}{rccc} \rho_\alpha : & S & \to & \Delta(C) \\ & s & \mapsto & \alpha \circ \rho_s^{-1} \end{array}$$

that associates to each $s \in S$ the distribution of consequences resulting from the choice of α in state s. As anticipated in the Introduction, Anscombe-Aumann acts are functions $f \in \Delta(C)^S$ from states to random consequences, and they are the prevalent objects of choice in the axiomatic literature on decision making under uncertainty.[19] The next proposition describes some properties of the relation between mixed actions and Anscombe-Aumann acts.

Proposition 2 *Let (A, S, C, ρ) be a decision framework. The map*

$$\begin{array}{rccc} F : & \Delta(A) & \to & \Delta(C)^S \\ & \alpha & \mapsto & \rho_\alpha \end{array}$$

has the following properties:

1. *For every $a \in A$ and every $s \in S$, $\rho_{\delta_a}(s) = \delta_{\rho(a,s)}$; that is, $F(a) = \rho_a$ under the identification of points and point-masses.*

2. *For every $\alpha, \beta \in \Delta(A)$ and every $q \in [0, 1]$, $\rho_{q\alpha+(1-q)\beta} = q\rho_\alpha + (1-q)\rho_\beta$; that is, F is affine.*

3. *$\{\rho_\alpha\}_{\alpha\in\Delta(A)} = \Delta(C)^S$ if and only if $\{\rho_a\}_{a\in A} = C^S$; that is, F is onto if and only if all Savage acts are conceivable.*

[19] See, again, Gilboa and Marinacci (2013).

The first point of this proposition shows that the relation between a mixed action α and the corresponding Anscombe-Aumann act ρ_α extends the relation between a pure action a and the corresponding Savage act ρ_a. Specifically, the Anscombe-Aumann act ρ_{δ_a} induced by the pure action a coincides with the Anscombe-Aumann act induced by the Savage act ρ_a. This observation allows to consistently extend to mixed actions the definition of realization equivalence of pure actions.

Definition 2 *Let (A, S, C, ρ) be a decision framework. Two mixed actions $\alpha, \beta \in \Delta(A)$ are* realization equivalent *if and only if they generate the same distribution of consequences in every state of the world, that is, $\alpha \approx \beta \overset{\text{def}}{\iff} \rho_\alpha = \rho_\beta$.*

By the first point of Proposition 2, given two pure actions $a, b \in A$ and $s \in S$,

$$\rho(a, s) = \rho(b, s) \iff \delta_{\rho(a,s)} = \delta_{\rho(b,s)} \iff \rho_{\delta_a}(s) = \rho_{\delta_b}(s) \tag{5}$$

That is, a and b are realization equivalent in the sense of the previous section (i.e., $\rho_a = \rho_b$) if and only if they are realization equivalent as mixed actions (i.e., $\rho_{\delta_a} = \rho_{\delta_b}$). In turn, Definition 2 allows to extend the notion of consequentialism to preferences $\succsim = \succsim_{\Delta(A)}$ between mixed actions. A decision maker is now consequentialist if and only if he is indifferent between mixed actions that generate the same distribution of consequences in every state of the world. Formally,

Mixed consequentialism *If $\alpha, \beta \in \Delta(A)$, then $\alpha \approx \beta$ implies $\alpha \sim \beta$.*

By (5), consequentialism is the restriction to pure actions of mixed consequentialism. Like consequentialism allows to immerse any decision problem with pure actions in the Savage framework ($a \overset{F}{\mapsto} \rho_a$), mixed consequentialism allows to immerse any decision problem with mixed actions in the Anscombe-Aumann framework ($\alpha \overset{F}{\mapsto} \rho_\alpha$). In fact, mixed consequentialism allows to define a binary relation $\succsim_F$ by

$$\rho_\alpha \succsim_F \rho_\beta \overset{\text{def}}{\iff} \alpha \succsim \beta \tag{6}$$

on the collection of Anscombe-Aumann acts $\{\rho_\alpha\}_{\alpha \in \Delta(A)} \subseteq \Delta(C)^S$. Affinity of F then allows to easily translate behavioral assumptions of $\succsim$ on $\Delta(A)$ into corresponding properties of $\succsim_F$ on $\{\rho_\alpha\}_{\alpha \in \Delta(A)}$, as detailed in Proposition 3 below and in Lemma 2 in Appendix B. Thanks to the 'transfer principle' between frameworks established by (6), in the mathematical derivation of choice models it remains possible take advantage of the analytical tractability of the Anscombe-Aumann setup, without incurring in its interpretational difficulties. To put it simply, the transfer principle makes it possible to conduct the behavioral analysis, in the main text of a decision theoretic paper, in terms of the more natural mixed actions, moving the Anscombe-Aumann paraphernalia to its appendix.

The last point of Proposition 2 shows in what sense our framework is never more demanding in terms of hypothetical comparisons than the Anscombe-Aumann one, and equally demanding only when all Savage acts are conceivable. In other words, the Anscombe-Aumann framework $\Delta(C)^S$ corresponds, via the transfer principle (6), to the mixed extension $\Delta\left(C^S\right)$ of the Savage framework.

Finally, note that also in a reduced decision framework mixed actions that are realization equivalent may well differ.[20] But, Proposition 2 implies that the framework cannot be further reduced by eliminating a pure action a which is realization equivalent to a mixed action β with support in $A \setminus \{a\}$.

Corollary 1 *Let (A, S, C, ρ) be a reduced decision framework. If $a \in A$ and $\beta \in \Delta(A)$, then $\delta_a \approx \beta \iff \beta = \delta_a$.*

The language developed so far also allows to connect decision analysis in the Anscombe-Aumann framework with statistical decision theory, as detailed in the working paper version.

[20] For example, (12) below is reduced, and the mixed actions $\alpha = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ and $\beta = \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$ are clearly distinct (they have disjoint support), but $\rho_\alpha = \rho_\beta$, so that they are realization equivalent.

4 Rationality and dominance

Up until this point, we have been studying two alternative decision theoretic frameworks and their relationships. We now focus our attention on the decision maker's preferences. Let (A, S, C, ρ) be a decision framework and $\succsim = \succsim_{\Delta(A)}$ be a binary relation on $\Delta(A)$ representing the decision maker's preferences.

Axiom A. 1 (Weak order) *$\succsim$ is complete and transitive.*

Although Karni, Maccheroni, and Marinacci (2013) show that completeness is not crucial for our analysis, we maintain it for the sake of simplicity. Inspired again by Luce and Raiffa (1957, p. 276), we also assume that

... our subject's preferences among ... outcomes, and among hypothetical lotteries with this outcomes as prizes, are consistent in the sense that they may be summarized by means of a utility function ...

Since the objects of choice are elements of $\Delta(A)$, not of C or $\Delta(C)$, this amounts to assume that all sure actions are conceivable and that the decision maker's preferences restricted to lotteries of sure actions (the mixed actions with support in the set of all sure actions) satisfy the axioms of von Neumann-Morgenstern's expected utility. For this reason, and for others that will become clear soon, reduced decision frameworks in which all sure actions are conceivable deserve a special name.

Definition 3 *A* Luce-Raiffa framework (A, S, C, ρ) *is a reduced decision framework in which all sure actions are conceivable.*

In these frameworks it becomes possible to elicit both the decision maker's preferences over consequences (Proposition 1.2) and his attitudes toward risk. In fact, the map

$$\begin{array}{cccc} \epsilon: & \Delta(C) & \to & \Delta(A) \\ & \sum\limits_{c\in C} \gamma(c)\, \delta_c & \mapsto & \sum\limits_{c\in C} \gamma(c)\, \delta_{cS} \end{array}$$

is an affine embedding of the set $\Delta(C)$ of random consequences onto the set $\Delta_\ell(A)$ of all mixed actions with support in the set $\{cS\}_{c\in C}$ of all sure actions.[21] For each $\gamma \in \Delta(C)$, by choosing $\epsilon(\gamma)$ the decision maker receives consequence c with objective probability $\gamma(c)$ regardless of which state obtains,[22] both γ and $\epsilon(\gamma)$ are formal descriptions of *lotteries* with consequences as prizes. Since ϵ is an affine bijection from $\Delta(C)$ to $\Delta_\ell(A)$, we can set

$$\gamma \succsim_{\Delta(C)} \zeta \overset{\text{def}}{\iff} \epsilon(\gamma) \succsim \epsilon(\zeta)$$

and infer the decision maker's preferences $\succsim_{\Delta(C)}$ between random consequences from his preferences $\succsim$ between mixed actions. In this way, ϵ becomes an isotone isomorphism between $\left(\Delta(C), \succsim_{\Delta(C)}\right)$ and $(\Delta_\ell(A), \succsim)$; for this reason, we often write γ instead of $\epsilon(\gamma)$, and $\gamma \succsim \zeta$ instead of $\gamma \succsim_{\Delta(C)} \zeta$.[23]

In this perspective, the utility function sought-after by Luce and Raiffa is delivered by the following axiom.

Axiom A. 2 (von Neumann-Morgenstern payoffs) *If $\gamma, \zeta, \xi \in \Delta_\ell(A)$, then*

1. $\{q \in [0,1] : q\gamma + (1-q)\zeta \succsim \xi\}$ *and* $\{q \in [0,1] : \xi \succsim q\gamma + (1-q)\zeta\}$ *are closed sets;*

[21] It is reduction that guarantees the well posedness of ϵ. Without reduction one should consider realization equivalence classes of mixed actions with support in the set of all sure actions. This is clearly possible, but leads to a notational cost which is not justified by the conceptual gain since mixed consequentialism will be maintained (see Proposition 3).

[22] In fact, $\rho_{\epsilon(\gamma)} \equiv \gamma$ as shown in Lemma 1 of Appendix B.

[23] The same happens in the Anscombe-Aumann framework where one writes γ not only to denote $\gamma \in \Delta(C)$, but also the constant act $\gamma_S \equiv \gamma$ (and $\gamma \succsim \zeta$ means $\gamma_S \succsim \zeta_S$ since the primitive preferences $\succsim$ are defined on $\Delta(C)^S$). In this case, the role of ϵ is played by the embedding $\gamma \hookrightarrow \gamma_S$ of random consequences onto constant Anscombe-Aumann acts.

2. $\gamma \sim \zeta$ *implies* $\frac{1}{2}\gamma + \frac{1}{2}\xi \sim \frac{1}{2}\zeta + \frac{1}{2}\xi$.

The preference $\succsim$ is a weak order on $\Delta_\ell(A) \simeq \Delta(C)$, and the above two requirements are the continuity and independence axioms of Hernstein and Milnor (1953) restricted to $\Delta_\ell(A)$. Hence, Axioms A.1 and A.2 imply the existence of a cardinally unique *payoff function* $u : C \to \mathbb{R}$ such that

$$\gamma \succsim \zeta \iff \sum_{c \in C} \gamma(c)\, u(c) \geq \sum_{c \in C} \zeta(c)\, u(c). \tag{7}$$

With respect to this payoff function, the (very weak) *dominance* relation of game theory

$$\alpha \geqslant_u \beta \overset{\text{def}}{\iff} \sum_{a \in A} \alpha(a)\, u(\rho(a,s)) \geq \sum_{a \in A} \beta(a)\, u(\rho(a,s)) \quad \forall s \in S$$

has a natural decision theoretic counterpart

$$\alpha \succcurlyeq_S \beta \overset{\text{def}}{\iff} \rho_\alpha(s) \succsim \rho_\beta(s) \quad \forall s \in S$$

also called *dominance*. According to both definitions, α dominates β if and only if the decision maker prefers the lottery generated by α to the lottery generated by β in every state. Axioms A.1 and A.2 guarantee that the two definitions of dominance are equivalent.

The next axiom requires that dominant actions be actually preferred.

Axiom A. 3 (Dominance) *If* $\alpha, \beta \in \Delta(A)$ *and* $\alpha \succcurlyeq_S \beta$, *then* $\alpha \succsim \beta$.

Under this additional axiom, the decision framework (A, S, C, ρ) can be identified with the zero-sum game (A, S, C, ρ, u) between decision maker and nature.[24] In fact, A.1-A.3 imply that two payoff equivalent actions are indifferent for the decision maker,[25] and the consequences c_{ij} in table (1) can be replaced by their utilities $u_{ij} = u(c_{ij})$ (cf. footnote 4).

In terms of choice behavior, the assumptions we made so far mean that the decision maker is rational in the sense of Arrow (1959) and chooses dominant actions whenever they are available. To see why this is the case, denote by $\wp(\Delta(A))$ the collection of all nonempty finite subsets of $\Delta(A)$. A *rational choice correspondence* $\Gamma : \wp(\Delta(A)) \to \wp(\Delta(A))$ maps every set $\mathcal{A} \in \wp(\Delta(A))$ into one of it subsets, $\Gamma(\mathcal{A}) \subseteq \mathcal{A}$, and satisfies the weak axiom of revealed preference:

WARP *If* $\mathcal{A} \subseteq \mathcal{B} \in \wp(\Delta(A))$ *and* $\mathcal{A} \cap \Gamma(\mathcal{B}) \neq \varnothing$, *then* $\Gamma(\mathcal{A}) = \mathcal{A} \cap \Gamma(\mathcal{B})$.

A binary relation $\succsim$ is said to be generated by a rational choice correspondence Γ if and only if $\alpha \succsim \beta$ is equivalent to $\alpha \in \Gamma(\{\alpha, \beta\})$. By Theorem 3 of Arrow (1959), $\succsim$ is a weak order and

$$\Gamma(\mathcal{A}) = \{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\} \quad \forall \mathcal{A} \in \wp(\Delta(A)).$$

In this case, it is natural to extend Γ to the family of all subsets $\mathcal{A}$ of $\Delta(A)$ for which $\{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\} \neq \varnothing$. Finally, each $u : C \to \mathbb{R}$ induces the dominance correspondence

$$\begin{array}{rccc} \Upsilon_u : & \wp(\Delta(A)) & \to & \wp(\Delta(A)) \cup \{\varnothing\} \\ & \mathcal{A} & \mapsto & \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta \ \forall \beta \in \mathcal{A}\} \end{array}$$

that associates to each set $\mathcal{A}$ of available alternatives the, possibly empty, set of dominant alternatives.

[24] As Luce and Raiffa (1957, p. 279) discuss, the choice of $-u$ as a payoff function for nature is best seen as purely formal.
[25] Formally, $\alpha \geqslant_u \beta$ and $\beta \geqslant_u \alpha$ imply $\alpha \sim \beta$.

Theorem 1 *Let (A, S, C, ρ) be a Luce-Raiffa framework and $\succsim$ be binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) $\succsim$ satisfies A.1, A.2, and A.3;

(ii) $\succsim$ is generated by a rational choice correspondence Γ for which there exists $u \in \mathbb{R}^C$ such that

$$\Upsilon_u(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) \quad \forall \mathcal{A} \in \wp(\Delta(A))$$

with equality if $\mathcal{A} \subseteq \Delta_\ell(A)$.

By Theorem 1, a rational decision maker first computes the expected payoffs of lotteries with consequences as prizes, and then ranks mixed actions according to dominance. At this point, he chooses dominant actions when they are available; in any case, his choices do not violate WARP.

The next result shows that this decision maker is consequentialist, so that the binary relation $\succsim_F$ on Anscombe-Aumann acts given by (6) is a well defined weak order satisfying risk independence and monotonicity,[26] that is, it is rational in the sense of Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011).

Proposition 3 *Let (A, S, C, ρ) be a Luce-Raiffa framework and $\succsim$ a binary relation on $\Delta(A)$ that satisfies A.1-A.3, and $\mathcal{F} = F(\Delta(A))$. Then:*

1. *$\succsim$ satisfies mixed consequentialism;*
2. *$\mathcal{F}$ is a convex subset of $\Delta(C)^S$ containing all constant Anscombe-Aumann acts;*
3. *$\succsim_F$ on $\mathcal{F}$ is a weak order satisfying risk independence and monotonicity.*

All this motivates the following definition.

Definition 4 *Let (A, S, C, ρ) be a Luce-Raiffa framework. A binary relation $\succsim$ on $\Delta(A)$ is a* rational preference *(*under uncertainty*) if and only if it satisfies A.1-A.3.*

5 Two classical criteria

We illustrate the notions introduced so far by establishing a behavioral characterization of two important rational criteria: classical maxminimization, due to Wald (1950), and subjective expected utility maximization, due to Savage (1954). Both criteria rely on a numerical representation of preferences. As for von Neumann-Morgenstern's expected utility, the existence of such a representation is ensured by the continuity axiom of Hernstein and Milnor, now assumed on the entire set $\Delta(A)$ rather than on $\Delta_\ell(A)$.

Axiom A. 4 (Continuity) *If $\alpha, \beta, \eta \in \Delta(A)$, then $\{q \in [0,1] : q\alpha + (1-q)\beta \succsim \eta\}$ and $\{q \in [0,1] : \eta \succsim q\alpha + (1-q)\beta\}$ are closed sets.*

Given a Luce-Raiffa framework (A, S, C, ρ) and a rational preference $\succsim$ on $\Delta(A)$, continuity implies that for every $\alpha \in \Delta(A)$ there exists $\alpha_\ell \in \Delta(C)$ such that $\alpha \sim \alpha_\ell$. Therefore, the functional

$$\begin{array}{rccc} V: & \Delta(A) & \to & \mathbb{R} \\ & \alpha & \mapsto & \sum_{c \in C} \alpha_\ell(c)\, u(c) \end{array}$$

[26] See Appendix A for a list of the most common axioms in the Anscombe-Aumann framework.

that associates to each mixed action its *equivalent expected payoff* $V(\alpha) = \mathbb{E}_{\alpha_\ell}[u]$ represents $\succsim$. In particular, it allows to associate to any decision problem $(\mathcal{A}, S, C, \rho)$ an *indirect (expected equivalent) payoff*

$$v(\mathcal{A}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha_\ell}[u] \tag{8}$$

and to describe the rational choice correspondence associated to $\succsim$ by

$$\Gamma(\mathcal{A}) = \arg\sup_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha_\ell}[u]$$

provided the supremum is attained.[27]

5.1 Classical maxminimization

A conservative criterion that a decision maker confronted with a decision problem $(\mathcal{A}, S, C, \rho)$ might adopt consists in choosing an action the (state-wise) lowest expected payoff of which is largest, that is, an element of

$$\Gamma(\mathcal{A}) = \arg\max_{\alpha \in \mathcal{A}} \min_{s \in S} \sum_{a \in A} \alpha(a)\, u(\rho(a, s)). \tag{9}$$

We call this procedure for choosing an action *classical maxminimization.* It may arise because the decision maker has no idea about the relative likelihood of the various states.[28] Behaviorally, the axiom that, on top of rationality and continuity, characterizes (9) is an extreme version of the 'default to certainty' axiom of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010).

Axiom A. 5 (Extreme caution) *If $\alpha \in \Delta(A)$, $\gamma \in \Delta_\ell(A)$, and $\alpha \not\geq_S \gamma$, then $\gamma \succ \alpha$.*

This axiom shows how conservative is the classical maxminimization criterion: if a mixed action does not dominate a lottery, then the lottery is strictly preferred.

Theorem 2 *Let (A, S, C, ρ) be a Luce-Raiffa framework and $\succsim$ be binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) $\succsim$ is a continuous rational preference that satisfies extreme caution;

(ii) there exists $u \in \mathbb{R}^C$ such that, if $\alpha, \beta \in \Delta(A)$,

$$\alpha \succsim \beta \iff \min_{s \in S} \sum_{a \in A} \alpha(a)\, u(\rho(a, s)) \geq \min_{s \in S} \sum_{a \in A} \beta(a)\, u(\rho(a, s)). \tag{10}$$

Relative to the original axiomatization of Milnor (1954), this axiomatization is simpler and arguably more intuitive.

5.2 Subjective expected utility

In game theory often rationality is identified with the adoption of the choice criterion

$$\Gamma(\mathcal{A}) = \arg\max_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha \times \mu}[u \circ \rho]$$

where μ is a subjective probability on S that the decision maker uses to assess the *subjective expected utility*

$$\mathbb{E}_{\alpha \times \mu}[u \circ \rho] = \sum_{s \in S} \mu(s) \sum_{a \in A} \alpha(a)\, u(\rho(a, s))$$

[27] One should write V_u and v_u instead of V and v since these functions obviously depend on u. The subscripts are omitted since u is cardinally unique.

[28] Or of what are the objectives and beliefs of his opponents (see Milnor, 1954, p. 49, and Osborne, 2003, p. 335).

of each mixed action $\alpha \in \Delta(A)$. This criterion, which evaluates each action by the expectation of its payoffs $u(\rho(a,s))$ with respect to the hybrid probability $\alpha(a)\mu(s)$,[29] is characterized by the independence axiom; this time assumed on the entire set $\Delta(A)$ rather than on $\Delta_\ell(A)$.

Axiom A. 6 (Independence) *If $\alpha, \beta, \eta \in \Delta(A)$, then $\alpha \sim \beta$ implies $\frac{1}{2}\alpha + \frac{1}{2}\eta \sim \frac{1}{2}\beta + \frac{1}{2}\eta$.*

Along with A.4, clearly A.6 implies A.2. A variation on the Expected Utility Theorem of Anscombe and Aumann, together with the techniques we developed so far, then delivers:

Theorem 3 *Let (A, S, C, ρ) be a Luce-Raiffa framework and $\succsim$ be binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) $\succsim$ is a continuous rational preference that satisfies independence;

(ii) there exist $u \in \mathbb{R}^C$ and $\mu \in \Delta(S)$ such that, if $\alpha, \beta \in \Delta(A)$,

$$\alpha \succsim \beta \iff \sum_{s\in S}\mu(s)\sum_{a\in A}\alpha(a)\,u(\rho(a,s)) \geq \sum_{s\in S}\mu(s)\sum_{a\in A}\beta(a)\,u(\rho(a,s)).$$

In general, μ is not unique. Like in Lehrer and Teper (2014), the set of Anscombe-Aumann acts generated by $\Delta(A)$ may be a proper subset of the whole set $\Delta(C)^S$ of all Anscombe-Aumann acts. In the Anscombe-Aumann framework, uniqueness holds under two additional assumptions:

- preferences are not trivial, so that $\succ_C$ is not empty (see Proposition 1);
- all acts are conceivable, so that $\succsim^*$ is a weak order on the class of all parts of S (see again Proposition 1).

The first assumption readily translates into the following axiom.

Axiom A. 7 (Nontriviality) *There exist $\alpha, \beta \in \Delta(A)$ such that $\alpha \succ \beta$.*

To obtain a complete $\succsim^*$, all bets have to be conceivable.

Definition 5 *A* Marschak-Radner framework *(A, S, C, ρ) is a reduced decision framework in which all bets are conceivable.*

Marschak and Radner (1972) assume that all acts be conceivable, but restrict their attention to pure actions. Their first chapter presents the counterpart of Savage's Expected Utility Theorem when pure actions, rather than Savage acts, are considered. Next theorem presents the counterpart of the Anscombe-Aumann's Expected Utility Theorem, when mixed actions, rather than Anscombe-Aumann acts, are considered.

Theorem 4 *Let (A, S, C, ρ) be a Marschak-Radner framework and $\succsim$ be binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) $\succsim$ is a nontrivial and continuous rational preference that satisfies independence;

(ii) there exist a nonconstant $u \in \mathbb{R}^C$ and $\mu \in \Delta(S)$ such that, if $\alpha, \beta \in \Delta(A)$,

$$\alpha \succsim \beta \iff \sum_{s\in S}\mu(s)\sum_{a\in A}\alpha(a)\,u(\rho(a,s)) \geq \sum_{s\in S}\mu(s)\sum_{a\in A}\beta(a)\,u(\rho(a,s)).$$

[29] We say that $\alpha(a)\mu(s)$ is hybrid because is the product of a chance $\alpha(a)$ and a belief $\mu(s)$. In the same vein, we use the term *payoff* for $u(\rho(a,s))$, *expected payoff* for the (objective) average $\sum_{a\in A}\alpha(a)\,u(\rho(a,s))$ of payoffs, and *expected utility* for the (subjective) average $\sum_{s\in S}\mu(s)\sum_{a\in A}\alpha(a)\,u(\rho(a,s))$ of expected payoffs.

In this case, μ is unique.

In a Bayesian decision theory perspective (with no data, see Berger, 1985, Ch. 1), the negative expected utility

$$r(\alpha, \mu) = -\mathbb{E}_{\alpha\times\mu}[u\circ\rho] = \int_S \mathbb{E}_{\rho_\alpha(s)}[-u]\, d\mu(s)$$

is called *Bayes risk* of (randomized action) α under (prior distribution) μ.[30]

6 Uncertainty aversion

6.1 Axiom and behavior

The difference between the two rational preferences described in the previous section, classical maxminimization and subjective expected utility maximization, is readily seen in the so called Ellsberg paradox (after Ellsberg, 1961). Consider a coin that a decision maker knows to be fair, as well as an urn that he knows to contain 100 black and white balls in unknown proportion (and so there may be from 0 to 100 black balls).[31] To bet on heads/tails means that the decision maker wins \$100 if the tossed coin lands on heads/tails (and nothing otherwise); similarly, to bet on black/white means that the decision maker wins \$100 if a ball drawn from the urn is black/white (and nothing otherwise).

Ellsberg's thought experiment suggests, and a number of behavioral experiments confirm, that many decision makers are indifferent between betting on either heads or tails and are also indifferent between betting on either black or white, but they strictly prefer to bet on the coin rather than on the urn. We can represent this preference pattern as

$$\text{bet on heads} \sim \text{bet on tails} \succ \text{bet on white} \sim \text{bet on black} \tag{11}$$

The urn draw is a version of decision problem (4), namely

ρ	B	W
a_1	\$0	\$100
a_2	\$100	\$0

where a_1 is the bet on white and a_2 is the bet on black. The corresponding Luce-Raiffa framework is obtained by adding the two sure actions b_1 and b_2 corresponding to consequences 100\$ and 0\$, respectively,

ρ	B	W
a_1	\$0	\$100
a_2	\$100	\$0
b_1	\$100	\$100
b_2	\$0	\$0

. (12)

In turn, the presence of sure actions allows to express the coin toss as a lottery delivering \$100 and \$0 with even chances, that is, $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} = \frac{1}{2}\delta_{b_2} + \frac{1}{2}\delta_{b_1}$ and (11) becomes

$$\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} \sim \frac{1}{2}\delta_{b_2} + \frac{1}{2}\delta_{b_1} \succ a_1 \sim a_2. \tag{13}$$

Consider now the following gamble proposed by Raiffa (1961): toss the coin, then bet on white if the coin lands on heads and bet on black otherwise. Formally, this gamble is represented by the mixed action $\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$,

[30] Again one should write r_u instead of r and again the subscript is omitted because of the cardinal uniqueness of u.

[31] A fair coin here is just a random device generating two outcomes with the same 1/2 chance. The original paper of Ellsberg models it as another urn that the decision maker knows to contain 50 white balls and 50 black balls.

which is easily seen to be realization equivalent to $\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}$.[32] Under mixed consequentialism, it follows

$$\frac{1}{2}\delta_{a_1}+\frac{1}{2}\delta_{a_2}\sim\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\succ a_1\sim a_2 \tag{14}$$

which is consistent with classical maxminimization and inconsistent with subjective expected utility.[33]

Also Raiffa (1961) uses the realization equivalence between the 'compound gamble' $\frac{1}{2}\delta_{a_1}+\frac{1}{2}\delta_{a_2}$ and the 'coin toss' $\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}$ to argue in favour of their indifference. From this indifference, he then concludes that the decision problem 'mixed urn draw' $(\Delta(\{a_1,a_2\}),\{B,W\},\rho)$ has the same indirect payoff of the problem 'coin toss' $\left(\left\{\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\right\},\{B,W\},\rho\right)$. Indeed, assuming wlog $u(100\$)=1$ and $u(0\$)=0$, according both to maxminimization

$$\max_{\alpha\in\Delta(\{a_1,a_2\})}\min_{s\in\{B,W\}}\sum_{a\in A}\alpha(a)\,u(\rho(a,s))=\frac{1}{2}=\max_{\beta\in\left\{\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\right\}}\min_{s\in\{B,W\}}\sum_{a\in A}\beta(a)\,u(\rho(a,s))$$

and to expected utility with uniform beliefs

$$\max_{\alpha\in\Delta(\{a_1,a_2\})}\sum_{s\in S}\frac{1}{|S|}\sum_{a\in A}\alpha(a)\,u(\rho(a,s))=\frac{1}{2}=\max_{\beta\in\left\{\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\right\}}\sum_{s\in S}\frac{1}{|S|}\sum_{a\in A}\beta(a)\,u(\rho(a,s))$$

The maxima are attained in both cases on the lhs at $\frac{1}{2}\delta_{a_1}+\frac{1}{2}\delta_{a_2}$ and on the rhs at $\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}$. But note that this observation, corresponding to the first part $\frac{1}{2}\delta_{a_1}+\frac{1}{2}\delta_{a_2}\sim\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}$ of (14), leaves its second part $\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\succ a_1\sim a_2$ normatively compelling because of the decision maker's ignorance about the relative likelihood of the states B and W. Jointly the two parts suggest that the 'mixed urn draw' problem has the same value as the 'coin toss' problem which has a greater value than the 'pure urn draw' problem $(\{a_1,a_2\},\{B,W\},\rho)$,[34] and this is exactly the normative insight of the Ellsberg paradox. Randomization has value since it eliminates the dependence of the probability of winning on the unknown composition of the urn; randomization, in fact, makes this probability a chance, thus hedging uncertainty. On the other hand, subjective expected utility cannot value randomization because of the simple mathematical fact that the expected utility of a mixed action is never greater than the maximum of the expected utilities of all pure actions in its support.

The preference for randomization that might emerge under uncertainty is called *uncertainty aversion*.

Axiom A. 8 (Uncertainty aversion) *If $\alpha,\beta\in\Delta(A)$, then $\alpha\sim\beta$ implies $\frac{1}{2}\alpha+\frac{1}{2}\beta\succsim\alpha$.*

This axiom, due to Schmeidler (1989), captures the idea that randomization (here in its simplest fifty-fifty form) may provide an hedge against epistemic uncertainty by trading it off for chance. Accordingly, decision makers who dislike uncertainty should (weakly) prefer to randomize. In this perspective, the observation of random choice behavior may be explained by the presence of uncertainty and aversion to it, as predicted by Raiffa in commenting Ellsberg.

6.2 Uncertainty averse representations

As shown by Theorem 5 below, continuous rational preferences that are uncertainty averse feature a representation $V:\Delta(A)\to\mathbb{R}$ of the form

$$V(\alpha)=\inf_{\sigma\in\mathcal{S}}R(\alpha,\sigma)\quad\forall\alpha\in\Delta(A) \tag{15}$$

where $R:\Delta(A)\times\Delta(S)\to(-\infty,\infty]$ is a suitable reward function whose first component is increasing in the expected utility $\mathbb{E}_{\alpha\times\sigma}[u\circ\rho]$ for each fixed σ. Intuitively, if the decision maker knew the probability σ of the

[32] If the color of the drawn ball is white (resp. black), then probability of winning by choosing this gamble is the chance that the coin toss assigns to betting on white (resp. black), and this chance is 1/2 since the coin is fair.

[33] Note that, under the assumptions of Theorem 3, $a_1\sim a_2$ implies $\mu(B)=\mu(W)=1/2$.

[34] See, again, Luce and Raiffa (1957, p. 279) and the discussion of Klibanoff (1992, p. 6).

states, he would maximize expected utility. Insufficient information about the environment, along with the need of taking decisions that perform well under different probabilistic scenarios $\sigma \in \mathcal{S} \subseteq \Delta(S)$, leads to a robust approach, that is, to maxminimization.

- *Classical maxminimization* is characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho]$$

for all $(\alpha, \sigma) \in \Delta(A) \times \Delta(S)$, so that

$$V(\alpha) = \min_{\sigma \in \Delta(S)} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] = \min_{s \in S} \sum_{a \in A} \alpha(a)\, u(\rho(a, s)). \tag{16}$$

- *Subjective expected utility* is characterized by

$$R(\alpha, \sigma) = \begin{cases} \mathbb{E}_{\alpha \times \mu}[u \circ \rho] & (\alpha, \sigma) \in \Delta(A) \times \{\mu\} \\ +\infty & \text{otherwise} \end{cases}$$

for some $\mu \in \Delta(S)$, so that

$$V(\alpha) = \mathbb{E}_{\alpha \times \mu}[u \circ \rho]. \tag{17}$$

- *Maxmin expected utility* (Gilboa and Schmeidler, 1989) is characterized by

$$R(\alpha, \sigma) = \begin{cases} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] & (\alpha, \sigma) \in \Delta(A) \times \mathcal{S} \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{S} \subseteq \Delta(S)$ is a compact set of probability distributions considered by the decision maker, so that

$$V(\alpha) = \min_{\sigma \in \mathcal{S}} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho]. \tag{18}$$

- *Variational preferences* (Maccheroni, Marinacci, and Rustichini, 2006) are characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] - c(\sigma) \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

where $c : \Delta(S) \to [0, \infty]$ is a lower semicontinuous cost function penalizing the probability distributions, so that

$$V(\alpha) = \min_{\sigma \in \Delta(S)} \{\mathbb{E}_{\alpha \times \sigma}[u \circ \rho] - c(\sigma)\}. \tag{19}$$

For example, denoting by $\mu \in \Delta(S)$ a reference probability and by $H(\sigma||\mu)$ the relative entropy of $\sigma \in \Delta(S)$ with respect to μ, the *multiplier preferences* of Hansen and Sargent (2001, 2008) correspond to the special case of variational preferences in which $c(\sigma)$ is proportional to $H(\sigma||\mu)$,[35] while their *constraint preferences* are maxmin expected utility preferences with $\mathcal{S} = \{\sigma \in \Delta(S) : H(\sigma||\mu) \leq \varepsilon\}$ for some $\varepsilon > 0$.

In order to obtain representation (15), due to Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), we need two final pieces of notation. First, given $u \in \mathbb{R}^C$, we denote by $U = \mathrm{co}(u(C))$ the smallest interval containing $u(C)$ and by $\mathcal{G}(U, \Delta(S))$ the set of functions $G : U \times \Delta(S) \to (-\infty, \infty]$ such that:

1. G is quasiconvex,

2. $\inf_{\sigma \in \Delta(S)} G(x, \sigma) = x$ for all $x \in U$,

3. G is increasing in the first component,

[35] See Strzalecki (2011) for an axiomatization.

4. the function $\varkappa \mapsto \inf_{\sigma \in \Delta(S)} G(\varkappa \cdot \sigma, \sigma)$ is continuous on U^S.

Second, for each $l \in -U$ and each $\sigma \in \Delta(S)$, we denote by $\mathcal{B}(l, \sigma)$ the set of all mixed actions that have Bayes risk level l under σ, that is,

$$\mathcal{B}(l, \sigma) = \{\beta \in \Delta(A) : r(\beta, \sigma) = l\}$$

and by $v(l, \sigma)$ the indirect payoff $v(\mathcal{B}(l, \sigma))$ of decision problem $(\mathcal{B}(l, \sigma), S, C, \rho)$, as defined in (8).

Theorem 5 *Let (A, S, C, ρ) be a Marschak-Radner framework and $\succsim$ be a binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) $\succsim$ is a nontrivial and continuous rational preference that satisfies uncertainty aversion;

(ii) there exist a nonconstant $u \in \mathbb{R}^C$ and $G \in \mathcal{G}(U, \Delta(S)) \to (-\infty, \infty]$ such that, if $\alpha, \beta \in \Delta(A)$,

$$\alpha \succsim \beta \iff \inf_{\sigma \in \Delta(S)} G(\mathbb{E}_{\alpha \times \sigma}[u \circ \rho], \sigma) \geq \inf_{\sigma \in \Delta(S)} G(\mathbb{E}_{\beta \times \sigma}[u \circ \rho], \sigma). \tag{20}$$

In this case, u is cardinally unique and, for each u, the minimal element of $\mathcal{G}(U, \Delta(S))$ satisfying (20) is the indirect payoff

$$G_u(x, \sigma) = v(-x, \sigma) \quad \forall (x, \sigma) \in U \times \Delta(S). \tag{21}$$

Continuous and uncertainty averse rational preferences are thus represented by

$$V(\alpha) = \inf_{\sigma \in \Delta(S)} G(\mathbb{E}_{\alpha \times \sigma}[u \circ \rho], \sigma) \quad \forall \alpha \in \Delta(A) \tag{22}$$

and (15) is obtained by setting

$$R(\alpha, \sigma) = G(\mathbb{E}_{\alpha \times \sigma}[u \circ \rho], \sigma) \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

and by choosing as $\mathcal{S}$ the projection on $\Delta(S)$ of the domain of R. In particular, when the minimal G_u described by (21) is considered,

$$R_u(\alpha, \sigma) = v(r(\alpha, \sigma), \sigma)$$

is the indirect payoff of the decision problem in which are available only the mixed actions with the same Bayes risk as α under σ. This is noteworthy since the indirect payoff $v(l, \sigma)$ can be seen as a comparative index of uncertainty aversion (see Cerreia-Vioglio et al., 2011).

Finally, Gilboa and Schmeidler (1989) show that in order to obtain the maxmin expected utility representation (18) it is necessary and sufficient to add an axiom guaranteeing the cardinal separation of payoffs and beliefs,[36] which we report in the form adopted by Maccheroni et al. (2006).

Axiom A. 9 (C-Independence) *If $\alpha, \beta \in \Delta(A)$, $\gamma, \zeta \in \Delta_\ell(A)$, and $p, q \in (0, 1]$, then*

$$p\alpha + (1-p)\gamma \succsim p\beta + (1-p)\gamma \Longrightarrow q\alpha + (1-q)\zeta \succsim q\beta + (1-q)\zeta.$$

It is worthwhile to mention that A.9 delivers maxmin expected utility in any Luce-Raiffa framework, that is, even if not all bets are conceivable.[37] The weakening of A.9 obtained by requiring $p = q$ is necessary and sufficient for the variational representation (19), as shown by Maccheroni et al. (2006).

[36] See Ghirardato, Maccheroni, and Marinacci (2005).

[37] Building on Gilboa et al. (2010), another way to obtain maxmin expected utility is replacing, in Theorem 2, 'extreme caution' with 'default to certainty' with respect to the unambiguous preference

$$\alpha \succsim^* \beta \iff q\alpha + (1-q)\eta \succsim q\beta + (1-q)\eta \quad \forall q \in [0, 1] \text{ and } \forall \eta \in \Delta(A).$$

See Ghirardato, Maccheroni, and Marinacci (2004).

7 Concluding remarks

7.1 Mixed Savage acts

After introducing the Anscombe-Aumann framework $\Delta(C)^S$, Kreps (1988) in Chapter 7 briefly considers the framework $\Delta(C^S)$ in which the objects of choice are lotteries with Savage acts as prizes. In the language adopted here, $\Delta(C^S)$ is the set of all mixed actions corresponding to the reduced decision framework (C^S, S, C, ϱ) of Savage in which all acts are conceivable and $\varrho(f, s) = f(s)$ (see Footnote 15 and point 3 of Proposition 2). Note that in this case, for each $\pi \in \Delta(C^S)$,

$$\varrho_\pi(c \mid s) = \pi\left(\left\{f \in C^S : \varrho(f, s) = c\right\}\right) = \pi\left(\left\{f \in C^S : f(s) = c\right\}\right) = \sum_{\{f \in \operatorname{supp}\pi : f(s) = c\}} \pi(f).$$

Therefore, ϱ_π is the Anscombe-Aumann act that Kreps calls $\phi(\pi)$ in Problem 3 (p. 111), where he also alludes at the role of mixed consequentialism.

His purpose is the foundation of subjective expected utility for the decision framework (C^S, S, C, ϱ). In this regard our analysis shows (Theorem 3) that it is actually unnecessary to assume that all acts are conceivable to obtain subjective expected utility; the exercise can be done in any Luce-Raiffa framework.

But, more importantly, our focus is different. Unlike Kreps, we are not after any axiomatic representation of preferences. The purpose of our analysis is, instead, to show that in applications there is little to lose and much to gain by replacing the framework of Anscombe and Aumann with that of Luce and Raiffa. In fact, tractability is preserved and interpretability is enhanced (as actions are often "few" and not naturally expressed as Savage acts), and the portability to the game theoretic/statistical/optimal control language is made immediate.

7.2 Independence and preference for randomization

In a Luce-Raiffa framework, the independence and uncertainty aversion axioms A.6 and A.8 are expressed for mixed actions, that is, for elements of $\Delta(A)$. Mathematically, these axioms have the same form as the independence and preference for randomization axioms of risk theory, which are expressed for lotteries. In general, we maintain independence on $\Delta_\ell(A) \simeq \Delta(C)$, the set of lotteries with consequences as prizes, and preference for randomization on $\Delta(A)$. This choice is motivated by the consequentialist theme of the paper and by the objective of remaining as close as possible to the uses and conventions of the Knightian uncertainty modelling. In particular, independence on $\Delta_\ell(A)$ is required by A.2.2, which is the analogue in a Luce-Raiffa framework of the standard risk independence axiom in an Anscombe-Aumann framework. The presence of states and the assumption of mixed consequentialism make independence on $\Delta(A)$ a strong requirement since it prevents payoff-hedging. Its restriction to $\Delta_\ell(A)$ is not vulnerable to this critique since hedging considerations are excluded by the state-independence of the distribution of consequences featured by the elements of $\Delta_\ell(A)$. On the contrary, hedging considerations are exactly those justifying uncertainty aversion on $\Delta(A)$.

In risk theory, preference for randomization may also capture aversion to uncertainty about the value of consequences, as discussed by Maccheroni (2002) and Cerreia-Vioglio (2009). This kind of uncertainty can also be viewed as uncertainty about a subjective state space, a concept introduced by Kreps (1979) and Dekel, Lipman and Rustichini (2001). Along these lines, we could also abandon independence on $\Delta_\ell(A)$, use the techniques of Cerreia-Vioglio (2009), and obtain a representation featuring both states of the world and subjective states.

Finally, observe that, while our analysis relies on the presence of randomization, an altogether different 'utilitarian' perspective on the Anscombe-Aumann framework that dispenses with randomization has been pursued by Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003).

7.3 Timing and commitment

The immersion $\alpha \overset{F}{\mapsto} \rho_\alpha$ of Section 3 may seem to associate an ex ante notion of randomization, featured by mixed actions, to an ex post one, featured by Anscombe-Aumann acts. Commitment, however, renders the distinction between ex ante and ex post empty. Specifically, in the Anscombe-Aumann framework an act $f : S \to \Delta(C)$ is a non-random object of choice with random consequences $f(s)$. Randomization is usually interpreted to occur *ex post*: the decision maker commits to f, 'observes' the realized state s, then 'observes' the consequence c generated by the random mechanism $f(s)$, and receives c.[38] In contrast, the mixed actions we consider are, by definition, random objects of choice. Randomization might be interpreted to occur either *ex ante* or *ex post*: in the first case, the decision maker commits to α, 'observes' the realized action a, then 'observes' the realized state s, and receives the consequence $c = \rho(a, s)$; in the second, the decision maker commits to α, 'observes' the realized state s, then 'observes' the realized action a, and receives the consequence $c = \rho(a, s)$.[39] Clearly the second interpretation conforms to the one of Anscombe-Aumann acts while the first does not. But commitment, that is, the impossibility of changing the action selected by the random device, makes the distinction immaterial.

Proposition 2.2 helps, *inter alia*, to further clarify this issue. Since each mixed action α can be written as a convex combination $\alpha = \sum_{a \in A} \alpha(a) \delta_a$ of point-masses, by affinity of F,

$$\rho_\alpha(s) = \sum_{a \in A} \alpha(a) \rho_{\delta_a}(s) \quad \forall s \in S \tag{23}$$

and $\rho_\alpha(s)$ is the chance distribution on C induced by the 'ex ante randomization' of actions a with probabilities $\alpha(a)$ if state s obtains. Consider the act $f_\alpha : S \to \Delta(C)$ given by

$$f_\alpha(s) = \sum_{a \in A} \alpha(a) \delta_{\rho(a,s)} \quad \forall s \in S \tag{24}$$

Now $f_\alpha(s)$ is the chance distribution on C induced by the 'ex post randomization' of consequences $\rho(a, s)$ with probabilities $\alpha(a)$ if state s obtains. Proposition 2.1 implies that $\rho_\alpha = f_\alpha$, thus showing that it is impossible to draw the distinction between 'ex ante' and 'ex post' in our abstract setup or in that of Fishburn (1970).[40]

In a richer setup, with two explicit layers of randomization, Anscombe and Aumann (1963) are able to formalize this issue and assume that 'it is immaterial whether the wheel is spun before or after the race'.[41] Here we do not pursue this matter anymore, but we refer to Sarin and Wakker (1997) and Wakker (2010, Ch. 4) for a further discussion on single-stage versus multi-stage perspectives on randomization. See also Eichberger, Grant, and Kelsey (2013) for a recent explicit dynamic choice model and Saito (2013) for a menu choice approach aimed at eliciting to what extent the decision maker believes in the hedging effects of randomization.

A Anscombe-Aumann axioms

Here we report the preferential axioms that are usually stated in the Anscombe-Aumann framework. In this section $\succsim_{\mathcal{F}}$ and $\succsim_{\mathcal{F}}^{\#}$ are binary relations on a convex subset $\mathcal{F}$ of $\Delta(C)^S$ that contains the set $\mathcal{C} \simeq \Delta(C)$ of all constant acts.

Axiom AA. 1 (Weak order) *$\succsim_{\mathcal{F}}$ is complete and transitive.*

Axiom AA. 2 (Risk independence) *If $\gamma, \zeta, \xi \in \mathcal{C}$, then $\gamma \sim_{\mathcal{F}} \zeta$ implies $\frac{1}{2}\gamma + \frac{1}{2}\xi \sim_{\mathcal{F}} \frac{1}{2}\zeta + \frac{1}{2}\xi$.*

[38] We say 'interpreted' since this timeline and the timed disclosure of information are unmodelled. In principle, one could think of the decision maker committing to f and receiving the outcome of the resulting process.

[39] Again the timeline and the timed disclosure of information are unmodelled. In principle, one could think of the decision maker committing to α and receiving the outcome of the resulting process.

[40] See Kuzmics (2012) on this issue.

[41] Seo (2009) studies the consequences of weakening this assumption.

Axiom AA. 3 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim_{\mathcal{F}} g(s)$ for all $s \in S$, then $f \succsim_{\mathcal{F}} g$.*

Axiom AA. 4 (Continuity) *If $f, g, h \in \mathcal{F}$, then $\{q \in [0,1] : qf + (1-q)\, g \succsim_{\mathcal{F}} h\}$ and $\{q \in [0,1] : h \succsim_{\mathcal{F}} qf + (1-q)\, g\}$ are closed sets.*

Axiom AA. 5 (Default to certainty) *If $f \in \mathcal{F}$, $\gamma \in \mathcal{C}$, and $f \not\succsim^{\#}_{\mathcal{F}} \gamma$, then $\gamma \succ_{\mathcal{F}} f$.*

Axiom AA. 6 (Independence) *If $f, g, h \in \mathcal{F}$, then $f \sim_{\mathcal{F}} g$ implies $\dfrac{1}{2}f + \dfrac{1}{2}h \sim_{\mathcal{F}} \dfrac{1}{2}g + \dfrac{1}{2}h$.*

Axiom AA. 7 (Non-triviality) *There exist $f, g \in \mathcal{F}$ such that $f \succ_{\mathcal{F}} g$.*

Axiom AA. 8 (Uncertainty aversion) *If $f, g \in \mathcal{F}$ and $q \in (0,1)$, $f \sim_{\mathcal{F}} g$ implies $qf + (1-q)\, g \succsim_{\mathcal{F}} f$.*

Axiom AA. 9 (C-Independence) *If $f, g \in \mathcal{F}$, $\gamma \in \mathcal{C}$, and $q \in (0,1]$, then*

$$f \succsim_{\mathcal{F}} g \iff qf + (1-q)\gamma \succsim_{\mathcal{F}} qg + (1-q)\gamma.$$

B Proofs

Throughout this appendix, if $\gamma \in \Delta(C)$ and $u \in \mathbb{R}^C$ we indifferently write

$$\sum_{c \in C} \gamma(c)\, u(c) \text{ or } \mathbb{E}_\gamma[u] \text{ or } u(\gamma).$$

Proof of Proposition 1. 1. For each $a \in A$, $\rho_a = \rho_a$ so that $a \approx a$ and by consequentialism $a \sim_A a$. Therefore $\succsim_A$ is reflexive and transitive, that is, a preorder. Conversely, let (A, S, C, ρ) be reduced and $\succsim_A$ be a preorder. For every $a, b \in A$, $a \approx b$ implies (because of reduction) $a = b$, and reflexivity of $\succsim_A$ delivers $a \sim_A b$. Therefore $\succsim_A$ satisfies consequentialism.

2. Let $c, d \in C$ and $a_c, b_c, a_d, b_d \in A$, not necessarily distinct, be such that

$$\rho(a_c, \cdot) = \rho(b_c, \cdot) \equiv c \text{ and } \rho(a_d, \cdot) = \rho(b_d, \cdot) \equiv d. \tag{25}$$

These sure actions exist since all sure actions are conceivable. By consequentialism, $\succsim_A$ is a preorder and (25) implies $a_c \sim_A b_c$ and $a_d \sim_A b_d$. Therefore, by transitivity of $\succsim_A$,

$$a_c \succsim_A a_d \iff b_c \succsim_A b_d$$

and $\succsim_C$ is well defined.[42] Reflexivity and transitivity of $\succsim_C$ descend from reflexivity and transitivity of $\succsim_A$.

3. The proof is similar to the one of the previous point, hence left to the reader. ■

Proof of Proposition 2. 1. Fix $s \in S$. For each $c \in C$,

$$\rho_{\delta_a}(c \mid s) = \delta_a(\{b \in A : \rho(b, s) = c\}) = \begin{cases} 1 & c = \rho(a, s) \\ 0 & \text{otherwise} \end{cases} = \delta_{\rho(a,s)}(c)$$

that is $\rho_{\delta_a}(s) = \delta_{\rho(a,s)}$. Write x instead of δ_x if either $x \in A$ or $x \in C$, then

$$F(a) = F(\delta_a) = \begin{bmatrix} \rho_{\delta_a}(s_1) \\ \rho_{\delta_a}(s_2) \\ \vdots \\ \rho_{\delta_a}(s_n) \end{bmatrix} = \begin{bmatrix} \delta_{\rho(a,s_1)} \\ \delta_{\rho(a,s_2)} \\ \vdots \\ \delta_{\rho(a,s_n)} \end{bmatrix} = \begin{bmatrix} \rho(a, s_1) \\ \rho(a, s_2) \\ \vdots \\ \rho(a, s_n) \end{bmatrix} = \rho_a$$

[42] Specifically, $\succsim_C = \{(c, d) \in C \times C : a_c \succsim a_d$ for some/all $a_c, a_d \in A$ such that $\rho_{a_c} \equiv c, \rho_{a_d} \equiv d\}$.

as desired.

2. Fix $s \in S$. For each $c \in C$,

$$\begin{aligned}\rho_{q\alpha+(1-q)\beta}(c \mid s) &= (q\alpha + (1-q)\beta)(\{a \in A : \rho(a,s) = c\}) \\ &= q\alpha(\{a \in A : \rho(a,s) = c\}) + (1-q)\beta(\{a \in A : \rho(a,s) = c\}) \\ &= q\rho_\alpha(c \mid s) + (1-q)\rho_\beta(c \mid s)\end{aligned}$$

that is, $\rho_{q\alpha+(1-q)\beta} = q\rho_\alpha + (1-q)\rho_\beta$.

3. Let D be an arbitrary nonempty subset of C. Then $\Delta(D)$ is convex in $\mathbb{R}^D$, and its extreme points are the point-masses $\{\delta_d\}_{d \in D}$. Therefore $\Delta(D)^S$ is convex too. Next we show that its extreme points are the vectors of point-masses. If $g \in \Delta(D)^S$ is not extreme there exist $g', g'' \in \Delta(D)^S$ with $g' \neq g''$, say $0 \leq g'(d \mid s) < g''(d \mid s) \leq 1$, and $q \in (0,1)$ such that $g = qg' + (1-q)g''$. Then $g(d \mid s) = qg'(d \mid s) + (1-q)g''(d \mid s) \in (0,1)$ and $g(s)$ is not a point-mass. Conversely, if $g \in \Delta(D)^S$ is not a vector of point-masses, then $g(s)$ is not a point-mass for some $s \in S$. Therefore, there exist $g'(s), g''(s) \in \Delta(D)$ with $g'(s) \neq g''(s)$ and $q \in (0,1)$ such that $g(s) = qg'(s) + (1-q)g''(s)$. But this implies that $g = qg' + (1-q)g''$ where g' (resp. g'') is obtained replacing the s-th element $g(s)$ of the vector g with $g'(s)$ (resp. $g''(s)$). Thus g is not extreme.

In particular, the generic extreme point of $\Delta(D)^S$ is

$$\begin{bmatrix}\delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)}\end{bmatrix} \quad \text{with} \quad \begin{bmatrix}e(s_1) \\ e(s_2) \\ \vdots \\ e(s_n)\end{bmatrix} = e \in D^S.$$

If part. Assume $\{\rho_a\}_{a \in A} = C^S$. For each $e \in C^S$, there exists $a_e \in A$ such that $\rho_{a_e} = e$. For each $f \in \Delta(C)^S$, set $D = \bigcup_{s \in S} \mathrm{supp} f(s)$. Then $\Delta(D)^S$ is compact in $\mathbb{R}^D$ since D is finite. By the Krein–Milman Theorem, there exists $\phi \in \Delta(D^S)$ such that

$$f = \sum_{e \in D^S} \phi(e) \begin{bmatrix}\delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)}\end{bmatrix} = \sum_{e \in D^S} \phi(e) \begin{bmatrix}\delta_{\rho(a_e,s_1)} \\ \delta_{\rho(a_e,s_2)} \\ \vdots \\ \delta_{\rho(a_e,s_n)}\end{bmatrix} = \sum_{e \in D^S} \phi(e) \begin{bmatrix}\rho_{\delta_{a_e}}(s_1) \\ \rho_{\delta_{a_e}}(s_2) \\ \vdots \\ \rho_{\delta_{a_e}}(s_n)\end{bmatrix} \tag{26}$$

where the last equality follows from point 1 of this proposition. By (26) and affinity of $\digamma$ (point 2 of this proposition),

$$f = \sum_{e \in D^S} \phi(e) \rho_{\delta_{a_e}} = \sum_{e \in D^S} \phi(e) \digamma(\delta_{a_e}) = \digamma\left(\sum_{e \in D^S} \phi(e) \delta_{a_e}\right)$$

where $\sum_{e \in D^S} \phi(e) \delta_{a_e} \in \Delta(A)$ since D^S is finite and $\Delta(A)$ is convex. Therefore, $\digamma$ is onto.

Only if part. Assume $\digamma(\Delta(A)) = \Delta(C)^S$. For each $e \in C^S$, there exists $\alpha \in \Delta(A)$ such that

$$\rho_\alpha(s) = \delta_{e(s)} \quad \forall s \in S. \tag{27}$$

Partition the finite support of α, into realization equivalence classes so that

$$\mathrm{supp}\alpha = \bigsqcup_{i=1}^{k} A_i$$

with $\rho_{a_i} = \rho_{a'_i}$ for all $a_i, a'_i \in A_i$ and all $i = 1, 2, ..., k$ and $\rho_{a_i} \neq \rho_{a_j}$ if $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$. Moreover, for each $i = 1, 2, ..., k$ arbitrarily select $b_i \in A_i$. By (27), for each $s \in S$,

$$\delta_{e(s)} = \sum_{a \in \mathrm{supp}\alpha} \alpha(a) \rho_{\delta_a}(s) = \sum_{i=1}^{k} \left(\sum_{a_i \in A_i} \alpha(a_i) \rho_{\delta_{a_i}}(s) \right)$$

but, for each $i = 1, 2, ..., k$, and all $a_i \in A_i$,

$$\rho_{\delta_{a_i}}(s) = \delta_{\rho(a_i,s)} = \delta_{\rho(b_i,s)} = \rho_{\delta_{b_i}}(s).$$

Therefore, setting $\beta(b_i) = \sum_{a_i \in A_i} \alpha(a_i)$ for all $i = 1, 2, ..., k$ and $\beta(a) = 0$ if $a \in A \setminus \{b_1, ..., b_k\}$, $\beta \in \Delta(A)$ is such that

$$\delta_{e(s)} = \sum_{i=1}^{k} \left(\sum_{a_i \in A_i} \alpha(a_i) \right) \rho_{\delta_{b_i}}(s) = \sum_{i=1}^{k} \beta(b_i) \rho_{\delta_{b_i}}(s) = \sum_{i=1}^{k} \beta(b_i) \delta_{\rho(b_i,s)}$$

and

$$\begin{bmatrix} \delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)} \end{bmatrix} = \sum_{i=1}^{k} \beta(b_i) \begin{bmatrix} \delta_{\rho(b_i,s_1)} \\ \delta_{\rho(b_i,s_2)} \\ \vdots \\ \delta_{\rho(b_i,s_n)} \end{bmatrix}. \tag{28}$$

The summands $\left[\delta_{\rho(b_i,s_1)} \ ... \ \delta_{\rho(b_i,s_n)}\right]^\intercal$ are distinct (extreme) points in $\Delta(C)^S$ and $\left[\delta_{e(s_1)} \ ... \ \delta_{e(s_1)}\right]^\intercal$ is an extreme point of $\Delta(C)^S$. Then $\beta = \delta_b$ for some $b \in \mathrm{supp}\beta \subseteq A$, and (28) implies $e = \rho_b$. The arbitrary choice of e allows to conclude $C^S \subseteq \{\rho_a\}_{a \in A}$ and the converse inclusion is trivial. ■

Proof of Corollary 1. If $\beta \neq \delta_a$ there exists $b \neq a$ such that $\beta(b) \neq 0$. Since the framework is reduced $\rho_b \neq \rho_a$ and there exists $s \in S$ such that $\rho(a,s) \neq \rho(b,s)$. Setting $c = \rho(b,s)$, it follows

$$\rho_\beta(c \mid s) \geq \beta(b) > 0 = \delta_{\rho(a,s)}(c) = \rho_{\delta_a}(c \mid s)$$

and hence $\rho_\beta \neq \rho_{\delta_a}$ and $\delta_a \not\approx \beta$. By contrapositive $\delta_a \approx \beta \Longrightarrow \beta = \delta_a$, and the converse is trivial. ■

Lemma 1 *Let (A, S, C, ρ) be a decision framework, $\alpha \in \Delta(A)$, $u \in \mathbb{R}^C$, and $\mu \in \Delta(S)$. Then:*

1. $\rho_\alpha(s) = \sum_{a \in A} \alpha(a) \delta_{\rho(a,s)}$ *for all $s \in S$;*
2. $u(\rho_\alpha(s)) = \mathbb{E}_{\rho_\alpha(s)}[u] = \sum_{a \in A} \alpha(a) u(\rho(a,s))$;
3. $\int_S u(\rho_\alpha(s)) d\mu(s) = \mathbb{E}_{\alpha \times \mu}[u \circ \rho]$.

Moreover, if (A, S, C, ρ) is a Luce-Raiffa framework, then

$$\begin{array}{cccc} \epsilon: & \Delta(C) & \to & \Delta_\ell(A) \\ & \sum_{c \in C} \gamma(c) \delta_c & \mapsto & \sum_{c \in C} \gamma(c) \delta_{cS} \end{array}$$

is affine and bijective. In particular:

4. *for each $\alpha = \sum_{c \in C} \alpha(cS) \delta_{cS} \in \Delta_\ell(A)$ the only $\gamma \in \Delta(C)$ such that $\epsilon(\gamma) = \alpha$ is $\epsilon^{-1}(\alpha) = \sum_{c \in C} \alpha(cS) \delta_c$;*
5. *for each $\gamma \in \Delta(C)$ and each $s \in S$, $\rho_{\epsilon(\gamma)}(s) = \gamma$.*

Proof. 1. Follows from points 1 and 2 of Proposition 2. Specifically,

$$\rho_\alpha = F\left(\sum_{a\in A}\alpha(a)\,\delta_a\right) = \sum_{a\in A}\alpha(a)\,F(\delta_a) = \sum_{a\in A}\alpha(a)\,\rho_{\delta_a}$$

and therefore for each $s \in S$

$$\rho_\alpha(s) = \sum_{a\in A}\alpha(a)\,\rho_{\delta_a}(s) = \sum_{a\in A}\alpha(a)\,\delta_{\rho(a,s)}.$$

2. By definition, for all $s \in S$,

$$u(\rho_\alpha(s)) = \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\sum\limits_{a\in A}\alpha(a)\delta_{\rho(a,s)}}[u] = \sum_{a\in A}\alpha(a)\,\mathbb{E}_{\delta_{\rho(a,s)}}[u] = \sum_{a\in A}\alpha(a)\,u(\rho(a,s)).$$

3. By definition,

$$\int_S u(\rho_\alpha(s))\,d\mu(s) = \sum_{s\in S}\mu(s)\sum_{a\in A}\alpha(a)\,u(\rho(a,s)) = \sum_{a\in A}\sum_{s\in S}\alpha(a)\,\mu(s)\,u(\rho(a,s)) = \mathbb{E}_{\alpha\times\mu}[u\circ\rho].$$

If (A, S, C, ρ) is a Luce-Raiffa framework, since all sure actions are conceivable for each $c \in C$ there exists a sure action $a_c \in A$ such that $\rho(a_c, \cdot) \equiv c$, reduction instead guarantees that such sure action is unique and we denote it by cS. In other words the map $c \mapsto cS$ is well defined. It is also easy to check that its range is the set of all sure actions and that the map is injective. Therefore $c \hookrightarrow cS$ is an embedding of C onto the set of all sure actions. The fact that ϵ is affine and bijective immediately follows.

4. Clearly $\gamma = \sum\limits_{c\in C}\alpha(cS)\,\delta_c \in \Delta(C)$ and $\gamma(c) = \alpha(cS)$ for all $c \in C$, by definition of ϵ it follows $\epsilon(\gamma) = \sum\limits_{c\in C}\gamma(c)\,\delta_{cS} = \alpha$.

5. By definition $\epsilon(\gamma) = \sum\limits_{c\in\mathrm{supp}\gamma}\gamma(c)\,\delta_{cS}$, therefore, for each $s \in S$,

$$\rho_{\epsilon(\gamma)}(s) = \rho_{\sum\limits_{c\in\mathrm{supp}\gamma}\gamma(c)\delta_{cS}}(s) = \sum_{c\in\mathrm{supp}\gamma}\gamma(c)\,\rho_{\delta_{cS}}(s) = \sum_{c\in\mathrm{supp}\gamma}\gamma(c)\,\delta_{\rho(cS,s)} = \sum_{c\in\mathrm{supp}\gamma}\gamma(c)\,\delta_c = \gamma. \tag{29}$$

■

Proof of Theorem 1. (i) $\Longrightarrow$ (ii). Since $\succsim$ satisfies A.1, then

$$\Gamma_{\succsim}(\mathcal{A}) = \{\alpha \in \mathcal{A} : \alpha \succsim \beta\ \forall\beta \in \mathcal{A}\} \quad \forall\mathcal{A} \in \wp(\Delta(A))$$

is a rational choice correspondence and generates $\succsim$. Since $\succsim$ also satisfies A.2, then $\succsim_{\Delta(C)}$ satisfies Axioms 1, 2, and 3 of Hernstein and Milnor (1953) and there exists $u \in \mathbb{R}^C$ such that, if $\gamma, \zeta \in \Delta(C)$, then

$$\gamma \succsim_{\Delta(C)} \zeta \iff \sum_{c\in C}\gamma(c)\,u(c) \geq \sum_{c\in C}\zeta(c)\,u(c)$$

and, by definition of $\succsim_{\Delta(C)}$ this means

$$\epsilon(\gamma) \succsim \epsilon(\zeta) \iff \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u].$$

Therefore, if $\alpha, \beta \in \Delta(A)$, then

$$\begin{aligned}
\alpha \succcurlyeq_S \beta &\iff \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s) \quad \forall s \in S \\
&\iff u(\rho_\alpha(s)) \geq u(\rho_\beta(s)) \quad \forall s \in S \\
&\iff \sum_{a\in A}\alpha(a)\,u(\rho(a,s)) \geq \sum_{a\in A}\beta(a)\,u(\rho(a,s)) \quad \forall s \in S \\
&\iff \alpha \geqslant_u \beta.
\end{aligned}$$

Now A.3 implies

$$\alpha \geqslant_u \beta \implies \alpha \succsim \beta \tag{30}$$

moreover, if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha = \epsilon(\gamma)$ and $\beta = \epsilon(\zeta)$ for some $\gamma, \zeta \in \Delta(C)$, and

$$\alpha \succsim \beta \iff \gamma \succsim_{\Delta(C)} \zeta \iff \rho_{\epsilon(\gamma)}(s) \succsim_{\Delta(C)} \rho_{\epsilon(\zeta)}(s) \quad \forall s \in S \iff \alpha \geqslant_u \beta$$

that is $\succsim$, $\succcurlyeq_S$, and $\geqslant_u$ coincide on $\Delta_\ell(A)$ where

$$\alpha \succsim \beta \iff \gamma \succsim_{\Delta(C)} \zeta \iff \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u] \iff \sum_{c \in C} \alpha(cS) u(c) \geq \sum_{c \in C} \beta(cS) u(c).$$

This immediately implies, for each $\mathcal{A} \in \wp(\Delta(A))$ such that $\mathcal{A} \subseteq \Delta_\ell(A)$,

$$\begin{aligned} \Gamma_{\succsim}(\mathcal{A}) &= \{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\} \\ &= \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta \ \forall \beta \in \mathcal{A}\} = \Upsilon_u(\mathcal{A}) \end{aligned}$$

For a generic $\mathcal{A} \in \wp(\Delta(A))$, by (30),

$$\begin{aligned} \Upsilon_u(\mathcal{A}) &= \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta \ \forall \beta \in \mathcal{A}\} \\ &\subseteq \{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\} = \Gamma_{\succsim}(\mathcal{A}). \end{aligned}$$

(ii) $\implies$ (i). If $\succsim$ is generated by a rational choice correspondence $\Gamma : \wp(\Delta(A)) \to \wp(\Delta(A))$, then $\succsim$ satisfies A.1. By (ii) there exists also $u \in \mathbb{R}^C$ such that:

$$\Upsilon_u(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) \text{ for all } \mathcal{A} \in \wp(\Delta(A)) \text{ and equality holds if } \mathcal{A} \subseteq \Delta_\ell(A).$$

Therefore, if $\alpha, \beta \in \Delta(A)$,

$$\begin{aligned} \alpha \succcurlyeq_S \beta &\iff \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s) \quad \forall s \in S \\ &\iff \epsilon(\rho_\alpha(s)) \succsim \epsilon(\rho_\beta(s)) \quad \forall s \in S \\ &\iff \epsilon(\rho_\alpha(s)) \in \Gamma(\{\epsilon(\rho_\alpha(s)), \epsilon(\rho_\beta(s))\}) \quad \forall s \in S \end{aligned}$$

but $\{\epsilon(\rho_\alpha(s)), \epsilon(\rho_\beta(s))\} \subseteq \Delta_\ell(A)$ for all $s \in S$, that is

$$\alpha \succcurlyeq_S \beta \iff \epsilon(\rho_\alpha(s)) \in \Upsilon_u(\{\epsilon(\rho_\alpha(s)), \epsilon(\rho_\beta(s))\}) \quad \forall s \in S \iff \epsilon(\rho_\alpha(s)) \geqslant_u \epsilon(\rho_\beta(s)) \quad \forall s \in S. \tag{31}$$

Moreover, given $\eta, \kappa \in \Delta(A)$

$$\eta \geqslant_u \kappa \iff \sum_{a \in A} \eta(a) u(\rho(a, w)) \geq \sum_{a \in A} \kappa(a) u(\rho(a, w)) \quad \forall w \in S \iff \mathbb{E}_{\rho_\eta(w)}[u] \geq \mathbb{E}_{\rho_\kappa(w)}[u] \quad \forall w \in S$$

and (31) becomes

$$\alpha \succcurlyeq_S \beta \iff \mathbb{E}_{\rho_{\epsilon(\rho_\alpha(s))}(w)}[u] \geq \mathbb{E}_{\rho_{\epsilon\left(\rho_\beta(s)\right)}(w)}[u] \quad \forall w, s \in S$$

but, $\rho_{\epsilon(\gamma)} \equiv \gamma$ for all $\gamma \in \Delta(C)$, therefore

$$\alpha \succcurlyeq_S \beta \iff \mathbb{E}_{\rho_\alpha(s)}[u] \geq \mathbb{E}_{\rho_\beta(s)}[u] \quad \forall w, s \in S \iff \sum_{a \in A} \alpha(a) u(\rho(a, s)) \geq \sum_{a \in A} \beta(a) u(\rho(a, s)) \quad \forall s \in S \tag{32}$$

$$\iff \alpha \geqslant_u \beta \implies \alpha \in \Upsilon_u(\{\alpha, \beta\}) \subseteq \Gamma(\{\alpha, \beta\}) \implies \alpha \succsim \beta$$

and $\succsim$ satisfies A.3.

Finally, if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha = \epsilon(\gamma)$ and $\beta = \epsilon(\zeta)$ for some $\gamma, \zeta \in \Delta(C)$, and

$$\alpha \succsim \beta \iff \epsilon(\gamma) \succsim \epsilon(\zeta) \iff \gamma \succsim_{\Delta(C)} \zeta \iff \rho_{\epsilon(\gamma)}(s) \succsim_{\Delta(C)} \rho_{\epsilon(\zeta)}(s) \quad \forall s \in S \iff \alpha \succcurlyeq_S \beta$$

but $\operatorname{supp}\alpha, \operatorname{supp}\beta \subseteq \{cS\}_{c \in C}$ and by (32)

$$\begin{aligned} \alpha \succsim \beta &\iff \sum_{a \in A} \alpha(a) u(\rho(a,s)) \geq \sum_{a \in A} \beta(a) u(\rho(a,s)) \quad \forall s \in S \\ &\iff \sum_{c \in C} \alpha(cS) u(\rho(cS,s)) \geq \sum_{c \in C} \beta(cS) u(\rho(cS,s)) \quad \forall s \in S \\ &\iff \sum_{c \in C} \alpha(cS) u(c) \geq \sum_{c \in C} \beta(cS) u(c) \end{aligned}$$

and so $\succsim$ satisfies Axioms 2 and 3 of Hernstein and Milnor (1953) on $\Delta_\ell(A)$, that is, A.2. ■

Recall that if $\gamma \in \Delta(C)$ and $u \in \mathbb{R}^C$ we indifferently write $\mathbb{E}_\gamma[u]$ or $u(\gamma)$; with a similar abuse, if $f \in \Delta(C)^S$ we denote by $u(f)$ the element of $\mathbb{R}^S$ defined by

$$u(f) = \begin{bmatrix} u(f(s_1)) \\ u(f(s_2)) \\ \vdots \\ u(f(s_n)) \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{f(s_1)}[u] \\ \mathbb{E}_{f(s_2)}[u] \\ \vdots \\ \mathbb{E}_{f(s_n)}[u] \end{bmatrix}.$$

Lemma 2 *Let (A, S, C, ρ) be a Luce-Raiffa framework, $\mathcal{F} = F(\Delta(A))$, $u \in \mathbb{R}^C$, and $\succsim$ be a preorder on $\Delta(A)$. Then:*

1. *$\mathcal{F}$ is a convex subset of $\Delta(C)^S$ containing all constant Anscombe-Aumann acts;*
2. *if $\succsim$ satisfies A.3, then $\succsim$ satisfies mixed consequentialism;*
3. *if all bets are conceivable, $\{u(f) : f \in \mathcal{F}\} = (\operatorname{co}(u(C)))^S = u(\Delta(C))^S$.*

Moreover, if $\succsim$ satisfies mixed consequentialism, then $\succsim_F$ is a well defined preorder on $\mathcal{F}$ and:

4. *for $N = 1, 3, 4, 6, 7, 9$, $\succsim_F$ satisfies AA.N if and only if $\succsim$ satisfies A.N;*
5. *AA.2 for $\succsim_F$ is implied by A.2 for $\succsim$ (and they are equivalent under A.4);*
6. *AA.5 for $\succsim_F$ is equivalent to A.5 for $\succsim$ when $\succsim_F^\#$ is defined by*

$$f \succsim_F^\# g \iff f(s) \succsim_F g(s) \quad \forall s \in S;$$

7. *AA.8 for $\succsim_F$ implies A.8 for $\succsim$ (and they are equivalent under A.1 and A.4).*

Proof. 1. Follows from affinity of F and the fact that $\rho_{\epsilon(\gamma)} \equiv \gamma$ for all $\gamma \in \Delta(C)$.

2. If $\alpha, \beta \in \Delta(A)$ and $\alpha \approx \beta$, then $\rho_\alpha(s) = \rho_\beta(s)$ for all $s \in S$; since $\succsim$ is a preorder $\epsilon(\rho_\alpha(s)) \succsim \epsilon(\rho_\beta(s))$, that is, $\rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s)$ for all $s \in S$, and A.3 implies $\alpha \succsim \beta$. By a symmetric argument $\beta \succsim \alpha$.

3. Obviously, $\{u(f) : f \in \mathcal{F}\} \subseteq (\operatorname{co}(u(C)))^S = u(\Delta(C))^S$. Set $U = \operatorname{co}(u(C))$ and assume $u(C)$ is not a singleton, otherwise the result is trivial. For every vector $\varkappa = [x_1 \ldots x_n]^\intercal \in U^S$ there exist $c \neq d$ in C and $q_1, q_2, \ldots, q_n \in [0,1]$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 u(c) + (1-q_1) u(d) \\ q_2 u(c) + (1-q_2) u(d) \\ \vdots \\ q_n u(c) + (1-q_n) u(d) \end{bmatrix} = \begin{bmatrix} u(q_1\delta_c + (1-q_1)\delta_d) \\ u(q_2\delta_c + (1-q_2)\delta_d) \\ \vdots \\ u(q_n\delta_c + (1-q_n)\delta_d) \end{bmatrix}.$$

Set $D = \{c, d\}$, set $B = \{a \in A : \rho_a(S) \subseteq \{c, d\}\}$, and consider the Luce-Raiffa framework (B, S, D, ρ). Since all the bets are conceivable in (A, S, C, ρ), then $\{\rho_b\}_{b \in B} = D^S$. In particular, by point 3 of Proposition 2 for each $f \in \Delta(D)^S$, there exists $\beta_f \in \Delta(B)$ such that

$$\beta_f(\{b \in B : \rho(b, s) = c\}) = f(c|s) \quad \forall s \in S.$$

Choose $f \in \Delta(D)^S$ such that $f(c \mid s_i) = q_i = 1 - f(d \mid s_i)$ for all $i = 1, ..., n$, and set $\alpha(a) = \beta_f(a)$ if $a \in B$ and $\alpha(a) = 0$ if $a \in A \setminus B$. Then, $\alpha \in \Delta(A)$ and, for all $i = 1, ..., n$,

$$\begin{aligned} \rho_\alpha(c \mid s_i) &= \beta_f(\{b \in B : \rho(b, s_i) = c\}) = f(c \mid s_i) = q_i \\ \rho_\alpha(d \mid s_i) &= f(d \mid s_i) = 1 - q_i \end{aligned}$$

so that $\rho_\alpha(s_i) = q_i \delta_c + (1 - q_i)\delta_d$ and $u(\rho_\alpha(s_i)) = q_i u(c) + (1 - q_i) u(d)$, that is, $u(\rho_\alpha) = \varkappa$.

Assume $\succsim$ satisfies mixed consequentialism. Let $f, g \in \mathcal{F}$ and $\alpha_f, \beta_f, \alpha_g, \beta_g \in \Delta(A)$, not necessarily distinct, be such that

$$\rho_{\alpha_f} = \rho_{\beta_f} = f \text{ and } \rho_{\alpha_g} = \rho_{\beta_g} = g. \tag{33}$$

These mixed actions exist since $\mathcal{F} = F(\Delta(A))$. By mixed consequentialism, (33) implies $\alpha_f \sim \beta_f$ and $\alpha_g \sim \beta_g$ Therefore, by transitivity of $\succsim$,

$$\alpha_f \succsim \alpha_g \iff \beta_f \succsim \beta_g$$

and $\succsim_F$ is well defined.[43] Reflexivity and transitivity of $\succsim_F$ descend from reflexivity and transitivity of $\succsim$.

The verification of points 4, 5, and 6 is routine.

7. Assume $\succsim_F$ satisfies AA.8. If $\alpha, \beta \in \Delta(A)$ and $\alpha \sim \beta$, then $\rho_\alpha \sim_F \rho_\beta$. By AA.8, $2^{-1}\rho_\alpha + 2^{-1}\rho_\beta \succsim_F \rho_\alpha$, therefore $\rho_{2^{-1}\alpha + 2^{-1}\beta} \succsim_F \rho_\alpha$ and $2^{-1}\alpha + 2^{-1}\beta \succsim \alpha$, so that $\succsim$ satisfies A.8.

Conversely, assume $\succsim$ satisfies A.1 and A.4. Next we show that A.8 implies that for all $\alpha, \beta \in \Delta(A)$ such that $\alpha \sim \beta$ and all $q \in (0, 1)$ it holds $q\alpha + (1 - q)\beta \succsim \alpha$. Per contra, assume there exist $\alpha, \beta \in \Delta(A)$ and $p \in (0, 1)$ such that $\alpha \sim \beta$ and $p\alpha + (1 - p)\beta \prec \alpha$. Set

$$T = \{t \in [0, 1] : t\alpha + (1 - t)\beta \prec \alpha\}.$$

Clearly $p \in T$. Moreover, by A.1 and A.4, T is open in $[0, 1]$ and hence there exists O open in $\mathbb{R}$ such that $T = O \cap [0, 1]$, but $T \subseteq (0, 1)$, therefore $T = O \cap (0, 1)$ is open in $\mathbb{R}$. Therefore there exists an open interval in T that contains p. The set

$$I = \bigcup_{p \ni (q,r) \subseteq T} (q, r)$$

is a union of pairwise overlapping open intervals, and so it is an open interval itself: $p \in I = (\bar{q}, \bar{r}) \subseteq T \subseteq (0, 1)$. If $\bar{q} \in T$, there would exist $\varepsilon > 0$ such that $(\bar{q} - \varepsilon, \bar{q} + \varepsilon) \subseteq T$, and then $p \in (\bar{q} - \varepsilon, \bar{r}) \subseteq T$, whence

$$(\bar{q} - \varepsilon, \bar{r}) \subseteq I = (\bar{q}, \bar{r})$$

a contradiction. Therefore $\bar{q} \notin T$ and (analogously) $\bar{r} \notin T$, that is,

$$\bar{q}\alpha + (1 - \bar{q})\beta \succsim \alpha \text{ and } \bar{r}\alpha + (1 - \bar{r})\beta \succsim \alpha.$$

Since $(\bar{q}, \bar{r}) \subseteq T$ is nonempty, eventually, $\bar{q} + n^{-1}$ and $\bar{r} - n^{-1}$ belong to T, and by A.4

$$\bar{q}\alpha + (1 - \bar{q})\beta \precsim \alpha \text{ and } \bar{r}\alpha + (1 - \bar{r})\beta \precsim \alpha.$$

[43] In fact, $\{(f, g) \mid \alpha \succsim \beta, \forall \alpha, \beta \in \Delta(A) : \rho_\alpha = f \text{ and } \rho_\beta = g\} = \{(f, g) \mid \exists \alpha, \beta \in \Delta(A) : \rho_\alpha = f,\ \rho_\beta = g, \text{ and } \alpha \succsim \beta\} \subseteq \mathcal{F} \times \mathcal{F}$ and the relation $\succsim_F$ they define on $\mathcal{F}$ is such that, if $\alpha, \beta \in \Delta(A)$ then $\alpha \succsim \beta \iff \rho_\alpha \succsim_F \rho_\beta$.

That is, $\bar{q}\alpha + (1-\bar{q})\beta \sim \bar{r}\alpha + (1-\bar{r})\beta \sim \alpha$, and A.8 implies

$$\left(\frac{\bar{q}}{2}+\frac{\bar{r}}{2}\right)\alpha + \left(1-\frac{\bar{q}}{2}-\frac{\bar{r}}{2}\right)\beta = \frac{1}{2}\left(\bar{q}\alpha + (1-\bar{q})\beta\right) + \frac{1}{2}\left(\bar{r}\alpha + (1-\bar{r})\beta\right) \succsim \bar{r}\alpha + (1-\bar{r})\beta \sim \alpha$$

but this is a contradiction since $2^{-1}\bar{q} + 2^{-1}\bar{r} \in (\bar{q}, \bar{r}) \subseteq T$.

Now let $f, g \in \mathcal{F}$ be such that $f \sim_F g$ and arbitrarily choose $q \in (0,1)$. Let $\alpha, \beta \in \Delta(A)$ be such that $f = \rho_\alpha$ and $g = \rho_\beta$, then $f \sim_F g$ implies $\alpha \sim \beta$, A.1, A.4, and A.8 imply $q\alpha + (1-q)\beta \succsim \alpha$, whence $\rho_{q\alpha+(1-q)\beta} \succsim_F \rho_\alpha$ and $qf + (1-q)g = q\rho_\alpha + (1-q)\rho_\beta \succsim_F \rho_\alpha = f$. As wanted. ■

Proof of Proposition 3. It immediately follows from Lemma 2. ■

Proofs of Theorems 2, 3, 4, and 5. Let $\succsim$ be a rational preference. As shown in the first part of the proof of Theorem 1, there exists $u \in \mathbb{R}^C$ such that:

- if $\gamma, \zeta \in \Delta(C)$, then $\gamma \succsim_{\Delta(C)} \zeta \iff \sum_{c\in C} \gamma(c) u(c) \geq \sum_{c \in C} \zeta(c) u(c)$;
- if $\alpha, \beta \in \Delta(A)$, then $\alpha \succcurlyeq_S \beta \iff \alpha \geqslant_u \beta \iff \sum_{a\in A} \alpha(a) u(\rho(a,s)) \geq \sum_{a\in A} \beta(a) u(\rho(a,s))$ for all $s \in S$;
- if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha \succsim \beta \iff \alpha \succcurlyeq_S \beta \iff \sum_{c\in C} \alpha(cS) u(c) \geq \sum_{c\in C} \beta(cS) u(c)$.

If moreover $\succsim$ is continuous, for every $\alpha \in \Delta(A)$ there exists $\alpha_\ell \in \Delta(C)$ such that $\alpha \sim \alpha_\ell$, more precisely, $\alpha \sim \epsilon(\alpha_\ell)$. In fact, choosing w and m in S so that

$$\rho_\alpha(m) \succsim_{\Delta(C)} \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\alpha(w) \quad \forall s \in S \tag{34}$$

it follows $\epsilon(\rho_\alpha(m)) \succcurlyeq_S \alpha \succcurlyeq_S \epsilon(\rho_\alpha(w))$. Whence $\beta = \epsilon(\rho_\alpha(m))$ and $\eta = \epsilon(\rho_\alpha(w))$ belong to $\Delta_\ell(A)$ and, by A.3, $\beta \succsim \alpha \succsim \eta$. Therefore, the nonempty and closed (by A.4) sets

$$\{q \in [0,1] : q\eta + (1-q)\beta \succsim \alpha\} \text{ and } \{q \in [0,1] : \alpha \succsim q\eta + (1-q)\beta\}$$

cover (by A.1) the connected set $[0,1]$. In particular, they cannot be disjoint, and there is $q_\ell \in [0,1]$ such that $q_\ell\eta + (1-q_\ell)\beta \sim \alpha$; convexity of $\Delta_\ell(A) = \epsilon(\Delta(C))$ implies that $q_\ell\eta + (1-q_\ell)\beta = \epsilon(\alpha_\ell)$ for some $\alpha_\ell \in \Delta(C)$.

If $\alpha, \beta \in \Delta(A)$, then

$$\alpha \succsim \beta \iff \epsilon(\alpha_\ell) \succsim \epsilon(\beta_\ell) \iff \alpha_\ell \succsim_{\Delta(C)} \beta_\ell \iff \sum_{c\in C} \alpha_\ell(c) u(c) \geq \sum_{c\in C} \beta_\ell(c) u(c)$$

that is, the functional

$$\begin{array}{cccc} V: & \Delta(A) & \to & \mathbb{R} \\ & \alpha & \mapsto & \sum_{c\in C} \alpha_\ell(c) u(c) \end{array}$$

represents $\succsim$ on $\Delta(A)$.[44]

For each $\alpha \in \Delta(A)$, by (34) and A.3, $\epsilon(\rho_\alpha(m)) \succsim \alpha \succsim \epsilon(\rho_\alpha(w))$ so that

$$\begin{aligned}
\max_{s\in S} \sum_{a\in A} \alpha(a) u(\rho(a,s)) &= \max_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\rho_\alpha(m)}[u] = V(\epsilon(\rho_\alpha(m))) \\
&\geq V(\alpha) \\
&\geq V(\epsilon(\rho_\alpha(w))) = \mathbb{E}_{\rho_\alpha(w)}[u] = \min_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] = \min_{s\in S} \sum_{a\in A} \alpha(a) u(\rho(a,s)).
\end{aligned}$$

[44] Notice that V is well defined since whenever $\alpha'_\ell, \alpha''_\ell \in \Delta_\ell(A)$ are such that $\epsilon(\alpha'_\ell) \sim \alpha$ and $\epsilon(\alpha''_\ell) \sim \alpha$, then $\alpha'_\ell \sim_{\Delta(C)} \alpha''_\ell$ and $\sum_{c\in C} \alpha'_\ell(c) u(c) = \sum_{c\in C} \alpha''_\ell(c) u(c)$. Also observe that if $\alpha = \epsilon(\gamma) \in \Delta_\ell(A)$, one can choose $\alpha_\ell = \gamma$ and obtain $V(\epsilon(\gamma)) = \sum_{c\in C} \gamma(c) u(c) = \mathbb{E}_\gamma[u]$.

Up until this point, we only assumed that $\succsim$ satisfies A.1-A.4.

Theorem 2. If, in addition, A.5 holds, then $\epsilon(\alpha_\ell) \precsim \alpha$ implies $\alpha \succcurlyeq_S \epsilon(\alpha_\ell)$, that is $\rho_\alpha(w) \succsim_{\Delta(C)} \alpha_\ell$, and

$$\min_{s\in S} \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) = \min_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\rho_\alpha(w)}[u] \geq \mathbb{E}_{\alpha_\ell}[u] = V(\alpha).$$

Summing up, $V(\alpha) = \min_{s\in S} \sum_{a\in A} \alpha(a)\, u(\rho(a,s))$ for all $\alpha \in \Delta(A)$, which proves (10). The converse is routine, hence omitted. □

If $\succsim$ satisfies A.1-A.4, by Lemma 2, $\succsim_F$ is well defined and satisfies AA.1-AA.4 on $\mathcal{F} = F(\Delta(A))$. For each $f = \rho_\alpha \in \mathcal{F}$, $\alpha \sim \epsilon(\alpha_\ell)$ implies $f \sim_F \rho_{\epsilon(\alpha_\ell)} = (\alpha_\ell)_S$. In the Anscombe-Aumann jargon, α_ℓ is a *certainty equivalent* γ_f of $f = \rho_\alpha$, more precisely,

$$\{\xi \in \Delta(C) : \alpha \sim \epsilon(\xi)\} = \{\gamma \in \Delta(C) : \rho_\alpha \sim \gamma_S\}.$$

Moreover, if $g = \rho_\beta \in \mathcal{F}$, then

$$f \succsim_F g \Longleftrightarrow \alpha \succsim \beta \iff \alpha_\ell \succsim_{\Delta(C)} \beta_\ell \Longleftrightarrow \mathbb{E}_{\alpha_\ell}[u] \geq \mathbb{E}_{\beta_\ell}[u] \iff u(\gamma_f) \geq u(\gamma_g).$$

In particular, for every $\gamma, \zeta \in \Delta(C)$,

$$\gamma_S \succsim_F \zeta_S \Longleftrightarrow \rho_{\epsilon(\gamma)} \succsim_F \rho_{\epsilon(\zeta)} \Longleftrightarrow \epsilon(\gamma) \succsim \epsilon(\zeta) \iff \gamma \succsim_{\Delta(C)} \zeta \Longleftrightarrow \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u] \iff u(\gamma) \geq u(\zeta).$$

Set $U = \mathrm{co}(u(C)) = u(\Delta(C))$ and $u(\mathcal{F}) = \{u(f) : f \in \mathcal{F}\} \subseteq U^S \subseteq \mathbb{R}^S$, and for each $\varkappa \in u(\mathcal{F})$ define

$$I(\varkappa) = u(\gamma_f) \quad \text{if } \varkappa = u(f).$$

Using the techniques of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) it can be shown that $I : u(\mathcal{F}) \to \mathbb{R}$ is well defined, monotone and normalized (that is $I(x1_S) = x$ for all $x \in U$). Moreover, for each $\alpha \in \Delta(A)$, $V(\alpha) = \mathbb{E}_{\alpha_\ell}[u] = u(\alpha_\ell) = I(u(\rho_\alpha))$ and if $f = \rho_\alpha, g = \rho_\beta \in \mathcal{F}$

$$f \succsim_F g \iff V(\alpha) \geq V(\beta) \iff I(u(\rho_\alpha)) \geq I(u(\rho_\beta)) \iff I(u(f)) \geq I(u(g)). \tag{35}$$

Theorems 3 and 4. If $\succsim$ satisfies A.6, then $\succsim_F$ satisfies AA.6 by Lemma 2, which, together with (35), implies I is affine. Theorem 3 follows by the finite dimensional versions of the Krein-Rutman Extension Theorem (see, e.g. Ok, 2007, p. 496) and the Riesz Representation Theorem. The additional assumptions of Theorem 4, nontriviality of $\succsim$ and conceivability of all bets, guarantee that u is nonconstant and μ is unique. □

Theorem 5. Follows from Theorem 3 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) and Lemma 2 again, which guarantees that $\{u(f) : f \in \mathcal{F}\} = U^S$ when all bets are conceivable and that $\succsim_F$ satisfies AA.8 when $\succsim$ satisfies A.1-A.4 and A.8. ■

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