WORKING PAPER SERIES

Elicitation of Multiattribute Value Functions through High Dimensional Model Representations

Francesca Beccacece, Emanuele Borgonovo, Greg Buzzard, Alessandra Cillo, Stanley Zionts

Working Paper n. 495

This Version: September 17, 2013

Elicitation of Multiattribute Value Functions through High Dimensional Model Representations

Francesca Beccacece

Department of Decision Sciences, Bocconi University, Milan, Italy

Emanuele Borgonovo*

Department of Decision Sciences, Bocconi University, Milan, Italy

Greg Buzzard

Department of Mathematics, Purdue University, United States

Alessandra Cillo

Department of Decision Sciences and IGIER, Bocconi University, Milan, Italy

Stanley Zionts

School of Management, State University of New York, Buffalo, United States

Abstract

This work addresses the early phases of the elicitation of multiattribute value functions proposing a practical method for assessing interactions and monotonicity. We exploit the link between multiattribute value functions and the theory of high dimensional model representations. The resulting elicitation method does not state any *a-priori* assumption on an individual's preference structure. We test the approach via an experiment in a riskless context in which subjects are asked to evaluate mobile phone packages that differ on three attributes.

Keywords: Multiattribute Utility Theory; High Dimensional Model Representations; Value Function Elicitation; Sparse Grid Interpolation.

1. Introduction

In many real-life situations decision-makers (DM) face decision problems with value trade-offs (Keeney and Raiffa, 1993). A DM might be selecting among transportation means based on attributes such as price, time, and comfort (Siskos and Yannacopoulos, 1985) or among jobs trading-off social contact and daily travel time (Tversky and Kahneman, 1991). In consumer research, several studies have shown that the DM's willingness to pay depends on the alternative attributes of the product under scrutiny (Bettman, 1979). In purchasing a car for instance, consumers are asked to trade-off price versus safety and environmental protection (Bettman et al., 1998). These problems fall within

---

the realm of multiattribute utility theory (MAUT) (Smith and von Winterfeldt, 2004; Wallenius et al., 2008). Wallenius et al. (2008) underline the significant growth in applications of MAUT, by stating that *the business world has become more competitive and less predictable, accentuating the importance of effective decision making and the use of decision support tools* (Wallenius et al., 2008, p. 1340). MAUT focuses on one of the main steps of the decision-analysis process: the definition and quantification of the DM's values and objectives.

An important distinction in this process is marked by whether the elicitation is carried over alternatives associated with certain or uncertain prospects. If the selection is among alternatives with uncertain prospects, following Keeney and Raiffa (1993), the resulting preference function is called a utility function. Conversely, under certainty, one refers to a value function. Several researchers have addressed the conditions under which a value function may exist and special forms of value functions can be used (Gorman, 1968; Krantz et al., 1971; Dyer and Sarin, 1979; Fishburn, 1970; Keeney and Raiffa, 1993). One of the key concepts in studying the form of value functions is preferential independence (*preference separability* in Gorman (1968)). To ease the appraisal, additive value functions are often assumed. Utility additive (UTA) methods use linear programming (LP) to represent preference models (Jacquet-Lagreze and Siskos, 1982; Greco et al., 2008, 2010; Figueira et al., 2009).

Deviations from additivity, however, are often encountered in the practice. Methods to better capture DM's preferences have been introduced. Recently, Kadzinsky et al. (2012) propose a method which can be seen as a new interactive UTA-like procedure. Their approach is based on an additive model but thanks to its interactive nature is able to *help the DM in the comprehension of the necessary and possible preference relations* (Kadzinsky et al., 2012, p. 544). Other authors have been dealing with interactions and dependencies in a more direct way. Sounderpandian (1991) considers the situation when preferential independence is absent and studies the value functions which are applicable to such cases. Angilella et al. (2004) and Angilella et al. (2010) model preferences with interactions among criteria via Choquet integral. Angilella et al. (2014) extend the Multi-criteria Satisfaction Analysis method to handle positive and negative synergies between pairs of attributes.

Our goal is to provide a method for assessing the strength of interactions in a DM's value function, without stating any a-priori assumption on the form of the function itself. The method is grounded in the high dimensional model representations (HMDR) theory (also known as functional ANOVA), a multivariate integral decomposition method that has its roots in the findings of (Efron and Stein, 1981; Rabitz and Alış, 1999; Li et al., 2002a; Sobol', 2003). We exploit the facts that *i)* any integrable multiattribute value function can be expanded as a sum of terms related to individual preferences for attributes, plus terms that quantify interactions among pairs, triplets of attributes, etc., and *ii)* the terms in the expansion retain several of the properties of the original function (Beccacece and Borgonovo, 2011), such as monotonicity. The methodology consists of two steps: 1) experimental elicitation of subjects' value functions over predetermined values of the attributes suggested by the adopted metamodelling method, and 2) reconstruction of the multiattribute value function through interpolation, guided by the HMDR principles [1].

---

[1]The literature on this subject is vast, and offers us families of methods such as smoothing spline ANOVA (Friedman, 1991; Lin and Zang,

We test the methodology via a laboratory experiment in which subjects are asked to value mobile phones packages evaluated by three attributes. The attributes are messages, minutes and presence or absence of a special number (a number for which an unlimited number of minutes is allowed).

We then discuss the insights gained by the reconstruction of subjects' value functions. Calculation of the interaction terms through high-order Sobol' indices allows us to appreciate the relevance of interactions. Calculation of the first order HDMR terms allows us to study monotonicity. At an aggregate level, we register that preferences are close to additive, with interactions growing in relevance in the presence of a special number. At an individual level, additivity does not hold in general. As to monotonicity, while results at an aggregate level show little violations, at an individual level we observe violations when high number of minutes or messages are present. A discussion of these findings is proposed.

The remainder of the paper is organized as follows. Section 2 provides a literature review. Section 3 introduces the theory and implements it. Section 4 is devoted to the experiment, while Section 5 shows the results. Section 6 concludes with suggestions for further research.

2. Literature Review

A seminal literature review about research activity in MAUT up to the early 1990s is offered by Dyer et al. (1992). They identify four major areas in which they predict significant future research contributions in multiple criteria decision making (MCDM) and MAUT. These four areas are decision support applications, incorporation of behavioral research, emphasis on robust decisions, and the role of heuristics. These suggestions are then verified sixteen years later in the literature review of Wallenius et al. (2008). Indeed, the authors find more than 5 thousand publications in MCDM or MAUT between 1992 and 2007. This number indicates both the importance and the intensity of the research activity in the field.

In a riskless choice context, the usefulness of value functions in complex decision making problems has motivated a considerable amount of formal theory development (for a review see Keeney and Raiffa (1993)). Debreu (1954) proves that preferential independence among attributes is a sufficient condition for the existence of an additive value function. Additivity is fundamental also in one of the most important elicitation procedures, namely the UTA method (Jacquet-Lagreze and Siskos, 1982).

Roy (1985) summarizes the problem to which UTA has been applied into: choice, ranking, sorting and description (Greco et al., 2010, 2008; Kadzinsky et al., 2012). Besides additivity, the UTA method assumes piecewise linear and monotonic marginal value functions. Piecewise-linearity is not required by the UTA^{GMS} method (Greco et al., 2008). UTA^{GMS} finds the largest set of additive value functions compatible with the preference information provided by the DM. There, we have two possible outcomes: the UTA^{GMS} method either does not find a solution or it finds multiple solutions. Alternative reasons can lead to the no-solution case. Among these, the fact that the DM's preferences do

---

2006), polynomial chaos expansion, sparse grid interpolation (Sudret, 2008; Buzzard, 2012), cut-HDMR (Li et al., 2002b).

not match the additive model. Conversely, when the preferences of the DM fit the additive model, then UTA$^{\text{GMS}}$ might find more than one compatible value function.

In some applications, if deviations from additivity do not lead to big biases, then they can be ignored (Dolan, 2000). However, it is often the case that separability is violated and interactions among attributes become relevant (Farquhar, 1977; Payne et al., 1984), so that interactions *are too large to be ignored* (Wakker et al., 2004, p. 219). The presence of interactions complicates the assessment. Kadzinsky et al. (2012) even though assuming an additive model introduce an approach which is able to better capture DM's preferences in a tailor-made fashion. They introduce a new interactive UTA-like procedure to select a single value function representing the entire set of compatible value functions. The general procedure for selection of the representative value function consists of four stages, which allow the DM to 1) say what advantage, in terms of the difference of values, the selected subset of alternatives should have over the remaining alternatives, 2) choose out of five pre-defined targets to be attained by the representative value function, 3) account for the intensity of preference in terms of comprehensive values of the alternatives, and 4) choose a type of the marginal value functions. Besides additivity, a property widely used in the multiattribute elicitation is monotonicity. Kadzinsky et al. (2012) assume a marginal monotone value function.

Several papers have proposed theoretical developments in case of absence of preferential independence (Gorman, 1968; Sounderpandian, 1991). Interaction among criteria has been accounted for within the Choquet integral framework using non-additive ordinal regression (Angilella et al., 2004) and non-additive robust ordinal regression (Angilella et al., 2010). As Angilella et al. (2004) state *it is also well known that in multiple criteria decision problems it is also important to consider positive interaction (synergy) or negative interaction among criteria (redundancy)* (Angilella et al., 2004, p. 735).

Although the riskless context is the main scope of this paper, we cannot omit a brief excursus on recent advances in preference elicitation under risk. In the risky context, Wakker et al. (2004) propose a methodology for determining levels of attributes that are as little as possible affected by interactions, so as to retain the advantages of additivity. Abbas and Howard (2005) propose a class of multiattribute utility functions called *attribute dominance utility functions* and show that they *can be constructed using products of normalized marginal-conditional utility assessments or using single-attribute utility assessments and a copula structure* (Abbas, 2009, p. 1368). To relax some conditions imposed in Abbas and Howard (2005) (such as the *n*-increasing condition which corresponds to multivariate risk-seeking (Richard, 1975)), Abbas (2009) introduces a more general copula structure, called multiattribute utility copula. As to monotonicity, also in the risky context, it stems naturally from the non-satiation principle, both in the single and multiattribute cases (Ingersoll, 1987; Tsetlin and Winkler, 2009) and is a widely adopted assumption (Abbas, 2009).

3. The Methodology

3.1. Theory

In this section, we lay out the mathematical aspects of the method. We represent a multiattribute value function as $v : I^n \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, where I denotes the unit interval $[0, 1]$, and I^n the n-dimensional unitary hypercube. Our sole assumption is that v is square integrable.

One traditional approach is to use a multivariate Taylor expansion around a reference value to approximate the dependence of output on the attributes. Instead, we propose the use of an integral-based expansion, the HDMR of Rabitz and Alış (1999) (Li et al., 2002a; Sobol', 2003).

We refer to Wang (2006) and Owen (2012) for thorough accounts. We follow the notation of these works and consider a group of r attributes, letting $z = \{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n\}$ denote the corresponding indices. The complementary set (i.e., the set of indices not in z) is denoted by $\sim z$, and is formally equivalent to $\{1, \ldots, n\} \setminus z$. For such z, $\mathbf{x}_z = (x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ denotes a group of r attributes, $\mathbf{x}_z \in I^r$. Under the assumption that v is measurable, with $\mathbf{x} \in I^n$, it can be proven (Sobol', 2003) that v can be written as

$$v(\mathbf{x}) = v_0 + \sum_{z \neq \varnothing} v_z(\mathbf{x}_z), \tag{1}$$

where $v_0 = \int_{I^n} v(\mathbf{x}) d\mathbf{x}$, $\sum_{z \neq \varnothing}$ denotes the sum over all nonempty subsets of indices, and the functions $v_z(\mathbf{x}_z)$ are determined recursively by

$$v_z(\mathbf{x}_z) = \int_{I^{n-r}} \left(v(\mathbf{x}_z, \mathbf{x}_{\sim z}) - \sum_{t \subset z} v_t(\mathbf{x}_t) \right) d\mathbf{x}_{\sim z}. \tag{2}$$

As shown in Sobol' (2003), the functions $v_z(\mathbf{x}_z)$ in eq. (2) are orthogonal with respect to the L^2 inner product using Lebesgue measure. The representation in eq. (1) expands $v(\mathbf{x})$ in such a way that the first order terms (functions depending on only one variable) represent how much of the deviation from the mean of the preferences can be ascribed to each attribute separately. The higher order terms represent the additional portion that emerges from interactions. In HDMR theory, one considers also the quantity $D = \int_{I^n} (v(\mathbf{x}) - v_0)^2 d\mathbf{x}$. Using orthogonality of the v_z, it follows that

$$D = \sum_{z \neq \varnothing} D_z, \quad \text{where} \quad D_z = \int_{I^r} \left[v_z(\mathbf{x}_z) \right]^2 d\mathbf{x}_z. \tag{3}$$

In eq. (3), D is the integral of the square deviations of v from the constant term v_0. Thus, D is decomposed into $2^n - 1$ terms which are in one-to-one correspondence with the nonconstant terms of the integral decomposition of $v(\mathbf{x})$. As such, eq. (1) is not representing or responding to uncertainty, but, instead, is obtained through a series of nested integration. We are replacing a Taylor expansion based on differentiation through an integral expansion. If one were to substitute the Lebesgue measure with any product measure (including Lebesgue) representative of the decision-maker's belief about $\mathbf{x}$, then D would be the variance of v, i.e., the second moment of the distribution of v. However, this interpretation goes beyond the mechanical decomposition originally conceived in Sobol´ (1993), and, in fact, the decomposition fails if correlations are present.

The consideration of eqs. (1), (2) and (3) is informative in many respects. In this work, two are of particular interest: (a) the quantification of the strength of interactions and (b) monotonicity. Concerning the strength of interactions, the quantity

$$\eta_z^r = \frac{D_z}{D} \tag{4}$$

represents the contribution of the term D_z in the decomposition of D and is called the interaction effect of order r, where r is the cardinality of z. Of particular relevance are the so-called main and total effects, i.e.,

$$\eta_i^1 = \frac{D_{\{i\}}}{D} \text{ and } \eta_i^T = \frac{\sum_{z:z\ni i} D_z}{D}. \tag{5}$$

In the formula for η_i^T the sum in the numerator includes all terms in the decomposition of v containing index i. Hence η_i^T is the fraction of D associated not only with x_i alone but also with its interactions with all the remaining factor groups. The difference

$$\eta_i^I = \eta_i^T - \eta_i^1 = \frac{(\sum_{z:z\ni i} D_z) - D_{\{i\}}}{D} \tag{6}$$

represents the strength of interactions associated with x_i. If v is additive, then it is readily seen that $\eta_i^I = 0$ for all $i = 1, 2, ..., n$. For numerical approximation purposes, $\eta_i^I \simeq 0$ for all $i = 1, 2, ..., n$, implies that

$$v(\mathbf{x}) \simeq v_0 + v_1(x_1) + v_2(x_2) + ... + v_n(x_n). \tag{7}$$

If $v(x)$ is an elicited value function, estimation of η_i^I then allows us to appraise how far we are from the additive assumption. If the difference between η_i^T and η_i^1 is non-negligible, then interactions emerge and we face non-additive preferences. However, if the difference is negligible, then we are close to additivity and the elicitation can be carried out with one of the methods that exploit this property.

Knowledge of eq. (2) can be utilized for assessing monotonicity as follows. Consider the univariate function of x_i obtained by averaging $v(\mathbf{x})$ over all attributes but x_i

$$g(x_i) = \int_{I^{n-1}} v(x_i, \mathbf{x}_{\sim\{i\}}) d\mathbf{x}_{\sim\{i\}.} \tag{8}$$

Then it is readily seen that

$$v_i(x_i) = g_i(x_i) - v_0. \tag{9}$$

Thus, $v_i(x_i)$ reproduces up to a constant term the average behavior of v as a function of x_i. In Beccacece and Borgonovo (2011), it is shown that if $v(\mathbf{x})$ is monotone, then all first order terms $v_i(x_i)$ retain the monotonicity of $v(\mathbf{x})$, independently of whether it is additive or not. Furthermore, if $v(\mathbf{x})$ is additive, it is exactly

$$v(\mathbf{x}) = \sum_{i=1}^{n} g_i(x_i). \tag{10}$$

Eqs. (9) and (10), then imply that the first order term $v_i(x_i)$ tells us the dependence of v as a function of x_i exactly. In fact, the monotonicity of $v_i(x_i)$ implies the monotonicity of $v(\mathbf{x})$ and the converse, provided that v is additive. If v is not additive, then the non-monotonicity of any of the $v_i(x_i)$ suffices to assert the non-monotonicity of $v(\mathbf{x})$.

6

In the example below we apply the integral decomposition to a two-attribute multilinear function (see Appendix B for the corresponding calculations).

Example 1. *Consider the following multilinear function, $v : I^2 \to \mathbb{R}$*

$$v(x_1, x_2) = k_1 x_1 + k_2 x_2 + k_{1,2} x_1 x_2. \tag{11}$$

Its integral decomposition results in the following terms

$$v_0 = k_1 \overline{x}_1 + k_2 \overline{x}_2 + k_{1,2} \overline{x}_1 \overline{x}_2 \tag{12}$$

where $\overline{x}_i = \int_0^1 x_i dx_i$, $i = 1, 2$. The first and second order terms, are, respectively

$$v_1(x_1) = (x_1 - \overline{x}_1)(k_1 + k_{1,2} \overline{x}_2) \tag{13}$$

$$v_2(x_2) = (x_2 - \overline{x}_2)(k_2 + k_{1,2} \overline{x}_1) \tag{14}$$

$$v_{1,2}(x_1, x_2) = k_{1,2}(x_1 - \overline{x}_1)(x_2 - \overline{x}_2). \tag{15}$$

The main effects are

$$\eta_1^1 = \frac{(k_1 + k_{1,2} \overline{x}_2)^2 D_1}{D} \quad and \quad \eta_2^1 = \frac{(k_2 + k_{1,2} \overline{x}_1)^2 D_2}{D} \tag{16}$$

where $D_i = \int_0^1 (x_i - \overline{x}_i)^2 dx_i$ and $D = \int_0^1 \int_0^1 (v(x_1, x_2) - v_0)^2 dx_1 dx_2$. The interaction indices are

$$\eta_1^I = \eta_2^I = \frac{k_{1,2} D_1 D_2}{D} \tag{17}$$

which leads to the total effects

$$\eta_1^T = \frac{(k_1 + k_{1,2} \overline{x}_2)^2 D_1 + k_{1,2} D_1 D_2}{D} \quad and \quad \eta_2^T = \frac{(k_2 + k_{1,2} \overline{x}_1)^2 D_2 + k_{1,2} D_1 D_2}{D}. \tag{18}$$

Eq. (15) confirms that the presence of $k_{1,2}$ is instrumental to trigger interactions: if $k_{1,2} = 0$, the function is additive and there are no interactions. However, the interaction term $v_{1,2}(x_1, x_2)$ is not simply equal to $k_{1,2} x_1 x_2$. It involves the deviations from $(\overline{x}_1, \overline{x}_2)$, which are absent in the term $k_{1,2} x_1 x_2$. The strength of the interactions is then quantified as a residual value over the individual effects. Concerning monotonicity, eqs. (13) and (14) suggest that $v(x_1, x_2)$ depends linearly on x_1 and x_2, on average. The "on average" can be omitted if $k_{1,2} = 0$. In fact, $v_1(x_1) = k_1 x_1 + const$ in the same way as $v(x_2) = k_2 x_2 + const$ if $k_{1,2} = 0$. Thus, the closer the function is to additivity, the better the average behavior approximates the exact behavior.

3.2. Implementation

The key to exploit the above-mentioned theoretical results is to approximate the mapping $v : \mathbf{x} \mapsto y$ from given data using one of the so-called surrogate model or metamodel methods that have been developed recently in the applied

statistical and mathematical literature. Metamodelling methods have developed greatly in association with the study of the behavior of scientific models produced through computer codes — we refer to the special issues of Bayarri et al. (2009), and Ratto et al. (2012).

The governing principle can be summarized intuitively as follows. The unknown input-output mapping is assumed to be square-integrable. Therefore, the mapping can be written in the form of eq. (1), where a suitable truncation order applies. To illustrate (and for notational clarity), let us stop the expansion at order 2. Then the governing equation is

$$v(\mathbf{x}) \simeq v_0 + \sum_{i=1}^{n} v_i(x_i) + \sum_{i<j}^{n} v_{i,j}(x_i, x_j). \tag{19}$$

In the case of a polynomial surrogate model, each of the terms in the integral expansion is then, in turn, written through polynomials in the form

$$v_i(x_i) \approx \sum_{r=1}^{h} \alpha_r^i \phi_r(x_i)$$
$$v_{i,j}(x_i, x_j) \approx \sum_{p=1}^{h'} \sum_{q=1}^{h''} \beta_{p,q}^{i,j} \phi_p(x_i) \phi_q(x_j) \tag{20}$$

where the $\phi_r(\cdot)$ functions are elements of an orthonormal basis, and $\alpha_r^i, \beta_{p,q}^{i,j}$ are corresponding coefficients to be determined numerically. In general, the specific choice of polynomials, ϕ_r, and the maximum degree of these polynomials, both in a single-variable and in combinations of variables, has a great impact on the ability of the resulting expansion to approximate a given function. A great deal of work has been done recently on the approximation properties of such expansions (Bungartz and Griebel, 2004; Barthelmann et al., 2000) as well as on their use for calculating sensitivity coefficients (Crestaux et al., 2009; Buzzard, 2012). To describe the construction of a surrogate model more completely, let $[\widehat{X}\ \widehat{Y}]$ denote a dataset, where $\widehat{X}$ is an $N \times n$ matrix containing the realizations of the attributes and $\widehat{Y}$ an $N \times 1$ vector with the corresponding values of Y. Depending on the number of degrees of freedom of the surrogate model versus the number of samples, the coefficients $\alpha_r^i, \beta_{p,q}^{i,j}$ are computed for best fitting the generated inputs ($\widehat{X}$) to the corresponding model output ($\widehat{Y}$). Then the functions $v_i(x_i)$, $v_{i,j}(x_i, x_j)$ can be estimated numerically using eq. (20) to obtain estimates $\widehat{v_i}(x_i)$, $\widehat{v_{i,j}}(x_i, x_j)$.

Many methods for approximating the functions v_i and $v_{i,j}$ are possible. For instance, polynomial chaos expansion is discussed in Sudret (2008). An alternative method, based on orthogonal polynomials, which builds over the cut-HDMR expansion is discussed into Rabitz and Alış (1999), Li et al. (2002a) and Ziehn and Tomlin (2009). Smoothing spline ANOVA methods are widely discussed in the statistical literature (Lin and Zang, 2006). Ratto and Pagano (2010) uses an approach based on the ACOSSO non-parametric regression method. All these methods use given data of the inputs ($\widehat{X}$) and the output ($\widehat{Y}$) to approximate $v(\mathbf{x})$.

We can then summarize the procedure in Table 1.

We observe that the methodology applies also if the dataset $\widetilde{XY}$ is given. This is the case if the dataset is obtained through, for instance, an online survey. If $\widetilde{XY}$ is not given, one needs to assess preferences through experiments, in which case both the choice of $\widehat{X}$ and the method for obtaining $\widehat{Y}$ must be considered carefully. This case shall be examined in this work in detail.

Table 1: Implementation steps

	Steps
1	Obtain or generate $\widehat{X}$
2	Obtain or generate $\widehat{Y}$
3	Compute $\widehat{v_i}(x_i), \widehat{v_{i,j}}(x_i, x_j)$
4	Compute $\widehat{\eta_i^2}, \widehat{\eta_i^T}, \widehat{\eta_i^l}$
5	Examine the results to study additivity and monotonicity

In this work, we follow the steps in Table 1 using the formulation in eq. (20) to guide the selection of sampling points (described further below). The sample points $\widehat{X}$ are chosen to be a nearly D-optimal set of points for polynomial regression, with maximum degree 4 in each variable separately and interaction terms up to degree 2. This polynomial basis is commonly used for sparse grid interpolation, which is known to have good approximation properties (Barthelmann et al., 2000). We then use cubic spline interpolation based on the Matlab routine "griddata" to approximate the functions v_i and $v_{i,j}$ for purposes of calculating sensitivity coefficients. Once the samples $\widehat{X}$ and the corresponding experimental values $\widehat{Y}$ are known, we approximate v using cubic splines and then use a quasi Monte Carlo method to evaluate the integrals in eq. (2) and (3) to obtain the sensitivity coefficients.

4. The Experiment

4.1. Subjects

Subjects were 65 students from Bocconi University, coming from various academic backgrounds (mainly Economics and Business Administration). They were paid a show-up fee of 10EUR. In addition, each subject had a 10% chance to be selected to play out one of his choices for real. Once the experiment was over, subjects drew a ticket from a nontransparent bag containing 10 tickets, one of which was a winning ticket. If the subject had a non-winning ticket, he received 10EUR and the experiment was over. If he had a winning ticket, he received 10EUR and he had to chose a number in a given range, whose upper bound was determined by the subject's number of questions. We then showed the subject the choice question corresponding to the selected number and the relative stored answer he provided. In case the package was chosen, the student was provided the corresponding monetary amount.

4.2. Procedures

The experiment was computer-based, run in Italian with 30 minute sessions of 8 subjects (on average). Two experimenters were present in each session. Instructions were read aloud and contained a warm up question. At the end of the instructions the subject provided age, gender, undergraduate/master degree.

Subjects had to select between pairs of alternatives. Alternative A was composed of a mobile phone package, lasting one month, with 3 attributes: messages, minutes, presence (or absence) of a special number (a number for which

an unlimited number of minutes is allowed). Alternative A was composed in such a way that subjects were familiar with the kind of questions that have been asked, given that *the rationale is that the students should be actively and seriuously involved in the decision problem* (Scheubrein and Zionts, 2006, p. 20). Alternative B was a sure amount. Subjects had to choose between A and B, by clicking on their preferred option. They had to confirm their choice. If they confirmed their choice, then the next question was displayed; if not, the choice was displayed anew. The confirmation question had the purpose of reducing the impact of response errors. For each package, we determined the value that would make the subject indifferent between A and B through a series of choices that zoomed into subjects' preferences. The iteration procedure to determine the indifference value is explained in detail in Appendix A. All choices were presented via a computer interface, an example of which is depicted in Figure 1.
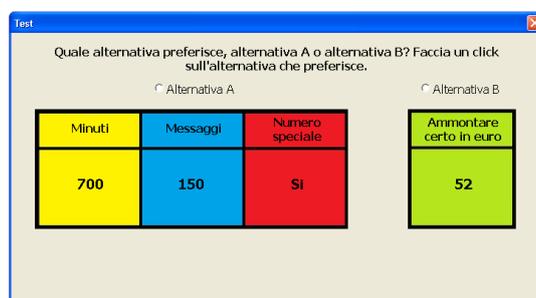

Figure 1: Screen view of the computer interface

Table 2 reports 26 points for which we want to determine the indifference values y_i for $i = 1, ..., 26$. The 26 points were chosen to form an approximately D-optimal set of points for polynomial and/or spline regression over 2 variables. At the end of the instruction, at the beginning of the experiment, subjects had to answer two training questions. These questions only served to monitor subjects complete understanding of the task to be performed.

It is well established in the experimental literature that subjects might deliver answers which are inconsistent with one another. Thus, a proper system for monitoring consistency is necessary. Best practices recommend that such system is implemented in the experiment setup as a tool in support of *internal validity of measurements*. We monitored subjects' consistency using two tests. In the first test, subjects were presented again with the third choice of 10 randomly chosen iterations. The third choice was repeated because the value of B in the third choice is generally close but not equal to the elicited indifference value, hence response errors are more likely (Bleichrodt et al., 2010). In the second test, subjects had to repeat the entire elicitations of y_{10}, y_{12}, y_{24}, y_{26}. These repetitions served for the purpose of giving an insight of the error made in the elicited indifference values.

Consistency checks are often also used to detect and hence, eliminate, subjects who are not approaching the experiment seriously. The consistency rate in the repetition of the third choice of a given iteration is in general high and around 70% (Stott, 2006). In the repetition of the full procedure, an absolute difference between the repeated point and the original one which is twice higher than the standard deviation reflects an inconsistent behavior (Baillon et al.,

10

2013). Violations of monotonicity are not rare, but a significant deviation from the average rate observed in the literature (Birnbaum and Thompson, 1996) can be interpreted as a subject's confusion in the task.

Once we started the elicitation of the indifference value of one package, we finished the entire bisection procedure. However, the consistency questions above described were interspersed all over so that subjects could not understand the bisection procedure.

5. Results

In total, 65 subjects were examined. 7 subjects were eliminated by the software because of inconsistency in executing the task. Indeed, if a subject clicked on the same option more than 5 times, the experiment ended automatically to avoid retaining subjects that had not understood the task.

For subjects 11, 32 and 40 the replication rates for the third choice of the 10 randomly chosen iterations are 40%, 50%, 50%, respectively. These rates are very low compared to the above-mentioned criterion. Adopting the criterion of the maximum acceptable deviation of the repeated point from the original one, we observe that subject 15 fails in 3 repetitions for points y_{12}, y_{24}, and y_{26}, while subject 24 fails in 2 repetitions for points y_{24} and y_{26}. Subject 1 has a violation of monotonicity rate of 53.8% for 13 tests (rate extremely high compared to the ones observed in the literature, see (Birnbaum and Thompson, 1996)). Hence, by excluding these six subjects, we are left with 52 subjects. Of these 52 subjects, the repeated third choice of 10 randomly selected iterations and the repetition of 4 entire elicitations are not distant from the original answers, displaying consistency. Precisely, the replication rate for the third choice of the 10 randomly iterations for 52 subjects is 91.3%, above the replication rates in the literature (Stott, 2006). The entirely repeated elicitations of y_{10}, y_{12}, y_{24}, y_{26} produce a paired-t test with p-values of 0.22, 0.6, 0.52, 0.23, respectively.

5.1. Interactions and Monotonicity

We analyze the aggregate data for a mean and median subject. For the analysis at an aggregate level, as predicted by the best practices in the experimental work, we exclude some points for some of the 52 subjects, because their assessment goes beyond a reasonable margin of error. We exclude those violations that are larger than 2 times the average standard deviation (8.6) of the four differences between the original and the repeated measurements. This occurs in 3.2% of the cases (4.4% in the 30 tests and 2.1% in the 13 tests), below what is commonly observed (for a review see Birnbaum (2008)). The excluded points are the ones deviating more from the median values for the 52 subjects. Table 2 displays aggregate mean, median and standard deviation values for the 52 subjects.

Figure 2 displays the results for mean data. The graphs in the first row show the conditional preference functions $v(x_1, x_2|x_3 = 0)$ and $v(x_1, x_2|x_3 = 1)$ as they result from interpolation, where x_1=Minutes, x_2=Messages, x_3=Special Number. Graphs a) and b) display the response surface obtained without and with special number, respectively. The graph with special number is not smooth, due to some weak violation of monotonicity, as it will be discussed next. Graph c) displays both graphs simultaneously.

0	Minutes	Messages	SN	Mean Values	Median Values
1	800	800	No	22.67(12.11)	20
2	1500	800	No	28.26(17.25)	24
3	800	1500	No	27.19(15.22)	26
4	200	800	No	18.12(8.68)	16
5	800	200	No	19.58(9.68)	18
6	1200	800	No	25.7(14.25)	24
7	800	1200	No	24.35(13.05)	24
8	500	800	No	21.18(11.01)	19
9	800	500	No	22.13(11.71)	20
10	1500	1500	No	31.96(18.89)	28
11	200	1500	No	18.88(8.43)	18
12	1500	200	No	26.31(15.91)	22
13	200	200	No	13.67(6.92)[	13
14	800	800	Yes	25.4(16.95)	24
15	1500	800	Yes	32.82(20.17)	28
16	800	1500	Yes	31.29(16.56)	30
17	200	800	Yes	21.44(10.43)	20
18	800	200	Yes	23.88(12.13)	22
19	1200	800	Yes	30.79(17.54)	28
20	800	1200	Yes	27.54(14)	28
21	500	800	Yes	25.68(13.42)	24
22	800	500	Yes	24.25(11.22)	24
23	1500	1500	Yes	37.67(21.35)	32
24	200	1500	Yes	24.2(13)	23
25	1500	200	Yes	30.57(19.1)	26
26	200	200	Yes	17.69(8.77)	15

Table 2: Selected packages. Mean and median values. In parenthesis the standard deviation. SN stands for presence of Special Number.
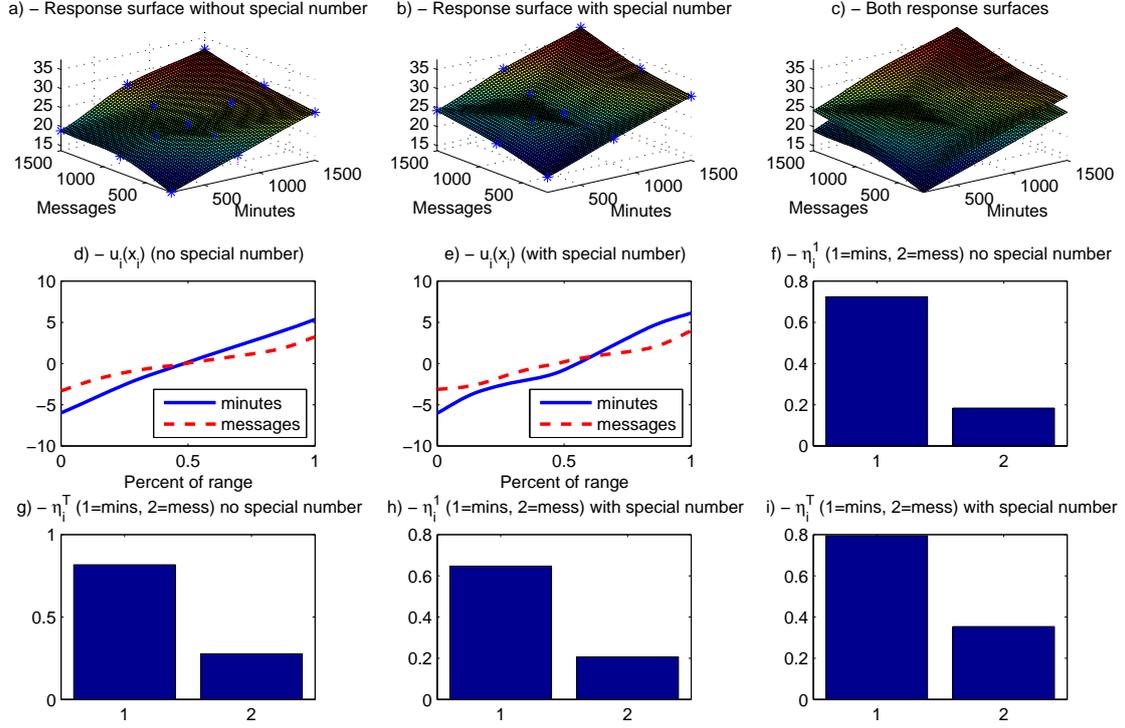
Figure 2: Results for mean values

Graphs *d*) and *e*) display the main effect functions $v_1(x_1)$ and $v_2(x_2)$. These functions show a monotone behavior.

Graph *f*) displays the estimates of the main effects for messages and minutes in the absence of the special number. We have $\widehat{\eta}_1^1 = 0.723$ and $\widehat{\eta}_2^1 = 0.184$. Thus, the attribute with the highest individual effect is minutes. Graph *g*) displays the corresponding total order sensitivity indices, $\widehat{\eta}_1^T$ and $\widehat{\eta}_2^T$. We have: $\widehat{\eta}_1^T = 0.817$ and $\widehat{\eta}_2^T = 0.277$, respectively. This result reveals that preferences are almost additive. In fact, the interaction effect is estimated at $\widehat{\eta}_1^I = \widehat{\eta}_2^I = 0.093$.

Graph *h*) displays the main effects in the presence of the special number. Their estimates are $\widehat{\eta}_1^1 = 0.646$ and $\widehat{\eta}_2^1 = 0.206$. Graph *i*) represents the total order sensitivity indices: $\widehat{\eta}_1^T = 0.794$ and $\widehat{\eta}_2^T = 0.354$. However, we register a higher value of the interaction effect, which, conditional on the presence of a special number, increases to $\widehat{\eta}_1^I = \widehat{\eta}_2^I = 0.148$. Thus, in the presence of a special number, preferences deviate further from additivity.

Figure 3 displays the results obtained utilizing median data. Graphs *a*) and *b*) display the response surface with and without the special number, respectively, while graph *c*) both surfaces. With median data, there is no smoothness of the surface also in case of no special number.

Graphs *d*) and *e*) display the Sobol' main effect functions without and with the special number. Graph *f*) displays the estimates of the main effects for messages and minutes in the absence of the special number. We have $\widehat{\eta}_1^1 = 0.581$ and $\widehat{\eta}_2^1 = 0.286$. Graph *g*) displays the corresponding total order sensitivity indices, $\widehat{\eta}_1^T = 0.715$ and $\widehat{\eta}_2^T = 0.419$, respectively. This result confirms minutes as the most important attribute. The estimated interaction effect is $\widehat{\eta}_1^I =$
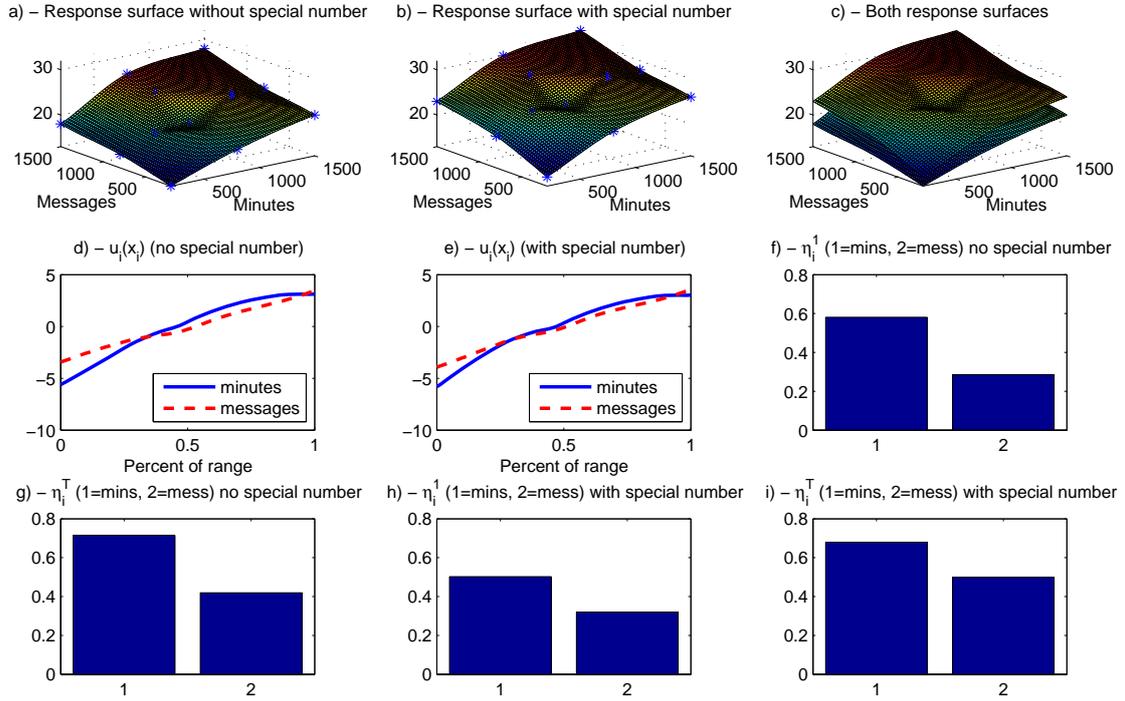
a) – Response surface without special number
b) – Response surface with special number
c) – Both response surfaces
d) – $u_i(x_i)$ (no special number)
e) – $u_i(x_i)$ (with special number)
f) – η_i^1 (1=mins, 2=mess) no special number
g) – η_i^T (1=mins, 2=mess) no special number
h) – η_i^1 (1=mins, 2=mess) with special number
i) – η_i^T (1=mins, 2=mess) with special number

Figure 3: Results for median values

$\widehat{\eta}_2^I = 0.134$, which is non-negligible.

Graph *h*) displays the main effects in the presence of a special number. Minutes are still more important than messages, but not as much as before. Indeed, the estimates are $\widehat{\eta}_1^1 = 0.502$ and $\widehat{\eta}_2^1 = 0.32$. This result is confirmed by the values of the total order sensitivity indices in graph *i*): $\widehat{\eta}_1^T = 0.68$ and $\widehat{\eta}_2^T = 0.5$. We also note that preferences become less additive: the interaction effect increases $\widehat{\eta}_1^I = \widehat{\eta}_2^I = 0.177$.

At an aggregate level, violations of monotonicity are more pronounced for median data than for mean data. Analysing the data, we observe that the presence of a special number seems to make preferences less smooth. Of the 13 pairwise comparisons between alternatives which differ only on the presence or absence of the special number, mean and median data always attribute a higher value to the alternative with the special number. On average, the presence of the special number is valued 4EUR more, both for mean and median data. Of the 30 pairwise comparisons between alternatives which differ only on one attribute, either minutes or messages, mean and median data seem to resist to violation of monotonicity, with few exceptions. In general the problem seems to appear with packages $(800, 500, 0)$ and $(800, 800, 0)$ for mean data and with packages $(800, 500, 1)$ and $(800, 800, 1)$ for median data. The increase of messages from 500 to 800 does not seem to be evaluated by subjects, independently of the presence or absence of the special number. A similar problem appears for median values of $(1200, 800, 1)$ and $(1500, 800, 1)$, and for median values of $(1200, 800, 0)$ and $(1500, 800, 0)$: the alternatives have the same value.

At an individual level, we observe that the interaction terms both in presence and absence of the special number

14

become significant ($p = 0$). Moreover, we cannot reject the null of their difference being equal (two-sample t test p=0.89).

At an individual level, violations of monotonicity are more frequent. Some violations of monotonicity are more common than others. Table 3 shows the alternatives for which more than 30% of subjects violate monotonicity.

	Alternative 1	Alternative 2	Violations of Monotonicity
7 vs 20	(800, 1200, 0)	(800, 1200, 1)	34.7%
4 vs 11	(200, 800, 0)	(200, 1500, 0)	32.7%
5 vs 9	(800, 200, 0)	(800, 500, 0)	30.8%
9 vs 1	(800, 500, 0)	(800, 800, 0)	40.8%
1 vs 7	(800, 800, 0)	(800, 1200, 0)	32.8%
12 vs 2	(1500, 200, 0)	(1500, 800, 0)	34.6%
8 vs 1	(500, 800, 0)	(800, 800, 0)	32.7%
6 vs 2	(1200, 800, 0)	(1500, 800, 0)	42.3%
18 vs 22	(800, 200, 1)	800, 500, 1)	30.8%
20 vs 16	(800, 1200, 1)	(800, 1500, 1)	30.8%
25 vs 15	(1500, 200, 1)	(1500, 800, 1)	40.4%

Table 3: Alternatives for which more than 30% of subjects violate monotonicity

As we can see from Table 3, violations of monotonicity occur more frequently when subjects are comparing alternatives that differ either in minutes or messages, then when they are comparing alternatives that differ by the presence or not of the special number.

Several of the above tests compare packages with high levels of minutes. Violations of monotonicity are common and there is a wide literature that supports such findings. The experiment was not constructed to test for violations of monotonicity, this would go beyond the scope of such work, rather to test if monotonicity holds. However, we can perform an analysis that allows us to state if some of these violations are statistically more significant than other ones. Hence, we perform a paired t-test for each of the above comparisons, to see if we can reject the null hypothesis that the values are equal. For most of them we cannot reject the null hypothesis. This might suggests that subjects were indifferent between the compared packages, and that an apparent violation of monotonicity might be due to error.

There are though some exceptions. For some comparisons we can reject the null that the values are equal. It seems that subjects, when facing a high level of minutes (500 seems to be perceived already as such), ignore the presence of the special number ((800, 1200, 0) vs (800, 1200, 1)), or a higher level of messages ((800, 200, 0) vs (800, 500, 0) and (1500, 200, 0) vs (1500, 500, 0)), or a higher level of minutes ((500; 800; 0) vs (800; 800; 0)), to the point that better packages are evaluated significantly less ($p < 0.02$). Most of these pairwise comparisons also display a high level of

messages, which can create similar problems. As before mentioned, the repeated elicitations provide values which do not differ from the original ones. Hence, also a comparison with the values obtained from the repeated elicitations would not provide a different answer. It is also true that if subjects evaluate wrongly a package this has impact on the other comparisons. Unfortunately, we are not able to control for this.

5.2. Discussion

At an aggregate level, results based on mean and median data are similar. There are three major similarities. First, minutes account for more variability in $v(\mathbf{x})$ than messages. This is reasonable to expect. Second, moving from packages without special number to packages with special number, the relevance of minutes decreases. This is still reasonable given that the presence of a special number can be naturally interpreted as if some minutes had been added to the package. Third, preferences become less additive in case of special number. The stronger effect of interactions is indeed an effect of the partial overlapping of the attributes minutes-special number, because offering a special number is equivalent to granting an unlimited number of minutes to the customer for specific calls.

The only difference between mean and median data is represented by the fact that two of the above-mentioned findings are less pronounced for median data. Indeed, minutes are still more important than messages, but in a less evident way. Furthermore, when shifting from absence to presence of special number, the interaction term increases its relevance.

At an individual level, the interaction terms become significant and we observe a higher rate of violations of monotonicity. Such violations appear to be more frequent for high levels of messages and minutes, as if above a given threshold value, subjects become insensitive to their increase. Indeed, Wathieu (2004) proposes a model to account for the fact that consumer's willingness to pay for a good is highest when the habitual level of consumption is moderate. Graphically, the willingness-to-pay function would have an interior maximum. If we agree on the fact that subjects are used to packages with a moderate level of minutes and messages, this model could explain the lower value assigned to packages with higher messages and minutes.

In experimental work we often observe differences between aggregate and individual data. However, the analysis of individual preferences provides critical insights for a mobile company when offering the packages.

6. Conclusions

We have presented an elicitation method for multiattribute value functions. The method rests on the HDMR theory and has the advantage of not stating any assumption on the DM's preferences structure. The interpolation of the value function exploits recent advances in the field of surrogate modelling which allows one to study whether preferences are additive and/or monotonic. The method does not pose restrictions on the type of data, which can be the result of surveys (internet surveys) or experiments (as discussed here). It is readily replicable and offers new reading glasses to investigate experimental results. It can be used as an initial screening exercise to detect consumers preferences. In fact, getting to know whether certain structural properties of multiattribute preferences hold in the initial phases of

the elicitation process might simplify the overall assessment procedure providing guidance on the methods to use in a subsequent phase of the elicitation.

We analysed data produced by experiments involving preference elicitation for mobile phone packages — a problem close to the sensitivity of the subjects involved in the experiment, namely, students. Application of the method has revealed (aside the overall consistency) that, at an aggregate level, little violations of monotonicity are registered and the presence of a special number increases the relevance of interactions and causes the value function of deviate from additivity. When data are analysed at an individual level, however, results reveal highly non-additive value functions. Also, violations of monotonicity are present for high levels of minutes and messages.

Finally, the extension of the method to a risky context is part of future research by the authors.

7. Appendix A

The initial value of B was the maximum between 12 and a random integer y in the interval $[x, x + 20]$, with x being a selected starting value for each point. There were two possible scenarios: (i) If A was chosen we increased y by 20 until B was chosen. We then halved the step size and decreased y by 10. If A [B] was subsequently chosen we once again halved the step size and increased [decreased] y by 5, etc. (ii) If B was chosen we decreased y by 20 until A was chosen. We then halved the step size and decreased y by 10. If B [A] was subsequently chosen we once again halved the step size and decreased [increased] y by 5, etc. The elicitation ended when the difference between the lowest value of y for which B was chosen and the highest value of y for which A was chosen was less than or equal to 5. It then took the midpoint, and stored it. If decreasing the number of 20 gave a negative number, then we decreased it by $y/2$. Decimals were not allowed to avoid confusion. We report an example below:

Iteration	y	Choice
1	50	A
2	70	B
3	60	A
4	55	B

Table 4: Example of the elicitation

Here the program ended since the difference is 5, took the midpoint 52.5 and stored 53.

8. Appendix B

We start with v_0. We have

$$v_0 = \int_0^1 \int_0^1 v(x_1, x_2) dx_1 dx_2 = k_1 \int_0^1 \int_0^1 x_1 dx_1 dx_2 + k_2 \int_0^1 \int_0^1 x_2 dx_1 dx_2 + k_{1,2} \int_0^1 \int_0^1 x_1 x_2 dx_1 dx_2 =$$
$$k_1 \bar{x}_1 + k_2 \bar{x}_2 + k_{1,2} \bar{x}_1 \bar{x}_2$$

17

For the first order terms, we have the non-orthogonalized functions

$$g_1(x_1) = \int_0^1 v(x_1, x_2)dx_2 = \int_0^1 (k_1x_1 + k_2x_2 + k_{1,2}x_1x_2)dx_2 = k_1x_1 + k_2\overline{x}_2 + k_{1,2}x_1\overline{x}_2$$

$$g_2(x_2) = \int_0^1 v(x_1, x_2)dx_1 = \int_0^1 (k_1x_1 + k_2x_2 + k_{1,2}x_1x_2)dx_1 = k_1\overline{x}_1 + k_2x_2 + k_{1,2}\overline{x}_1x_2$$

Subtracting v_0, we obtain the orthogonalized first order functions

$$v_1(x_1) = k_1x_1 + k_2\overline{x}_2 + k_{1,2}x_1\overline{x}_2 - (k_1\overline{x}_1 + k_2\overline{x}_2 + k_{1,2}\overline{x}_1\overline{x}_2) =$$
$$k_1x_1 + k_2\overline{x}_2 + k_{1,2}x_1\overline{x}_2 - k_1\overline{x}_1 - k_2\overline{x}_2 - k_{1,2}\overline{x}_1\overline{x}_2 =$$
$$k_1x_1 + k_{1,2}x_1\overline{x}_2 - k_{1,2}\overline{x}_1\overline{x}_2 = k_1x_1 - k_1\overline{x}_1 + k_{1,2}x_1\overline{x}_2 - k_{1,2}\overline{x}_1\overline{x}_2 =$$
$$k_1(x_1 - \overline{x}_1) + k_{1,2}\overline{x}_2(x_1 - \overline{x}_1) = (x_1 - \overline{x}_1)(k_1 + k_{1,2}\overline{x}_2)$$

The function $v_2(x_2)$ is found in a similar way, obtaining

$$v_2(x_2) = (x_2 - \overline{x}_2)(k_2 + k_{1,2}\overline{x}_1)$$

Finally, the interaction term $v_{1,2}(x_1, x_2)$ is given by the difference between $v(x_1, x_2) - v_1(x_1) - v_2(x_2) - v_0$. We have

$$v_{1,2}(x_1, x_2) = k_1x_1 + k_2x_2 + k_{1,2}x_1x_2+$$
$$-((x_1 - \overline{x}_1)(k_1 + k_{1,2}\overline{x}_2)) - ((x_2 - \overline{x}_2)(k_2 + k_{1,2}\overline{x}_1)) - (k_1\overline{x}_1 + k_2\overline{x}_2 + k_{1,2}\overline{x}_1\overline{x}_2)$$
$$= k_{1,2}(x_1 - \overline{x}_1)(x_2 - \overline{x}_2).$$

For the global sensitivity indices, we exploit eq. (3)

$$\eta_1^2 = \int_0^1 (x_1 - \overline{x}_1)^2(k_1 + k_{1,2}\overline{x}_2)^2dx_1 = \frac{(k_1 + k_{1,2}\overline{x}_2)^2D_1}{D}$$

$$\eta_2^2 = \int_0^1 (x_2 - \overline{x}_2)^2(k_2 + k_{1,2}\overline{x}_1)^2dx_2 = \frac{(k_2 + k_{1,2}\overline{x}_1)^2D_2}{D}$$

$$\eta_1^I = \eta_2^I = \frac{k_{1,2}D_1D_2}{D}.$$

where $D_i = \int_0^1 (x_i - \overline{x}_i)^2dx_i$ and $D = \int_0^1 \int_0^1 (v(x_1, x_2) - v_0)^2dx_1dx_2$. Hence, the total contributions are given by

$$\eta_1^T = \frac{(k_1 + k_{1,2}\overline{x}_2)^2D_1 + k_{1,2}D_1D_2}{D}$$

and

$$\eta_2^T = \frac{(k_2 + k_{1,2}\overline{x}_1)^2D_1 + k_{1,2}D_1D_2}{D}$$

so that

$$\eta_1^I = \eta_2^I = \frac{k_{1,2}D_1D_2}{D}.$$

References

Abbas, A., 2009. Multiattribute utility copulas. Operations Research 57 (6), 1367–1383.

Abbas, A., Howard, R., 2005. Attribute dominance utility. Decision Analysis 2 (4), 185–206.

Angilella, S., Corrente, S., Greco, S., Slowinski, S., 2014. Musa-int: Multicriteria customer satisfaction analysis with interacting criteria. Omega 42, 189–200.

Angilella, S., Greco, S., Lamantia, F., Matarazzo, B., 2004. Assessing non-additive utility for multicriteria decision aid. European Journal of Operational Research 158, 734–744.

Angilella, S., Greco, S., Matarazzo, B., 2010. Non-additive robust ordinal regression: A multiple criteria decision model based on the choquet integral. European Journal of Operational Research 201 (1), 277–288.

Baillon, A., Bleichrodt, H., Cillo, A., 2013. A tailor-made test of intransitive choice. Working Paper.

Barthelmann, V., Novak, E., Ritter, K., 2000. High dimensional polynomial interpolation on sparse grids. Advances in Computational Mathematics 12 (4), 273–288.

Bayarri, M. J., Berger, J., Steinberg, D., 2009. Special issue on computer modeling. Technometrics 51 (4), 353.

Beccacece, F., Borgonovo, E., 2011. Functional ANOVA, ultramodularity and monotonicity: Applications in multiattribute utility theory. European Journal of Operational Research 210 (2), 326 – 335.

Bettman, J., 1979. An information processing theory of consumer choice. Addison-Wesley, Reading, MA.

Bettman, J., Luce, M., Payne, J., 1998. Constructing consumer choice processes. Journal of Consumer Research 25 (3), 187–217.

Birnbaum, M., Thompson, A., 1996. Violations of monotonicity in choices between gambles and certain cash. The American Journal of Psychology 109 (4), 501–523.

Bleichrodt, H., Cillo, A., Diecidue, E., 2010. A quantitative elicitation of regret theory. Management Science 56 (1), 161–175.

Bungartz, H.-J., Griebel, M., 4 2004. Sparse grids. Acta Numerica 13, 147–269.

Buzzard, G., 2012. Global sensitivity analysis using sparse grid interpolation and polynomial chaos. Reliability Engineering & System Safety 107, 82–89.

Crestaux, T., Le Maître, O., Martinez, J.-M., 2009. Polynomial chaos expansion for sensitivity analysis. Reliability Engineering & System Safety 94 (7), 1161–1172.

Debreu, G., 1954. Representation of a preference ordering by a numerical function, decision processes Edition. Wiley, NY, USA.

Dolan, P., 2000. The Measurement of Health-Related Quality of Life for Use in Resource Allocation Decisions in Health Care, handbook of health economics Edition. Elsevier, Amsterdam, The Netherlands.

Dyer, J., Fishburn, P., Steuer, R., Wallenius, J., Zionts, S., 1992. Multiple criteria decision making, multiattribute utility theory: The next ten years. Management Science 38, 645–654.

Dyer, J., Sarin, R., 1979. Measurable multiattribute value function. Operations Research 27, 810–822.

Efron, B., Stein, C., 1981. The jackknife estimate of variance. Annals of Statistics 9 (3), 586–596.

Farquhar, P., 1977. A Survey of Multiattribute Utility Theory and Applications. Multiple Criteria Decision Making. North Holland, Amsterdam.

Figueira, J., Greco, S., R., S., 2009. Building a set of additive value functions representing a reference preorder and intensities of preference: Grip method. European Journal of Operational Research 195, 460–486.

Fishburn, P., 1970. Utility theory for decision making. Wiley, NY, USA.

Friedman, J., 1991. Multivariate adaptive regression splines (with discussion). Annals of Statistics 19 (1), 1–67.

Gorman, W., 1968. The structure of utility functions. The Review of Economic Studies 35 (4), 367–390.

Greco, S., Mousseau, V., Slowinski, R., 2008. Ordinal regression revisited: Multiple criteria ranking with a set of additive value functions. European Journal of Operational Research 191, 415–435.

Greco, S., Mousseau, V., Slowinsky, R., 2010. Multiple criteria sorting with a set of additive value functions. European Journal of Operational Research 207, 1455–1470.

Ingersoll, J., 1987. Theory of financial decision making. Cambridge University Press, Cambridge, UK (ISBN 0-847-67359-6).

Jacquet-Lagreze, E., Siskos, Y., 1982. Assessing a set of additive utility functions for multicriteria decision making: the UTA method. European

Journal of Operational Research 10, 151–164.

Kadzinsky, M., Greco, S., Slowinsky, R., 2012. Selection of a representative value function in robust multiple criteria ranking. European Journal of Operational Research 217, 541–553.

Keeney, R., Raiffa, H., 1993. Decisions with Multiple Objectives. ISBN 0-521-43883-7. Rowman & Littlefield Publishers, Inc., Totowa, U.S.A.

Krantz, D., Luce, R., Suppes, P., Tversky, A., 1971. Foundations of measurement. Academic Press, NY, USA.

Li, G., Wang, S., Rabitz, H., 2002a. Practical approaches to construct RS-HDMR component functions. J. Phys. Chem. 106, 8721–8733.

Li, G., Wang, S.-W., Rabitz, H., Wang, S., Jaffé, P., 2002b. Global uncertainty assessments by high dimensional model representations (HDMR). Chemical Engineering Science 57, 4445–4460.

Lin, Y., Zang, H., 2006. Component selection and smoothing in multivariate nonparametric regression. Annals of Statistics 35 (5), 2272–2297.

Owen, A., 2012. Variance components and generalized Sobol' indices. Stanford, http://stat.stanford.edu/ owen/reports/newsobol.pdf.

Payne, J., Laughhunn, D., Crum, R., 1984. Multiattribute risky choice behavior: The editing of complex prospects. Management Science 30 (11), 1350–1361.

Rabitz, H., Alış, Ö. F., 1999. General foundations of high-dimensional model representations. J. Math. Chem. 25 (2–3), 197–233.

Ratto, M., Castelletti, A., Pagano, A., 2012. Emulation techniques for the reduction and sensitivity analysis of complex environmental models. Environmental Modelling & Software 34, 1–4.

Ratto, M., Pagano, A., 2010. Using recursive algorithms for the efficient identification of smoothing spline anova models. Advances in Statistical Analysis 94, 367–388.

Richard, S., 1975. Multivariate risk aversion, utility undependence and separable utility functions. Management Science 22 (1), 12–21.

Roy, B., 1985. Methodologie multicritere d'aide a la decision. Economica, Paris, France., 423 pages.

Scheubrein, R., Zionts, S., 2006. A problem structuring front end for a multiple criteria decision support system. Computers & Operations Research 33, 18–31.

Siskos, Y., Yannacopoulos, D., 1985. Utastar: An ordinal regression method for building additive value functions. Investigacao Operacional 5 (1), 39–53.

Smith, J., von Winterfeldt, D., 2004. Decision analysis in management science. Management Science 50 (5), 561–574.

Sobol´, I., 1993. Sensitivity estimates for nonlinear mathematical models. Mathematical Modelling & Computational Experiments 1, 407–414.

Sobol', I. M., 2003. Theorem and examples in high dimensional model representation. Reliability Engineering and System Safety 79, 187–193.

Sounderpandian, J., 1991. Value functions when decision criteria are not totally substitutable. Operations Research 39 (4), 592–600.

Stott, H. P., 2006. Cumulative prospect theory's functional managerie. Journal of Risk and Uncertainty 32, 101–130.

Sudret, B., jul 2008. Global sensitivity analysis using polynomial chaos expansions. Reliability Engineering & System Safety 93 (7), 964–979.

Tsetlin, I., Winkler, R., 2009. Multiattribute utility satisfying a preference for combining good with bad. Management Science 55 (12), 1942–1952.

Tversky, A., Kahneman, D., 1991. Loss aversion in riskless choice: A reference-dependent model. The Quarterly Journal of Economics 106 (4), 1039–1061.

Wakker, P., Jansen, S., Stiggelbout, A., 2004. Anchor levels as a new tool for the theory and measurement of multiattribute utility. Decision Analysis 1 (4), 217–234.

Wallenius, J., Dyer, J. S., Fishburn, P. C., Steuer, R. E., Zionts, S., Deb, K., 2008. Multiple criteria decision making, multiattribute utility theory: Recent accomplishments and what lies ahead. Management Science 54 (7), 1336–1349.

Wang, X., 2006. On the effects of dimension reduction techniques on some high-dimensional problems in finance. Operations Research 54 (6), 1063–1078.

Wathieu, L., 2004. Consumer habituation. Management Science 50 (5), 587–596.

Ziehn, T., Tomlin, A., 2009. GUI-HDMR - a software tool for global sensitivity analysis of complex models 24, 775–785.