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S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini

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IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
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The Structure of Variational Preferences*

S. Cerreia-Vioglio,[a] F. Maccheroni,[a] M. Marinacci,[a] and A. Rustichini[a,b]
[a]IGIER and Università Bocconi and [b]University of Minnesota

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Abstract

Maccheroni, Marinacci, and Rustichini [17], in an Anscombe-Aumann framework, axiomatically characterize preferences that are represented by the variational utility functional

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F},$$

where u is a utility function on outcomes and c is an index of uncertainty aversion. In this paper, for a given variational preference, we study the class $\mathcal{C}$ of functions c that represent V. Inter alia, we show that this set is fully characterized by a minimal and a maximal element, $c^\star$ and $d^\star$. The function $c^\star$, also identified by Maccheroni, Marinacci, and Rustichini [17], fully characterizes the decision maker's attitude toward uncertainty, while the novel function $d^\star$ characterizes the uncertainty perceived by the decision maker.

1 Introduction

In this paper, we study the functional structure of variational preferences. This class of binary relations was introduced by Maccheroni, Marinacci, and Rustichini [17] (henceforth, MMR). In an Anscombe and Aumann framework, a binary relation $\succsim$ over the set of acts $\mathcal{F}$ is said to be a variational preference if and only if it admits the following representation

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F}, \tag{1}$$

where u is an affine utility index, Δ is the set of probabilities, and $c : \Delta \to [0, \infty]$ is a grounded, lower semicontinuous, and convex function. In other words, each variational preference is characterized by a pair (u, c), where u is a utility index over consequences and c is an index of uncertainty aversion.

For a given variational preference $\succsim$ and a given u, we study the set of all functions $c : \Delta \to [0, \infty]$ which are grounded, lower semicontinuous, convex, and such that the corresponding V, given by (1), represents $\succsim$. We denote this set by $\mathcal{C}$. MMR showed that if $\succsim$ also satisfies an unboundedness axiom, then the function c in (1) is unique; that is, $\mathcal{C}$ is a singleton. Without such an axiom, $\mathcal{C}$ is no longer a singleton. Our analysis sheds light on the structure of $\mathcal{C}$ when it contains more than one element. In Theorem 1, we show that $\mathcal{C}$ is a convex set and a complete lattice. In particular, $\mathcal{C}$ admits a minimum and a maximum element, denoted by $c^\star$ and $d^\star$.

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From a decision theoretic point of view, the function $c^\star$ is the function identified by MMR, which captures the decision maker's uncertainty attitudes (see [17, Proposition 8]). The function $d^\star$ is a novel object; we show it characterizes the revealed unambiguous preference as defined by Ghirardato, Maccheroni, and Marinacci [10].[1]

As a consequence of our main result, we show that each lower semicontinuous and convex function c which is such that $c^\star \leq c \leq d^\star$ also satisfies (1), and thus represents $\succsim$ (Corollary 1). From a conceptual and formal point of view, these observations suggest that variational representations of preferences are characterized by a triple $(u, c^\star, d^\star)$, which reduces to a pair (u, c) in the unbounded case, more thoroughly studied by MMR.

2 Preliminaries

2.1 Decision Theoretic Set Up

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*, that is, of Σ-measurable functions $f : S \to X$ that take finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume that X is a convex subset of a vector space. This is the case, for instance, if X is the set of all lotteries on a set of outcomes, as in the classic setting of Anscombe and Aumann [1]. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$ as follows: For each $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$.

We model the decision maker's *preferences* on $\mathcal{F}$ by a binary relation $\succsim$. Given such a binary relation $\succsim$, $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$. Finally, we denote by $\mathcal{F}_{\text{int}}$ the set of acts

$$\{f \in \mathcal{F} : \exists x, y \in X \text{ s.t. } x \succ f(s) \succ y \qquad \forall s \in S\}.$$

2.2 Mathematical Preliminaries

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions, so that $u(f) \in B_0(\Sigma)$ whenever $u : X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an interval $K \subseteq \mathbb{R}$, we denote by $B_0(\Sigma, K)$ the set of all real-valued Σ-measurable simple functions that take values in the interval K. Note that, if $K = \mathbb{R}$, then $B_0(\Sigma, \mathbb{R}) = B_0(\Sigma)$.

The (sup)norm dual space of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ; it is a (weak*) compact and convex subset of $ba(\Sigma)$. The set Δ is endowed with the relative weak* topology.

Given a function $c : \Delta \to [0, \infty]$, we say that c is grounded if and only if $\min_{p \in \Delta} c(p) = 0$. We denote the effective domain of c by

$$\operatorname{dom} c = \{p \in \Delta : c(p) < \infty\}.$$

2.3 Variational Preferences

We consider three nested classes of preferences: Anscombe-Aumann expected utility preferences, Gilboa-Schmeidler preferences, and variational preferences à la MMR. Before formally defining them, we provide

[1] This result is also based on an equivalence between Greenberg–Pierskalla differentials and Clarke's differentials, which is established in Theorem 2.

the axioms that characterize these preferences. For a thorough discussion of these assumptions, we refer the interested reader to [1], [12], and [17].

Axiom A. 1 (Weak Order) *The binary relation $\succsim$ is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom A. 3 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.*

Axiom A. 4 (Independence) *If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \succsim g \Rightarrow \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h.$$

Definition 1 *Let $\succsim$ be a binary relation on $\mathcal{F}$. We say that $\succsim$ is an Anscombe-Aumann expected utility preference if and only if it satisfies Weak Order, Monotonicity, Continuity, and Independence.*

By Anscombe and Aumann [1], $\succsim$ is an Anscombe-Aumann expected utility preference if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$ and a unique $p \in \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \int u(f)\, dp \qquad \forall f \in \mathcal{F},$$

represents $\succsim$.[2]

Gilboa-Schmeidler preferences differ from expected utility ones in terms of the Independence assumption. In fact, Gilboa and Schmeidler weaken the Independence assumption and replace it with the following two postulates (see also [17, Lemma 1]):

Axiom A. 5 (C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha, \beta \in (0,1)$,*

$$\alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x \Rightarrow \beta f + (1-\beta) y \succsim \beta g + (1-\beta) y.$$

Axiom A. 6 (Uncertainty Aversion) *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha) g \succsim f$.*

Definition 2 *Let $\succsim$ be a binary relation on $\mathcal{F}$. We say that $\succsim$ is a Gilboa-Schmeidler preference if and only if it satisfies Weak Order, Monotonicity, Continuity, C-Independence, and Uncertainty Aversion.*

By Gilboa and Schmeidler [12], a binary relation $\succsim$ is a Gilboa-Schmeidler preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a unique closed and convex set $C \subseteq \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in C} \int u(f)\, dp \qquad \forall f \in \mathcal{F},$$

represents $\succsim$.

Finally, Maccheroni, Marinacci, and Rustichini [17] consider binary relations $\succsim$ on $\mathcal{F}$ that satisfy an even weaker assumption of Independence.

Axiom A. 7 (Weak C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x \Rightarrow \alpha f + (1-\alpha) y \succsim \alpha g + (1-\alpha) y.$$

[2] That is, $f \succsim g$ if and only if $V(f) \geq V(g)$.

Definition 3 *Let $\succsim$ be a binary relation on $\mathcal{F}$. We say that $\succsim$ is a variational preference if and only if it satisfies Weak Order, Monotonicity, Continuity, Weak C-Independence, and Uncertainty Aversion.*

By MMR [17, Theorem 3], a binary relation $\succsim$ is a variational preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a grounded, lower semicontinuous, and convex function $c : \Delta \to [0, \infty]$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F}, \tag{2}$$

represents $\succsim$.

Given a binary relation $\succsim$ on $\mathcal{F}$, we define $\succsim^*$ as the *revealed unambiguous preference* of Ghirardato, Maccheroni, and Marinacci [10]:

$$f \succsim^* g \quad \iff \quad \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \qquad \forall \lambda \in (0,1], \forall h \in \mathcal{F}.$$

By Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [5], if $\succsim$ is a variational preference, then there exists a unique closed and convex set C^* such that

$$f \succsim^* g \quad \iff \quad \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C^*.$$

The binary relation $\succsim^*$ is a Bewley preference (see Bewley [3]). Typically, $\succsim^*$ is interpreted as including the rankings for which the decision maker is sure, and the set C^* is interpreted as the uncertainty perceived by the decision maker.

3 Results

In this section, we consider a variational preference $\succsim$ represented by a function V defined as in (2). Given V and u, object of our study is the set $\mathcal{C} = \mathcal{C}(V, u)$ defined as the set of all functions $c : \Delta \to [0, \infty]$ which are grounded, lower semicontinuous, convex, and further satisfy (2). An object that will play an important role in what follows is the correspondence $\pi_u : \mathcal{F}_{\text{int}} \rightrightarrows \Delta$:

$$\pi_u(f) = \left\{ p \in \Delta : \int u(f)\, dp \geq \int u(g)\, dp \implies f \succsim g \right\} \qquad \forall f \in \mathcal{F}_{\text{int}}.$$

The set $\pi_u(f)$ consists of the beliefs that rationalize the decision maker's preferences at f. These sets of local beliefs have been studied by Rigotti, Shannon, and Strzalecki [18] and by Hanany and Klibanoff [15].

In the next result, we show that $\mathcal{C}$ is a convex set and a complete lattice. In particular, $\mathcal{C}$ admits a minimum and a maximum element, $c^\star$ and $d^\star$. From a decision theoretic point of view, the function $c^\star$ is the function identified by MMR (see Remark 2 and Appendix B) and shown to capture the decision maker's uncertainty attitudes (see [17, Proposition 8]). On the other hand, the function $d^\star$ is a novel object that characterizes $\succsim^*$. Finally, point (v) shows that all functions $c \in \mathcal{C}$ coincide on the collection of local beliefs.

We conclude by computing $d^\star$ when $\succsim$ is a Gilboa-Schmeidler preference and when $\succsim$ is an Anscombe-Aumann expected utility preference.

Theorem 1 *Let $\succsim$ be a variational preference on $\mathcal{F}$. The following statements are true:*

(i) There exist $c^\star, d^\star \in \mathcal{C}$ such that

$$c^\star \leq c \leq d^\star \qquad \forall c \in \mathcal{C}.$$

(ii) $\mathcal{C}$ *is a convex set and a complete lattice.*

(iii) If $u(X)$ *is unbounded, then* $\mathcal{C}$ *is a singleton and* $c^\star = d^\star$.

(iv) $\mathrm{cl}(\mathrm{dom}\, d^\star) = C^*$.

(v) For each $f \in \mathcal{F}_{\mathrm{int}}$, *for each* $p \in \pi_u(f)$, *and for each* $c \in \mathcal{C}$,

$$c^\star(p) = c(p) = d^\star(p).$$

In particular,

$$C^* = \bar{\mathrm{co}}\left(\bigcup_{f\in\mathcal{F}_{\mathrm{int}}} \pi_u(f)\right). \tag{3}$$

Remark 1 In the unbounded case, and for the more general class of uncertainty averse preferences, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6, Proposition 11] show that the collection of all sets $\pi_u(f)$ characterizes $\succsim^*$. Equation (3) in point (v) extends this finding, for the class of variational preferences, to the bounded case. This fact could be proven in two ways: (a) by direct methods, as we do in Appendix B, (b) by showing (loosely speaking) that the set of Greenberg-Pierskalla differentials $\pi_u(f)$ coincides with the set of normalized Clarke's differentials (see our Theorem 2), and then invoking Ghirardato and Siniscalchi [11, Theorem 2], who show that C^* coincides with the closed convex hull of all normalized Clarke's differentials.

Remark 2 Recall that $c^\star : \Delta \to [0,\infty]$ is such that

$$c^\star(p) = \sup_{f\in\mathcal{F}}\left\{u(x_f) - \int u(f)\,dp\right\} \qquad \forall p \in \Delta, \tag{4}$$

where $x_f \sim f$ for all $f \in \mathcal{F}$ (see [17, Theorem 3]).

The next result confirms the intuition that most of the structure of a variational preference is captured by the functions $c^\star$ and $d^\star$ associated to its representation. In fact, given a function $c : \Delta \to [0,\infty]$, it is enough to check whether it is lower semicontinuous, convex, and such that $c^\star \le c \le d^\star$ to conclude that $c \in \mathcal{C}$, that is, that c is also grounded and represents $\succsim$ as in (2).

Corollary 1 *Let* $\succsim$ *be a variational preference on* $\mathcal{F}$ *and* $c : \Delta \to [0,\infty]$ *a lower semicontinuous and convex function. The following statements are equivalent:*

(i) $c^\star \le c \le d^\star$;

(ii) $c \in \mathcal{C}$.

The next results confirm that $d^\star$ captures an important behavioral trait of the decision maker. Its effective domain, aside from coinciding (up to closure) with the set of probabilities characterizing the revealed unambiguous preference $\succsim^*$, is the smallest closed convex set over which any function $c \in \mathcal{C}$ can be restricted to in (2).

Corollary 2 *Let* $\succsim$ *be a variational preference on* $\mathcal{F}$. *If* $c \in \mathcal{C}$, *then* C^* *is the smallest closed and convex subset of* Δ *such that the* min *in (2) can be restricted to.*

In particular, for each $c \in \mathcal{C}$ and for each $f \in \mathcal{F}$

$$V(f) = \min_{p\in C^*}\left\{\int u(f)\,dp + c(p)\right\}.$$

Corollary 3 *If $\succsim$ is a Gilboa-Schmeidler preference on $\mathcal{F}$, then $d^\star = \delta_C$.*[3]

Note that if the interval $u(X)$ is bounded, by (4), then $c^\star(p) \leq \text{length}\, u(X) < \infty$ for all $p \in \Delta$. Therefore, unless $C = C^* = \Delta$, we have that $c^\star \neq d^\star$ and so $c^\star \neq \delta_C$.

Corollary 4 *If $\succsim$ is an Anscombe-Aumann expected utility preference on $\mathcal{F}$, then $d^\star = \delta_{\{p\}}$.*

Summing up, the triple $(u, c^\star, d^\star)$ characterizes the representation of variational preferences. Under unboundedness, the triple reduces to the pair $(u, c^\star)$. This latter case was the center of the analysis of [17].

A Nonsmooth Differentials

Recall that we denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions and by $ba(\Sigma)$ the set of all bounded and finitely additive set functions. We endow the former set with the supnorm. Thus, the latter set can be identified with the norm dual of $B_0(\Sigma)$. Recall also that we endow $ba(\Sigma)$ and any of its subsets with the weak* topology. We denote by $\langle \cdot, \cdot \rangle : B_0(\Sigma) \times ba(\Sigma) \to \mathbb{R}$ the dual pairing. The function $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \varphi, p \rangle = \int \varphi dp \qquad \forall (\varphi, p) \in B_0(\Sigma) \times ba(\Sigma).$$

In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we denote by k both the real number and the constant function on S that takes value k. Finally, we denote by K an interval of $\mathbb{R}$ such that $0 \in \text{int}\, K$.

Consider a functional $I : B_0(\Sigma, K) \to \mathbb{R}$ and $\psi \in B_0(\Sigma, \text{int}\, K)$. Define the

(i) *Clarke upper (directional) derivative* $I^\circ(\psi, \cdot) : B_0(\Sigma) \to [-\infty, \infty]$ at ψ by:

$$I^\circ(\psi; \xi) = \limsup_{\substack{\varphi \to \psi \\ t \downarrow 0}} \frac{I(\varphi + t\xi) - I(\varphi)}{t} \qquad \forall \xi \in B_0(\Sigma),$$

(ii) *Clarke lower (directional) derivative* $I_\circ(\psi, \cdot) : B_0(\Sigma) \to [-\infty, +\infty]$ at ψ by:

$$I_\circ(\psi; \xi) = \liminf_{\substack{\varphi \to \psi \\ t \downarrow 0}} \frac{I(\varphi + t\xi) - I(\varphi)}{t} \qquad \forall \xi \in B_0(\Sigma),$$

It is easy to check that

$$I_\circ(\psi; \xi) \leq I^\circ(\psi; \xi) \qquad \forall \psi \in B_0(\Sigma, \text{int}\, K) \text{ and } \forall \xi \in B_0(\Sigma). \tag{5}$$

The *Clarke differential* $\partial^\circ I(\psi)$ at ψ is defined by

$$\partial^\circ I(\psi) = \{p \in ba(\Sigma) : \forall \varphi \in B_0(\Sigma) \quad \langle \varphi, p \rangle \leq I^\circ(\psi; \varphi)\}.$$

At this level of generality, we can also define another notion of superdifferential:

$$\partial_{GP} I(\psi) = \{p \in \Delta : \forall \varphi \in B_0(\Sigma, K) \quad \langle \varphi, p \rangle \leq \langle \psi, p \rangle \implies I(\varphi) \leq I(\psi)\}.$$

[3] Recall that $\delta_D : \Delta \to [0, \infty]$ is such that $\delta_D(p) = 0$ if $p \in D$ and $\delta_D(p) = \infty$ otherwise.

This notion of differential is common in Quasiconvex Analysis and it is due to Greenberg and Pierskalla [13]. If I is a normalized and concave niveloid,[4] we can also define a third notion of superdifferential

$$\partial I(\psi) = \{p \in \Delta : \forall \varphi \in B_0(\Sigma, K) \quad I(\varphi) - I(\psi) \leq \langle \varphi, p \rangle - \langle \psi, p \rangle\}.$$

This is the standard notion of superdifferential which is common in Convex Analysis. The following lemma can be found, for example, in Ghirardato, Maccheroni, and Marinacci [10] (see also Jahn [16, Chapter 3, Section 5] and Clarke [8, Proposition 2.1.2]).

Lemma 1 *Let K be an open interval. If $I : B_0(\Sigma, K) \to \mathbb{R}$ is locally Lipschitz, then $\partial^\circ I(\psi)$ is a nonempty, convex, and compact subset of $ba(\Sigma)$ at each $\psi \in B_0(\Sigma, K)$ with*

$$\partial^\circ I(\psi) = \{p \in ba(\Sigma) : \forall \varphi \in B_0(\Sigma) \quad \langle \varphi, p \rangle \geq I_\circ(\psi; \varphi)\}. \tag{6}$$

In particular, if I is monotone, then $\partial^\circ I(\psi) \subseteq ba_+(\Sigma)$ for all $\psi \in B_0(\Sigma, K)$.

Theorem 2 *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is monotone, continuous, locally Lipschitz on $B_0(\Sigma, \operatorname{int} K)$, and quasi-concave, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$*

$$\left\{\frac{p}{\|p\|} : 0 \neq p \in \partial^\circ I(\psi)\right\} \subseteq \partial_{GP} I(\psi). \tag{7}$$

Moreover, if $0 \notin \partial^\circ I(\psi)$, then equality holds in (7).

Proof. Fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. Consider $p \in ba(\Sigma)$ such that $0 \neq p \in \partial^\circ I(\psi)$. By Lemma 1 and since I is monotone, $\bar{p} = p/\|p\| \in \Delta$. Consider now $\varphi \in B_0(\Sigma, K)$. We prove three facts:

1. $\langle \varphi, \bar{p} \rangle < \langle \psi, \bar{p} \rangle \implies I(\varphi) \leq I(\psi)$. Define $\varepsilon = \langle \psi - \varphi, p \rangle$. By assumption, we have that $\varepsilon > 0$. Since $p \in \partial^\circ I(\psi)$, $I^\circ(\psi; \psi - \varphi) \geq \langle \psi - \varphi, p \rangle = \varepsilon > 0$. By definition of $I^\circ(\psi; \psi - \varphi)$, we have that there exist $\{t_n\}_{n \in \mathbb{N}} \subseteq (0, \infty)$ and $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq B_0(\Sigma)$ such that

$$\frac{I(\varphi_n + t_n(\psi - \varphi)) - I(\varphi_n)}{t_n} \to I^\circ(\psi; \psi - \varphi)$$

where $0 < t_n \to 0$ and $\varphi_n \to \psi$. It follows that for n large enough

$$\frac{I(\varphi_n + t_n(\psi - \varphi)) - I(\varphi_n)}{t_n} \geq \frac{\varepsilon}{2}.$$

Since I is locally Lipschitz, we have that for n large enough

$$\begin{aligned}
\frac{\varepsilon}{2} &\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{I(\varphi_n + t_n(\psi - \varphi)) - I(\varphi_n + t_n(\varphi_n - \varphi))}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{K\|\varphi_n + t_n(\psi - \varphi) - \varphi_n - t_n(\varphi_n - \varphi)\|}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{K\|t_n(\psi - \varphi_n)\|}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + K\|\psi - \varphi_n\|.
\end{aligned}$$

[4] See [7] for a definition of niveloid. Recall that a normalized niveloid is such that

1. $I(k) = k$ for all $k \in K$;
2. I is monotone;
3. $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma, K)$ and $k \in K$ such that $\varphi + k \in B_0(\Sigma, K)$.

This implies that for n large enough

$$\frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} \geq \frac{\varepsilon}{4} > 0.$$

We can finally conclude that for n large enough

$$I(\varphi_n + t_n(\varphi_n - \varphi)) > I(\varphi_n). \tag{8}$$

Define $\alpha_n = (1+t_n)^{-1} \in (0,1)$ for all $n \in \mathbb{N}$. Note that $\varphi_n = \alpha_n(\varphi_n + t_n(\varphi_n - \varphi)) + (1-\alpha_n)\varphi$ for all $n \in \mathbb{N}$. Since I is quasiconcave and by (8), we have that for n large enough

$$I(\varphi_n) \geq \min\{I(\varphi_n + t_n(\varphi_n - \varphi)), I(\varphi)\} = I(\varphi).$$

Since I is continuous, it follows that $I(\varphi) \leq \lim_n I(\varphi_n) = I(\psi)$.

2. $\langle \varphi, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ and $\varphi \in B_0(S, \operatorname{int} K) \implies I(\varphi) \leq I(\psi)$. Since $p \neq 0$, there exists $\phi \in B_0(\Sigma)$ such that $\langle \phi, \bar{p} \rangle < 0$. Define
$$\varphi_n = \varphi + \frac{1}{n}\phi \qquad \forall n \in \mathbb{N}.$$
Since $\varphi \in B_0(S, \operatorname{int} K)$, note that $\{\varphi_n\}_{n\in\mathbb{N}}$ eventually belongs to $B_0(\Sigma, \operatorname{int} K)$. It is also immediate to see that $\langle \varphi_n, \bar{p} \rangle < \langle \psi, \bar{p} \rangle$ for all $n \in \mathbb{N}$. By point 1 and since I is continuous, we have that $I(\varphi) = \lim_n I(\varphi_n) \leq I(\psi)$.

3. $\langle \varphi, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ and $\varphi \notin B_0(S, \operatorname{int} K) \implies I(\varphi) \leq I(\psi)$. Define
$$\varphi_n = \left(1 - \frac{1}{n}\right)\varphi + \frac{1}{n}\psi \qquad \forall n \in \mathbb{N}.$$
Since $\varphi \in B_0(S, K)$ and $\psi \in B_0(S, \operatorname{int} K)$, note that $\{\varphi_n\}_{n\in\mathbb{N}}$ belongs to $B_0(\Sigma, \operatorname{int} K)$. Moreover, we also have that $\langle \varphi_n, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ for all $n \in \mathbb{N}$. By point 2 and since I is continuous, we have that $I(\varphi) = \lim_n I(\varphi_n) \leq I(\psi)$.

By points 1, 2, and 3, we proved that, if $\varphi \in B_0(\Sigma, K)$ is such that $\langle \varphi, \bar{p} \rangle \leq \langle \psi, \bar{p} \rangle$, then $I(\varphi) \leq I(\psi)$. Thus, $p/\|p\| = \bar{p} \in \partial_{GP} I(\psi)$, proving (7).

Suppose $0 \notin \partial^\circ I(\psi)$. Let $\bar{p} \in \partial_{GP} I(\psi)$. If $\xi \in \ker \bar{p} = \{\varphi \in B_0(\Sigma) : \langle \varphi, \bar{p} \rangle = 0\}$, then we have that $\langle \psi + t\xi, \bar{p} \rangle \leq \langle \psi, \bar{p} \rangle$ for all $t \geq 0$. Since $\bar{p} \in \partial_{GP} I(\psi)$, this implies that $I(\psi + t\xi) \leq I(\psi)$ for all $t \geq 0$. By definition of $I_\circ(\psi; \xi)$, it follows that $I_\circ(\psi; \xi) \leq 0$. By the Hahn-Banach Theorem, there exists a continuous linear functional $p : B_0(\Sigma) \to \mathbb{R}$ such that $\langle \xi, p \rangle \geq I_\circ(\psi; \xi)$ for all $\xi \in B_0(\Sigma)$ and $\langle \xi, p \rangle = 0$ for all $\xi \in \ker \bar{p}$. In particular, we have that $\ker \bar{p} \subseteq \ker p$. By Lemma 1, we can conclude that $p \in \partial^\circ I(\psi)$. By Lemma 1 and since $0 \notin \partial^\circ I(\psi)$ and I is monotone, it follows that $0 \neq p \geq 0$. By the Fundamental Theorem of Duality (see [2, Theorem 5.91]) and since $\ker \bar{p} \subseteq \ker p$ and $0 \neq p \geq 0$, there exists $\alpha > 0$ such that $p = \alpha \bar{p}$. We can conclude that $\alpha = \|p\|$ and $\bar{p} = p/\|p\|$, proving the equality in (7) when $0 \notin \partial^\circ I(\psi)$. ■

Lemma 2 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. If $J : B_0(\Sigma) \to \mathbb{R}$ is a normalized and concave niveloid which extends I, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$*

$$\partial_{GP} I(\psi) = \partial_{GP} J(\psi).$$

Proof. Fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. Consider $p \in \partial_{GP} J(\psi)$. It follows that for each $\varphi \in B_0(\Sigma)$

$$\langle \varphi, p \rangle \leq \langle \psi, p \rangle \quad \implies \quad J(\varphi) \leq J(\psi).$$

Since J extends I, we have that for each $\varphi \in B_0(\Sigma, K)$

$$\langle \varphi, p\rangle \leq \langle \psi, p\rangle \quad \Longrightarrow \quad I(\varphi) = J(\varphi) \leq J(\psi) = I(\psi),$$

proving that $p \in \partial_{GP} I(\psi)$. On the other hand, consider $p \in \partial_{GP} I(\psi)$. By contradiction, assume that $p \notin \partial_{GP} J(\psi)$. It follows that there exists $\bar{\varphi} \in B_0(\Sigma)$ such that

$$\langle \bar{\varphi}, p\rangle \leq \langle \psi, p\rangle \quad \text{and} \quad J(\bar{\varphi}) > J(\psi).$$

Since $\psi \in B_0(\Sigma, \operatorname{int} K)$, there exists $\lambda \in (0,1)$ such that $\lambda\bar{\varphi} + (1-\lambda)\psi \in B_0(\Sigma, \operatorname{int} K)$. It is immediate to check that $\langle \lambda\bar{\varphi} + (1-\lambda)\psi, p\rangle \leq \langle \psi, p\rangle$. Since J is concave and extends I, we also have that

$$I(\lambda\bar{\varphi} + (1-\lambda)\psi) = J(\lambda\bar{\varphi} + (1-\lambda)\psi) \geq \lambda J(\bar{\varphi}) + (1-\lambda) J(\psi) > J(\psi) = I(\psi),$$

a contradiction with $p \in \partial_{GP} I(\psi)$. ■

Corollary 5 *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is a normalized and concave niveloid, then*

$$\partial^\circ I(\psi) = \partial I(\psi) = \partial_{GP} I(\psi) \qquad \forall \psi \in B_0(\Sigma, \operatorname{int} K).$$

Proof. Since I is a niveloid, I is monotone and Lipschitz continuous. Next, fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. It is immediate to see that

$$\partial_{GP} I(\psi) \subseteq \partial I(\psi) \subseteq \partial^\circ I(\psi). \tag{9}$$

Next, consider $p \in \partial^\circ I(\psi)$. By Lemma 1, $p \geq 0$ and since I is a normalized and concave niveloid, then $1 = I^\circ(\psi; 1) \geq \langle 1, p\rangle \geq I_\circ(\psi; 1) = 1$, proving that $\langle 1, p\rangle = 1$. It follows that $0 \neq p \in \Delta$ and $\|p\| = 1$. This implies that $\left\{\frac{p}{\|p\|} : 0 \neq p \in \partial^\circ I(\psi)\right\} = \partial^\circ I(\psi)$ and $0 \notin \partial^\circ I(\psi)$. By Theorem 2, we can also conclude that

$$\partial^\circ I(\psi) = \left\{\frac{p}{\|p\|} : 0 \neq p \in \partial^\circ I(\psi)\right\} = \partial_{GP} I(\psi). \tag{10}$$

Since ψ was arbitrarily chosen and by (9), the statement follows. ■

B Proofs

Next, we study a normalized and concave niveloid $I : B_0(\Sigma, K) \to \mathbb{R}$. We denote by β an element in $\operatorname{int} K$. Note that the constant function β belongs to $B_0(\Sigma, \operatorname{int} K)$.

Proposition 1 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized niveloid. I is concave if and only if there exists a grounded, lower semicontinuous, and convex function $c : \Delta \to [0, \infty]$ such that*

$$I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p\rangle + c(p)\} \qquad \forall \varphi \in B_0(\Sigma, K). \tag{11}$$

Moreover, there exists a minimal function $c^\star : \Delta \to [0, \infty]$, defined by

$$c^\star(p) = \sup_{\varphi \in B_0(\Sigma, K)} \{I(\varphi) - \langle \varphi, p\rangle\} \qquad \forall p \in \Delta,$$

which is grounded, lower semicontinuous, convex, and satisfies (11).

Proof. For a proof see Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini [7]. ■

Given I, define as $\mathcal{C} = \mathcal{C}(I)$ the class of functions $c : \Delta \to [0, \infty]$ such that c is grounded, lower semicontinuous, convex, and represents I as in (11). By Proposition 1, if I is a normalized and concave niveloid, then $\mathcal{C}$ is nonempty. Given $c \in [0, \infty]^{\Delta}$ and $\psi \in B_0(\Sigma, K)$, we define $M_c(\psi)$ by

$$M_c(\psi) = \{p \in \Delta : \langle \psi, p \rangle + c(p) = I(\psi)\}.$$

If $c \in \mathcal{C}$, then $M_c(\psi) \neq \emptyset$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$.

Proposition 2 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. If $c \in \mathcal{C}$, then $M_c(\psi) = \partial_{GP} I(\psi)$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. In particular,* $\arg\min c = M_c(\beta) = \partial_{GP} I(\beta)$*, that is,*

$$c(p) = 0 \textit{ if and only if } p \in \partial_{GP} I(\beta).$$

Proof. Pick c in $\mathcal{C}$. Define $J_c : B_0(\Sigma) \to \mathbb{R}$ by $J_c(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + c(p)\}$ for all $\varphi \in B_0(\Sigma)$. It is immediate to verify that the concave conjugate of J_c, $J_c^* : ba(\Sigma) \to [-\infty, \infty)$, is such that $J_c^*(p) = -c(p)$ if $p \in \Delta$ and $J_c^*(p) = -\infty$ otherwise. Pick ψ in $B_0(\Sigma, \operatorname{int} K)$. By [4, Proposition 4.4.1], Corollary 5, and Lemma 2, it follows that

$$M_c(\psi) = \partial J_c(\psi) = \partial_{GP} J(\psi) = \partial_{GP} I(\psi), \tag{12}$$

proving the main statement. Finally, consider the case $\psi = \beta$. By (12), we have that $M_c(\beta) = \partial_{GP} I(\beta)$. Next, consider $p \in \arg\min c$, that is, since c is grounded, consider $p \in \Delta$ such that $c(p) = 0$. It follows that $I(\beta) = \beta = \langle \beta, p \rangle = \langle \beta, p \rangle + c(p)$, proving that $p \in M_c(\beta)$. On the other hand, if $p \in M_c(\beta)$, then $\beta = I(\beta) = \langle \beta, p \rangle + c(p) = \beta + c(p)$, proving that $c(p) = 0$ and $p \in \arg\min c$. ■

Remark 3 *By Proposition 2, we can conclude that if $c \in \mathcal{C}$, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$ the set $\partial_{GP} I(\psi)$ is nonempty since $\emptyset \neq M_c(\psi) = \partial_{GP} I(\psi)$.*

Theorem 3 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. The following statements are true:*

1. *$\mathcal{C}$ is a complete lattice.*
2. *There exist $c^\star, d^\star \in \mathcal{C}$ such that $c^\star \leq c \leq d^\star$ for all $c \in \mathcal{C}$.*
3. *If c is lower semicontinuous, convex, and such that $c^\star \leq c \leq d^\star$, then $c \in \mathcal{C}$.*
4. *$\mathcal{C}$ is a convex set.*
5. $\operatorname{cl}(\operatorname{dom} d^\star) = \overline{\operatorname{co}} \left(\bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi) \right)$.
6. *For each $\psi \in B_0(\Sigma, \operatorname{int} K)$, for each $p \in \partial_{GP} I(\psi)$, and for each $c \in \mathcal{C}$*
$$c^\star(p) = c(p) = d^\star(p).$$

Proof. Given I, consider $\mathcal{C}$. By Proposition 1, it follows that $\mathcal{C} \neq \emptyset$.

1. Consider a nonempty subset $\{c_\gamma\}_{\gamma \in \Gamma} \subseteq \mathcal{C}$. Define $d : \Delta \to [0, \infty]$ by
$$d(p) = \sup_{\gamma \in \Gamma} c_\gamma(p) \qquad \forall p \in \Delta. \tag{13}$$

Step 1: d is lower semicontinuous and convex.

Proof of the Step. By (13) and since each element c in $\mathcal{C}$ is lower semicontinuous and convex, the statement follows (see [2, Lemma 2.41] and [14, Lemma 4.26]). $\square$

Step 2: For each $\psi \in B_0(\Sigma, \operatorname{int} K)$ we have that $M_d(\psi) = \partial_{GP} I(\psi) \neq \emptyset$. In particular, d is grounded.

Proof of the Step. Consider $\bar{p} \in \partial_{GP} I(\psi)$. By Proposition 2, we have that $\langle \psi, \bar{p} \rangle + c(\bar{p}) = I(\psi)$ for all $c \in \mathcal{C}$. In particular, $c_\gamma(\bar{p}) = I(\psi) - \langle \psi, \bar{p} \rangle$ for all $\gamma \in \Gamma$. Since d is defined as the pointwise supremum over $\{c_\gamma\}_{\gamma \in \Gamma}$, we can conclude that $d(\bar{p}) = I(\psi) - \langle \psi, \bar{p} \rangle$, that is, $\langle \psi, \bar{p} \rangle + d(\bar{p}) = I(\psi)$, proving that $\bar{p} \in M_d(\psi)$. Since $\{c_\gamma\}_{\gamma \in \Gamma}$ is nonempty, consider $c_{\bar{\gamma}} \in \{c_\gamma\}_{\gamma \in \Gamma}$. Consider $\bar{p} \in M_d(\psi)$. Since $c_{\bar{\gamma}}$ represents I as in (11) and by construction of d, it follows that

$$I(\psi) = \langle \psi, \bar{p} \rangle + d(\bar{p}) \geq \langle \psi, \bar{p} \rangle + c_{\bar{\gamma}}(\bar{p}) \geq \min_{p \in \Delta} \{\langle \psi, p \rangle + c_{\bar{\gamma}}(p)\} = I(\psi),$$

proving that $\bar{p} \in M_{c_{\bar{\gamma}}}(\psi)$. By Proposition 2, this implies that $\bar{p} \in \partial_{GP} I(\psi)$. From the previous part of the proof, it follows that $M_d(\beta) = \partial_{GP} I(\beta) \neq \emptyset$. Since I is normalized, if $\bar{p} \in M_d(\beta)$, then $\beta = I(\beta) = \langle \beta, \bar{p} \rangle + d(\bar{p}) = \beta + d(\bar{p})$, that is, $d(\bar{p}) = 0$, proving that d is grounded. $\square$

Step 3: $I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + d(p)\}$ for all $\varphi \in B_0(\Sigma, K)$, that is, d satisfies (11).

Proof of the Step. Consider $\varphi \in B_0(\Sigma, \operatorname{int} K)$. Since $\{c_\gamma\}_{\gamma \in \Gamma}$ is nonempty, consider $c_{\bar{\gamma}} \in \{c_\gamma\}_{\gamma \in \Gamma}$. By definition of d, it follows that $\langle \varphi, p \rangle + c_{\bar{\gamma}}(p) \leq \langle \varphi, p \rangle + d(p)$ for all $p \in \Delta$. By Step 2, we have that there exists $\bar{p} \in M_d(\varphi)$, that is, $I(\varphi) = \langle \varphi, \bar{p} \rangle + d(\bar{p})$. We can conclude that

$$\min_{p \in \Delta} \{\langle \varphi, p \rangle + d(p)\} \leq \langle \varphi, \bar{p} \rangle + d(\bar{p}) = I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + c_{\bar{\gamma}}(p)\} \leq \min_{p \in \Delta} \{\langle \varphi, p \rangle + d(p)\},$$

proving that I coincides to J_d on $B_0(\Sigma, \operatorname{int} K)$. Since both functionals are Lipschitz continuous on $B_0(\Sigma, K)$, the statement follows. $\square$

Step 4: $\mathcal{C}$ is a complete lattice, that is, given a nonempty subset $\{c_\alpha\}_{\alpha \in A} \subseteq \mathcal{C}$ there exists $\hat{c}$ and $\hat{d}$ such that

$$\hat{c} \leq c_\alpha \leq \hat{d} \qquad \forall \alpha \in A$$

where $\hat{c}$ is the greatest lower bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$ and $\hat{d}$ is the least upper bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$.

Proof of the Step. Define $\hat{c}, \hat{d} : \Delta \to [0, \infty]$ by

$$\hat{c}(p) = \sup_{c' \in \{c \in \mathcal{C} : \forall \alpha \in A \ c \leq c_\alpha\}} c'(p) \text{ and } \hat{d}(p) = \sup_{\alpha \in A} c_\alpha(p) \qquad \forall p \in \Delta.$$

In the first case, we have that $\{c_\gamma\}_{\gamma \in \Gamma} = \{c \in \mathcal{C} : \forall \alpha \in A \ c \leq c_\alpha\} \ni c^\star$ and $\hat{c} = d$ where d is defined as in (13). In the second case, we have that $\{c_\gamma\}_{\gamma \in \Gamma} = \{c_\alpha\}_{\alpha \in A}$ and $\hat{d} = d$ where d is again defined as in (13). In light of this observation and Steps 1-3, we have that both $\hat{c}$ and $\hat{d}$ are elements of $\mathcal{C}$. By construction, it is immediate to see that $\hat{c} \leq c_\alpha \leq \hat{d}$ for all $\alpha \in A$ and $\hat{c}$ is the greatest lower bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$ and $\hat{d}$ is the least upper bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$. $\square$

Step 4 proves point 1 of the statement.

2. Since $\mathcal{C}$ is a complete lattice, it follows that there exist a minimum and a maximum element. By Proposition 1, the minimum element is $c^\star$. By the proof of point 1, the maximum element $d^\star : \Delta \to [0, \infty]$ is defined by

$$d^\star(p) = \sup_{c \in \mathcal{C}} c(p) \qquad \forall p \in \Delta.^5$$

[5] $d^\star$ can also be obtained as the Fenchel-Moreau biconjugate of the function $c^\star + \delta_R$ where $R = \bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi)$.

3. Consider a lower semicontinuous and convex function $c : \Delta \to [0, \infty]$ such that $c^\star \leq c \leq d^\star$. By Proposition 2 and since $c^\star, d^\star \in \mathcal{C}$, we have that $c^\star(p) = 0 = d^\star(p)$ if and only if $p \in \partial_{GP} I(\beta)$. Since $\partial_{GP} I(\beta) \neq \emptyset$, consider $\bar{p} \in \partial_{GP} I(\beta)$. It follows that $0 = c^\star(\bar{p}) \leq c(\bar{p}) \leq d^\star(\bar{p}) = 0$, that is, c is grounded. Finally, observe that for each $\varphi \in B_0(\Sigma, K)$ we have that

$$I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + c^\star(p)\} \leq \min_{p \in \Delta} \{\langle \varphi, p \rangle + c(p)\} \leq \min_{p \in \Delta} \{\langle \varphi, p \rangle + d^\star(p)\} = I(\varphi),$$

proving that c satisfies (11), that is, $c \in \mathcal{C}$.

4. Consider $c_1, c_2 \in \mathcal{C}$ and fix $\lambda \in (0,1)$. Define $c_\lambda = \lambda c_1 + (1-\lambda) c_2$. The convex linear combination of lower semicontinuous and convex functions is lower semicontinuous and convex (see Clarke [9, Propositions 2.13 and 2.20]). Finally, by point 2, we have that $c^\star \leq c_1, c_2 \leq d^\star$. This implies that $c^\star \leq c_\lambda \leq d^\star$. By point 3, we can conclude that $c_\lambda \in \mathcal{C}$.

5. Fix $\varphi \in B_0(\Sigma, \operatorname{int} K)$. By Proposition 2, recall that $M_{d^\star}(\psi) = \partial_{GP} I(\psi)$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. Define D as

$$D = \overline{\text{co}} \left(\bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi) \right).$$

Define also $d : \Delta \to [0, \infty]$ by $d(p) = d^\star(p)$ if $p \in D$ and $d(p) = \infty$ if $p \notin D$. In other words, $d(p) = \sup\{d^\star(p), \delta_D(p)\}$ for all $p \in \Delta$. It is immediate to check that d is grounded, lower semicontinuous, convex, and such that $d \geq d^\star$. On the other hand, by definition, we have that $M_d(\varphi) \supseteq \partial_{GP} I(\varphi) \neq \emptyset$. It follows that $I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + d(p)\}$. Since φ was chosen to be generic, this shows that d satisfies (11) on $B_0(\Sigma, \operatorname{int} K)$. By a simple continuity argument, we have that d satisfies (11) on $B_0(\Sigma, K)$, thus $d \in \mathcal{C}$. By construction of $d^\star$, we can conclude that $d \leq d^\star$, that is, $d = d^\star$. In turn, this yields $\operatorname{dom} d^\star \subseteq D$. Since D is closed, we have that $\operatorname{cl}(\operatorname{dom} d^\star) \subseteq D$. In order to derive the opposite inclusion, observe that $M_{d^\star}(\psi) \subseteq \operatorname{dom} d^\star$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. By Proposition 2, we can conclude that

$$\bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi) = \bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} M_{d^\star}(\psi) \subseteq \operatorname{dom} d^\star.$$

Since $d^\star$ is convex, $\operatorname{dom} d^\star$ is convex. This implies that

$$\text{co} \left(\bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi) \right) \subseteq \operatorname{dom} d^\star.$$

By taking the closure, we obtain that $D \subseteq \operatorname{cl}(\operatorname{dom} d^\star)$, proving the statement.

6. Consider $\psi \in B_0(\Sigma, \operatorname{int} K)$, $p \in \partial_{GP} I(\psi)$, and $c \in \mathcal{C}$. By point 2, we have that $c^\star, d^\star \in \mathcal{C}$ and $c^\star \leq c \leq d^\star$. By Proposition 2, we also have that $M_{c^\star}(\psi) = \partial_{GP} I(\psi) = M_{d^\star}(\psi)$. It follows that

$$\langle \psi, p \rangle + c^\star(p) = I(\psi) = \langle \psi, p \rangle + d^\star(p).$$

Since $c^\star \leq c \leq d^\star$, this implies that

$$c^\star(p) = c(p) = d^\star(p).$$

Since ψ, p, and c were arbitrarily chosen, the statement follows. ■

Before proving the results of Section 3, we need some extra notation and an extra ancillary fact. Given a functional $I : B_0(\Sigma, K) \to \mathbb{R}$, we define $\succcurlyeq^\circ$ to be the binary relation on $B_0(\Sigma, K)$ such that

$$\varphi \succcurlyeq^\circ \psi \iff I(\lambda \varphi + (1-\lambda)\phi) \geq I(\lambda \psi + (1-\lambda)\phi) \ \ \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma, K). \tag{14}$$

By [5], it follows $\succcurlyeq^\circ$ is an affine (conic) preorder (see also [10, Appendix A]).

Proposition 3 *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is a normalized and concave niveloid, then*

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad \int \varphi dp \geq \int \psi dp \qquad \forall p \in \mathrm{cl}\,(\mathrm{dom}\, d^\star)\,.$$

Proof. Define D as in the previous proof. We proceed by steps.

Step 1: Let $\varphi_1, \varphi_2 \in B_0(\Sigma, \mathrm{int}\, K)$. (i) implies (ii), (ii) implies (iii), and (iii) implies (iv) where
(i) $\varphi_1 \succcurlyeq^\circ \varphi_2$;
(ii) For each $\phi \in B_0(\Sigma, \mathrm{int}\, K)$ and for each $\lambda \in (0, 1]$

$$I(\lambda\varphi_1 + (1-\lambda)\phi) \geq I(\lambda\varphi_2 + (1-\lambda)\phi)\,;$$

(iii) For each $\phi \in B_0(\Sigma, \mathrm{int}\, K)$ and for each $\lambda \in (0, 1]$ there exists $\delta_{\lambda,\phi} > 0$ such that for each $\delta' \in (0, \delta_{\lambda,\phi})$

$$I(\lambda\varphi_1 + (1-\lambda)\phi + \delta') > I(\lambda\varphi_2 + (1-\lambda)\phi)\,;$$

(iv) For each $\psi \in B_0(\Sigma, \mathrm{int}\, K)$ and for each $p \in \partial_{GP} I(\psi)$

$$\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle\,.$$

Proof of the Step. (i) implies (ii). It follows by the definition of $\succcurlyeq^\circ$.

(ii) implies (iii). Consider $\phi \in B_0(\Sigma, \mathrm{int}\, K)$ and $\lambda \in (0, 1]$. Since $\varphi_1, \varphi_2, \phi \in B_0(\Sigma, \mathrm{int}\, K)$, it follows that $\lambda\varphi_1 + (1-\lambda)\phi, \lambda\varphi_2 + (1-\lambda)\phi \in B_0(\Sigma, \mathrm{int}\, K)$. This implies that there exists $\delta_{\lambda,\phi} > 0$ such that $\lambda\varphi_1 + (1-\lambda)\phi + \delta' \in B_0(\Sigma, \mathrm{int}\, K)$ for all $\delta' \in (0, \delta_{\lambda,\phi})$. Since (ii) holds and I is a normalized niveloid, we have that

$$I(\lambda\varphi_1 + (1-\lambda)\phi + \delta') = I(\lambda\varphi_1 + (1-\lambda)\phi) + \delta' > I(\lambda\varphi_2 + (1-\lambda)\phi)\,,$$

proving the statement.

(iii) implies (iv). Consider now $\psi \in B_0(\Sigma, \mathrm{int}\, K)$ and $p \in \partial_{GP} I(\psi)$. Since $\psi \in B_0(\Sigma, \mathrm{int}\, K)$, there exists $\phi \in B_0(\Sigma, \mathrm{int}\, K)$ and $\lambda \in (0, 1)$ such that $\psi = \lambda\varphi_2 + (1-\lambda)\phi$. By definition of $\partial_{GP} I(\psi)$, we have that for each $\psi' \in B_0(\Sigma, K)$

$$I(\psi') > I(\psi) \quad \Longrightarrow \quad \langle \psi', p\rangle > \langle \psi, p\rangle\,.$$

Since $I(\lambda\varphi_1 + (1-\lambda)\phi + \delta') > I(\lambda\varphi_2 + (1-\lambda)\phi) = I(\psi)$ for all $\delta' \in (0, \delta_{\lambda,\phi})$, it follows that

$$\langle \lambda\varphi_1 + (1-\lambda)\phi + \delta', p\rangle > \langle \psi, p\rangle = \langle \lambda\varphi_2 + (1-\lambda)\phi, p\rangle \qquad \forall \delta' \in (0, \delta_{\lambda,\phi})\,.$$

Since $\lambda \in (0, 1)$, it follows that $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$. □

Step 2: Let $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. $\varphi_1 \succcurlyeq^\circ \varphi_2$ only if $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$ for all $p \in D$.

Proof of the Step. Consider $\beta \in \mathrm{int}\, I$. It follows that $\frac{1}{2}\varphi_1 + \frac{1}{2}\beta, \frac{1}{2}\varphi_2 + \frac{1}{2}\beta \in B_0(\Sigma, \mathrm{int}\, K)$ and $\frac{1}{2}\varphi_1 + \frac{1}{2}\beta \succcurlyeq^\circ \frac{1}{2}\varphi_2 + \frac{1}{2}\beta$. By Step 1, we have that

$$\frac{1}{2}\langle \varphi_1, p\rangle + \frac{1}{2}\beta = \left\langle \frac{1}{2}\varphi_1 + \frac{1}{2}\beta, p\right\rangle \geq \left\langle \frac{1}{2}\varphi_2 + \frac{1}{2}\beta, p\right\rangle = \frac{1}{2}\langle \varphi_2, p\rangle + \frac{1}{2}\beta \quad \forall p \in \bigcup_{\psi \in B_0(\Sigma, \mathrm{int}\, K)} \partial_{GP} I(\psi)\,.$$

It follows that $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$ for all $p \in D$, proving the statement. □

Step 3: Let $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. If $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$ for all $p \in D$, then $\varphi_1 \succcurlyeq^\circ \varphi_2$.

Proof of the Step. By point 5 of Theorem 3, we have that $D = \mathrm{cl}\,(\mathrm{dom}\, d^\star)$. First, observe that

$$I(\psi) = \min_{p \in \Delta}\{\langle \psi, p\rangle + d^\star(p)\} = \min_{p \in D}\{\langle \psi, p\rangle + d^\star(p)\} \qquad \forall \psi \in B_0(\Sigma, K)\,. \tag{15}$$

Consider $\phi \in B_0(\Sigma, K)$ and $\lambda \in (0,1]$. If $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$ for all $p \in D$, then $\langle \lambda\varphi_1 + (1-\lambda)\phi, p\rangle \geq \langle \lambda\varphi_2 + (1-\lambda)\phi, p\rangle$ for all $p \in D$. It follows that

$$\langle \lambda\varphi_1 + (1-\lambda)\phi, p\rangle + d^\star(p) \geq \langle \lambda\varphi_2 + (1-\lambda)\phi, p\rangle + d^\star(p) \qquad \forall p \in D.$$

By (15), it follows that $I(\lambda\varphi_1 + (1-\lambda)\phi) \geq I(\lambda\varphi_2 + (1-\lambda)\phi)$. Since ϕ and λ were arbitrarily chosen, it follows that $\varphi_1 \succcurlyeq^\circ \varphi_2$. □

By point 5 of Theorem 3 and from Steps 2 and 3, it follows that $\varphi_1 \succcurlyeq^\circ \varphi_2$ if and only if $\langle \varphi_1, p\rangle \geq \langle \varphi_2, p\rangle$ for all $p \in D = \mathrm{cl}(\mathrm{dom}\, d^\star)$, proving the statement. ■

Proof of Theorem 1 and Corollary 1. By [17, Lemma 28 and Theorem 3] and since $\succsim$ is a variational preference, there exist a nonconstant affine function $u : X \to \mathbb{R}$ and a normalized and concave niveloid $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Without loss of generality, we can assume that $0 \in \mathrm{int}\, u(X)$. It is immediate to verify that $\mathcal{C}(V, u) = \mathcal{C}(I)$, $u(\mathcal{F}_{\mathrm{int}}) = B_0(\Sigma, \mathrm{int}\, u(X))$, $\pi_u(f) = \partial_{GP} I(u(f))$ for all $f \in \mathcal{F}_{\mathrm{int}}$. Thus, points (i), (ii), and the first part of (v) follow from points 2, 4, 1, and 6 of Theorem 3. Point (iii) follows from [17, Proposition 6]. Next, observe that $f \succsim^* g$ if and only if $u(f) \succcurlyeq^\circ u(g)$. By Proposition 3, it follows that

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in \mathrm{cl}(\mathrm{dom}\, d^\star),$$

proving that $C^* = \mathrm{cl}(\mathrm{dom}\, d^\star)$ and point (iv) of Theorem 1. Thus, the second part of point (v) follows from point 5 of Theorem 3. Corollary 1 follows from points 2 and 3 of Theorem 3. ■

Proof of Corollary 3 and Corollary 4. We retain the notation of the proof of Theorem 1. By [12], if $\succsim$ is a Gilboa-Schmeidler preference, then there exists a closed and convex set $C \subseteq \Delta$ such that I can be chosen to be

$$I(\varphi) = \min_{p \in C} \int \varphi dp \qquad \forall \varphi \in B_0(\Sigma, u(X)).$$

This implies that $\delta_C \in \mathcal{C}$. By Theorem 1, we have that $\delta_C \leq d^\star$. At the same time, $\partial_{GP} I(\beta) = C$. By Proposition 2, this implies that $d^\star(p) = 0$ for all $p \in C$. Thus, we can conclude that $d^\star \leq \delta_C$, that is, $d^\star = \delta_C$. Since Anscombe-Aumann expected utility preferences are a particular case of Gilboa-Schmeidler preferences with $C = \{p\}$, Corollary 4 also follows. ■

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