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October, 2014

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Abstract

We build a model of a limit order book and examine the consequences of adding a dark pool. Starting with an illiquid book, we show that book and consolidated fill rates and volume increase, but the spread widens, depth declines and welfare deteriorates. When book liquidity increases, more orders migrate to the dark pool and large traders' welfare improves; but while the spread-increase is dampened, the depth-reduction is amplified and small traders are still worse off. All effects are stronger for a continuous than for a periodic dark pool and when the tick size is large.

1 Introduction

Dark pools are Alternative Trading Systems (ATSs) that do not provide their best-priced orders for inclusion in the consolidated quotation data. They offer subscribers venues where anonymous, undisplayed orders interact away from the lit market yet execute at prices no worse than the National Best Bid Offer (NBBO). Dark pools today represent a considerable fraction of volume (Figure 1). In the U.S. there are over 50 dark pools, and the 19 of them for which data is available (from Rosenblatt Securities Inc.) account for more than 14% of consolidated volume. In Europe the 16 dark markets which report to Rosenblatt account for approximately 4.5% of volume, and in Canada they represent 2% of volume.

[**Insert Figure** 1 **here**]

The rising market share of dark trading recently prompted three major U.S. exchanges to publicly urge the Securities and Exchange Commission (SEC) to put rules in place to curb dark pool trading. Exchange officials are concerned that dark pools divert volume away from lit venues, rather than attracting new order flow to the market. With declining trading volumes world-wide, such a diversion of order flow is a real threat to exchanges' bottom line. Consequently, it is important for exchanges to understand which factors cause order flow to go dark, and under what circumstances dark pools are likely to primarily divert volume away from lit venues as opposed to create more opportunities for trades to take place.

Regulators are concerned about the effects of dark trading on market quality and welfare and the effects of dark trading on the informational efficiency of prices. Dark pools may influence total welfare as a reduction in pre-trade transparency impacts the quality of lit markets and hence trading costs. Dark pools may also affect the distribution of welfare between retail and institutional investors, as dark markets are primarily used by institutional traders. Moreover, regulators are concerned about the effects of the introduction of dark markets on the informational efficiency of the pricing process. If a significant fraction of trading migrates to dark pools, the ability of traders to discover the fundamental value of the asset by looking at quotes and transaction prices on the lit market might be adversely affected.

In this paper we build a theoretical model that enables us to address the concerns raised by exchanges and regulators in a realistic market setting. Specifically, we populate our model with large and small fully rational traders who form their optimal trading strategies based on their private valuations. All traders in our model can choose to submit market or limit orders to a transparent limit order book (LOB) with a discrete price grid. In addition, large traders may submit orders to a dark pool and can use a mixture of market and limit orders, and of dark and transparent orders. If sufficient two-sided trading interest is routed to the dark pool, orders are executed at a price derived from the NBBO. The dark pool can either execute orders periodically or it can execute orders continuously, meaning that traders can simultaneously access the lit and the dark market. In the latter setting, we introduce an additional order type, Immediate-or-Cancel (IOC) orders, that is broadly used in today's markets. We use this rich setup to address the concerns raised by exchange officials and regulators, market participants and media about order migration, market quality, and welfare.

We first investigate to what extent orders migrate away from the lit market following the introduction of a dark pool. We also discuss whether this migration is associated with an overall increase in trading volume. Second, we study what factors are important for determining the extent to which the dark platform attracts order flow away from the lit market. This topic is the focus of existing empirical research on dark pools, and our model can help researchers better design future empirical studies. Finally, we tackle the concerns expressed by regulators by studying how the introduction of a dark pool affects the quality of the lit market as well as the distribution of welfare between small and large traders.

Our theoretical model contributes to the literature in several ways. Previous models of dark trading focus on the comparison between a lit dealer market (DM) where public traders are restricted to using market orders and a periodic dark crossing network (e.g., Degryse, Van Achter, and Wuyts, DVW 2009; and Zhu, 2014). By contrast, we model a LOB where all traders rationally decide whether to supply or demand liquidity and whether they will route their order to the lit or the dark market (or both). In our model, the state of the book observed by traders when they come to the market, as well as the price and priority rules that govern trading, influence traders' strategic choice between trading venues and between order types, and hence affect market quality and welfare. This feature is consistent with empirical evidence that shows how order submission strategies depend on the state of the order book and the price/time priority rules (Griffiths, Smith, Turnbull, and White, 2000; Handa, Schwartz, and Tiwari, 2003; and Ranaldo, 2004) and that dark trading is affected by spread, depth and tick-to-price ratio (Buti, Rindi, and Werner, 2011; and

Ready, 2013).[1]

Models of LOB markets in stationary equilibrium, either make simplifying assumptions that severely restrict traders' ability to choose order type freely (e.g., Foucault, 1999; and Foucault, Kadan, and Kandel, 2005) or assume that prices are continuous (Rosu, 2009 and 2014) which means that the cost of stepping ahead of a competing order is miniscule (in other words, there is no price/time priority).[2] By contrast, in our model a discrete price grid and strict price/time priority are crucial for the development of traders' optimal strategies. To assign a role to priority rules, the model must embed a positive minimum price increment, i.e., a tick size. The tick size guarantees that the pricing rule is discretionary thus allowing trades to take place at discrete prices. More importantly, it forces traders to price improve by a significant economic amount, which guarantees that price and time priority are enforced. It is precisely these market structure features that create a LOB, and give rise to the fundamental trade-off between price opportunity costs and non-execution costs.

Our model runs for four periods and starts with an empty LOB and an empty dark pool. Starting with an empty LOB is challenging because it takes at least two periods for a non-marketable limit order to have a chance to execute as it requires the arrival of a trader who wants to trade in the opposite direction and chooses to submit a marketable order. However, it turns out that a four-period model is sufficiently long for liquidity to endogenously build up both in the LOB and in the dark pool to generate rich cross-sectional as well as dynamic implications from our model. We generate cross-sectional implications by starting from each possible level of liquidity that result from the first period's order submission decisions and by analyzing the remaining three periods of the model. By comparing the results starting with an illiquid book and four periods of trading to the results starting with an illiquid book and three periods of trading remaining, we are able to illustrate how the model's predictions are affected by the time horizon. All our results are amplified when we compare the 4-period with the 3-period model.

Finally, we extend the previous literature that focuses on crossing networks by modeling dark pools that execute continuously, and where traders can simultaneously access lit and dark venues

[1] LOBs around the world are governed by a discretionary pricing rule, and by price and time priority rules.

[2] Foucault (1999) assumes that the book is always either empty or full, and traders cannot compete to provide liquidity. Foucault, Kadan and Kandel (2005) adopt a set of simplifying assumptions that force limit orders to price improve by narrowing the spread by at least one tick, that dictate that buyers and sellers alternate with certainty, and prohibit traders from accessing two markets simultaneously. Rosu's (2009 and 2014) frameworks are not suitable to discuss competition between dark and lit markets as he assumes that prices are continuous.

using a sophisticated order type, IOC orders, which are first sent to the dark and, if not immediately executed, automatically routed to the LOB as market orders. Several dark pools offer this type of functionality, for example Sigma X in the U.S. and Match Now in Canada, and our goal is to model a realistic dark venue that captures the most important features of how real life dark pools interact with a transparent limit order market. The most active types of dark pools in the U.S., Europe and Canada are Independent/Agency and Bank/Broker pools (Figure 1). The Independent/Agency pools, like ITG POSIT, are run by independent agency brokers and offer periodic executions at the midpoint of the primary market inside spread, which in the U.S. generally coincides with the NBBO. The Bank/Broker pools are instead operated by banks and are used both for agency and proprietary trading. These pools generally offer continuous rather than periodic execution and they execute at prices no worse than the NBBO.[3] We review the extensive theoretical literature related to our model in Section 2.

Our theoretical model builds on Parlour (1998), but in the spirit of Buti and Rindi (2013) we extend her model to include a price grid, a dark pool and additional order types. We also differentiate between small and large traders and only allow the latter to access the dark pool. We need a LOB with a price grid to distinguish among books which differ in spread and depth. We also need additional order types because when large traders use both the lit and the dark venue at the same time, they rely on orders that are more sophisticated than simple market and limit orders. Therefore, we introduce IOC orders and we also allow traders to split their orders between the lit and the dark market.

We start by modeling a benchmark LOB where large and small traders decide whether to submit a market order, a limit order, a combination of the two or to refrain from trading based on the information they infer about future execution probabilities from the current state of the LOB. We then introduce a dark pool which executes periodically at the prevailing LOB midpoint and which gathers orders from large traders. Traders compare the potential price improvement (midquote price) in the dark pool to the trading opportunities on the LOB. This protocol allows us to identify factors which determine dark pool market share and also allows us to show the effects of the introduction of a typical Independent/Agency pool on market quality and traders' welfare.

[3] Within the Bank/Broker category of dark pools, the Market Maker pools are characterized by the fact that liquidity can only be provided by the manager of the pool, whereas the Consortium-Sponsored pools are owned by several banks which already own their dark pool and use the Consortium-Sponsored pools as trading venues of last resort. Finally, Exchange-Based dark pools are owned by exchanges and offer continuous execution.

We then model the same LOB but this time competing with a dark pool that offers continuous execution like the Bank/Broker and Exchange-Based pools discussed above. This protocol allows market participants not only to demand liquidity by sending orders to the dark venue, but also to supply and demand liquidity simultaneously on both trading platforms. This very rich set of strategies enables us to provide policy prescriptions for the group of Bank/Broker dark pools that executes 57%, 67% and 87% of dark volumes in the U.S., Europe and Canada respectively, and by extension also for the Exchange-Based dark pools for which official data is not available.[4]

By comparing results from the benchmark LOB model without a dark pool to the results from the model with a LOB competing with a dark pool, we are able to address the concerns raised by exchange officials, regulators, market participant, and media discussed above. We show that the introduction of a dark pool to a LOB market results in higher consolidated fill rates and volume, but also higher LOB fill rates and volume. A dark pool always attracts orders away from the LOB, but the consequences for LOB fill rates, volume and market quality depend whether it is predominately limit or market orders that leave the book. When limit orders leave the LOB, the provision of liquidity decreases and this leads to a reduction in market depth and to a widening of the spread. By contrast, a departure of market orders has a positive effect on both depth and spread as market orders subtract liquidity from the book. When a dark pool is introduced, it is always a mixture of market and limit orders that migrate away from the lit market. When the LOB is illiquid as it is the case initially in our four-period model, traders primarily use limit orders absent a dark pool. When a dark pool is available, it is mainly limit orders that migrate to the dark venue. In addition, remaining traders are more likely to switch from limit to market orders as the execution probability of limit orders decline in the presence of a dark pool. As a result, LOB fill rates and share volume increase. Hence, our model suggests that exchanges are actually better off in the presence of dark pools because the higher volume allows them to harvest additional trading fees.

As it is primarily limit orders that migrate to the dark venue in an illiquid market, the increase in trading volume is associated with a wider LOB spread and lower LOB depth. This suggests that

[4] What is important is that the price in the dark pool is derived from the NBBO, not whether or not trades execute at the midquote. However, for tractability, our dark pools always execute at the midquote of the NBBO. Our framework does not include competition among different dark venues and it is therefore inadequate to model the Consortium-Sponsored dark pools that are sometimes used by banks to look for the execution of orders that do not find any matching interest in their main dark pool. This group of dark pools, however, executes only a minimal part of the dark volumes (Figure 1).

the concerns raised by regulators that dark trading may undermine the liquidity of the lit market book are warranted. However, note the reason for lit market depth to decline and spreads to widen is that more marketable orders are submitted to the LOB, resulting in higher fill rates and volume. Our model therefore illustrates that there is a trade-off between displayed liquidity and trading volume. Ultimately, the question then becomes whether traders are better or worse off. The fill rate of their orders increases, but this occurs at worse trading conditions so that welfare both for small and large traders deteriorates.

The four-period model also shows that the magnitude of the effects of an introduction of a dark pool on market quality and welfare depends on the dark pool trading protocol (periodic or continuous) and the tick size. We find that dark pool activity is higher for a continuous than for a periodic dark pool, and the effects of dark pool activity on LOB and consolidated fill rates and volume are stronger for the continuous dark pool. With a continuous dark pool more orders and volume migrate to the dark both because executions take place at each trading round and because traders use the new order that allows them to access simultaneously the LOB and the dark pool. An order submitted to the dark pool has more chances of getting executed under the continuous protocol, which encourages dark order submissions. This may explain the popularity of the mostly continuous Bank/Broker pools relative to the mostly periodic Independent/Agency pools in Figure 1. We also find that dark pool activity is increasing in the tick size, and the effects of dark pool activity on LOB and consolidated fill rates and volume, market quality, and welfare are stronger when the tick size is large. Intuitively, a larger tick size provides traders with additional incentives to use limit orders so that when the dark market is introduced more limit orders go dark. Moreover, with a larger tick size market orders become more expensive and the value of being able to cross at the midquote, or more generally inside the NBBO, increases.

Since we start with an empty LOB and an empty dark pool, we cannot derive cross-sectional predictions based on the four-period model. However, the discrete time nature of the model allows us to analyze the equilibrium trader order submission strategies from period two onwards. In other words, we can study each node of the decision tree separately, starting from each possible equilibrium outcome in period one.[5]

[5] Note that as this truncated model has fewer future periods, the value of having access to a dark pool is lower and the effects on traders' optimal LOB strategies are therefore smaller.

We use this feature of our model to compare the effect of introducing a dark pool for stocks with different levels of initial liquidity. We find that dark pool fill rates and volumes are higher for stocks that are more liquid. For stocks with greater depth at the inside and/or narrower spread, there is more competition for the provision of liquidity. This implies that a limit order submitted to the LOB has to be more aggressive to gain priority over the orders already on the book. As a result, the possibility of obtaining a midquote execution in the dark pool becomes relatively more attractive. Moreover, as liquidity in the lit market increases, more orders migrate to the dark venue and the execution probability of dark orders increases thus making these orders more profitable. Consequently, our model predicts that order migration and dark pool market share increase in liquidity. This prediction is confirmed in recent empirical work on dark pool data by Buti, Rindi, and Werner (2011) and Ready (2013).

As dark pool activity increases, both LOB and consolidated fill rates and volume decline for liquid stocks but interestingly they actually increase for illiquid stocks. When market orders move to the dark venue in our setting, fewer trades take place; whereas when limit orders move to the dark, more trades take place. The reason is that market orders have certainty of execution in the LOB, but when they move to the dark to get a better price their execution probability declines and fewer trades occur. By contrast, limit orders only move to the dark venue if they expect a higher execution probability since they actually get a worse price in the dark venue than in the lit market. As a result more trades take place when limit orders go dark. Because traders tend to make a greater use of market orders at the expense of limit orders in deeper books, the fill rates and volume are lower when the book is liquid. Similarly, because traders are more likely to use limit orders in shallower books, trades and share volume are higher when the book is illiquid. Hence, while the introduction of a dark pool hurts lit market fill rates and volume for liquid books, it actually helps boost lit market fill rates and volume for illiquid books.

This truncated version of the model also shows that the magnitude of the effects of an introduction of a dark pool on market quality and welfare depends on the initial liquidity of the book. As mentioned above, the average effect of the introduction of a dark pool is wider spreads and lower depth. However, we find that the spread widens less and depth declines more for liquid stocks compared to illiquid stocks. For liquid books, traders generally tend to use more market orders, and fewer traders therefore switch from limit to market orders when a dark pool is introduced. As a result, the drain of lit market liquidity is smaller and spreads widen less. However, the limit order queue is long, so limit orders migrate to the dark venue, resulting in a decline in LOB depth.

When assessing the welfare implications of introducing a dark pool in the context of our model, it is important to recognize that there is a trade-off between the increased fill rates and share volume which enhance welfare and the deteriorating market quality which is detrimental for welfare. Second, large and small traders do not face the same opportunities, and we therefore should consider both the overall welfare and the welfare of each trader group. The four-period model which starts with an illiquid book shows that the benefits from the increase in fill rates and share volume are not sufficient to out-weigh the deterioration of market quality, and overall welfare declines. With deteriorating LOB market quality, it is not surprising that small traders who have access only to the lit market suffer a welfare loss. However, in an illiquid market large traders' use of the dark pool is limited. Therefore gains from having access to the dark venue do not out-weigh the cost of worse LOB market quality and large traders' welfare also declines. In the context of the truncated model, we do find that when initial liquidity is high, large traders use the dark pool more extensively and experience welfare gains large enough to out-weigh the losses faced by the small traders who are restricted to trade in the lit market. Further, the results show that these effects are all amplified when the dark pool has continuous as opposed to periodic executions and when the tick size is large.

Our model is an attempt to capture a realistic market where a lit LOB competes with a dark pool which uses a derivative pricing rule and trades either continuously or periodically. Traders in our model have access to a wide range of order types, and they form optimal trading strategies based on the inference they make about future trading opportunities both in the lit and the dark markets based on the market conditions as reflected in the LOB. In this setup, it is very challenging to also include asymmetric information, and therefore we are unable to discuss the effects of dark trading on the informational efficiency of prices and on order toxicity, the extent to which orders move prices due to their information content. For a discussion about the role of information asymmetries in the context of dark pools, we refer to Ye (2011) and Zhu (2014). We also note that any setting that generates order migration from a lit market to a dark pool will potentially change the composition of traders in the lit venue, and hence may affect the price discovery process. However, most ATSs control access and actively screen out toxic order flow. Even absent the screening, Zhu (2014) predicts that primarily uninformed traders migrate to the dark pool and that price discovery in the lit market improves as a result.

Note that our predictions are very different from what would be obtained if the lit market were modeled as a DM as for instance in DVW (2009) and Zhu (2014). In their models, traders who are patient and unwilling to pay the spread cannot submit limit orders and hence either stay out of the market or move to the dark pool to execute at the midquote. By contrast, patient traders in our model do not need to move to the dark as they can post their limit orders on the LOB. As a result, we find less order migration to the dark venue than what is predicted by DVW and Zhu. Our model also generates very different predictions about the factors that drive orders to go dark. DVW and Zhu find that the smaller the spread, the fewer orders go dark because the price improvement offered by the dark pool is small. When instead the spread is large, traders are more likely to route their orders to the dark venue since it offers a larger price improvement compared to dealer quotes. Our model predicts the opposite, i.e., that dark pools are more actively used for liquid stocks. DVW and Zhu also conclude that dark trading is beneficial for stocks with larger spread, i.e., illiquid stocks. We find instead that it is precisely when trading illiquid stocks that small traders loose the most.

The paper is organized as follows. In Section 2 we review the related literature. In Section 3 we present both the benchmark framework and the framework with a dark pool, be it periodic or continuous. In Section 4 we report the results on factors that affect order flows and dark pool market share, and in Section 5 on the effects on market quality and welfare. Section 6 is dedicated to the model's empirical implications and Section 7 to the conclusions and policy implications. All proofs are in the Appendix.

2 Literature Review

The literature on multimarket competition is extensive.[6] As we model competition between a LOB and a dark pool, our paper is related in particular to the branch of the literature which deals with competition between trading venues with different pre-trade transparency and focuses on the interaction between crossing networks (CN) and DM. The paper which is closest to ours is DVW (2009), who investigate the interaction of a CN and a DM and show that the composition and

[6] Works on competition among trading venues include: Barclay, Hendershott, and McCormick (2003), Baruch, Karolyi, and Lemmon (2007), Bessembinder and Kaufman (1997), Easley, Kiefer, and O'Hara (1996), Karolyi (2006), Lee (1993), Pagano (1989), Reiss and Werner (2004) and Subrahmanyam (1997).

dynamics of the order flow on both systems depend on the level of transparency.[7] However, as we discuss in depth later in the paper, our contribution differs substantially from DVW (2009). First of all, we consider the interaction between a LOB -rather than a DM- and a dark venue, so that in our model traders can both demand liquidity (via market orders) and compete for the provision of liquidity (via limit orders). Second, in addition to a dark CN, we consider a dark pool with a continuous execution system where traders have simultaneous access both to the LOB and to the dark pool.

Another related paper is Zhu (2014) who uses the Glosten and Milgrom (1985) model to show that when the dark market is introduced to a DM, price discovery on the lit venue improves. The reason is that informed traders choose to send their market orders to the DM and not to the CN because they would all submit orders on the same side in the dark venue and no executions would take place. By contrast, in our LOB model traders can act as liquidity suppliers and earn the spread. We conjecture that if we were to extend our model to include asymmetric information, there would be no reason for informed traders to concentrate in the LOB and avoid the dark pool. The reason is that unlike a DM, the LOB does not offer an infinite supply of liquidity. We also conjecture that dark pool trading would not necessarily cause a wider spread even if asymmetric information were introduced since informed traders can use limit orders in our model. This is especially likely to be the case in books with limited supply of liquidity at the inside LOB spread. Also note that Ye (2011) finds opposite results on price discovery by modeling competition between a Kyle (1985) auction market and a dark pool. Ye assumes that only informed traders but not noise traders can strategically opt to trade in the dark pool, and finds that dark pools harm price discovery. In our model it would be unlikely for informed traders to concentrate in the dark pool because rational uninformed traders would exit the pool. Hence, informed traders in our setting would use both the dark pool and the LOB.

Our model is also closely related to Foucault and Menkveld (2008) who focus on the competition between two transparent LOBs. They show that when brokers can apply Smart Order Routing Technology (SORT), the execution probability of limit orders (i.e., the liquidity provision) in the incumbent LOB increases. In our model traders can use IOC instructions to route orders and we

[7] Hendershott and Mendelson (2000) model the interaction between a CN and a DM and show costs and benefits of order flow fragmentation. Donges and Heinemann (2004) model intermarket competition as a coordination game among traders and investigate when a DM and a CN can coexist; Foster, Gervais and Ramaswamy (2007) show that a volume-conditional order-crossing mechanism next to a DM Pareto improves the welfare of additional traders.

suggest that this routing technology enhances the competition from the new trading venue.

Finally, dark pools are currently competing with other dark options offered by exchanges to market participants and this provides a link to the recent literature on hidden orders. In Buti and Rindi (2013) and Moinas (2010) traders active in a LOB can choose between disclosed and undisclosed orders, whereas in our model they can choose between lit and dark trading venues.[8]

Because dark pools are characterized by limited or no pre-trade transparency, our model is also related to the vast literature on anonymity and transparency.[9] In particular the recent paper by Boulatov and George (2013) shows that in a Kyle (1989) setting the quality of the market in a dark regime is better than in a transparent one. The reason is that more informed agents are drawn into providing liquidity, and they trade more aggressively as liquidity providers than as liquidity demanders. Our model differs from Boulatov and George's because it focuses on the trader's endogenous choice between a dark and a visible market, rather than on trading in a setting that can be either dark or transparent.

3 The Model

Existing dark pools can be classified into two broad categories, Periodic Dark Pools ($PDPs$) and Continuous Dark Pools ($CDPs$). $PDPs$ cross orders periodically, for example every hour on the hour. Clients can submit orders directly to the dark pool. Moreover, both liquidity demanders and liquidity suppliers can split their orders between the dark and the lit venue. However, traders have to wait until the next cross to see if their orders are executed. $CDPs$ cross orders continuously and allow investors to use more sophisticated trading strategies; for example, liquidity demanders may submit orders to the CDP that can immediately bounce back to the lit market in case of non-fill or partial execution.

In this Section we present a model of a LOB with both small and large traders and we use it as a benchmark protocol. We then add a dark pool and investigate the competition between the LOB and either a PDP or a CDP.

[8] See also the experimental papers by Bloomfield, O'Hara, and Saar (2014), and Gozluklu (2009).

[9] See for example the theoretical works by Admati and Pfleiderer (1991), Baruch (2005), Fishman and Longstaff (1992), Forster and George (1992), Madhavan (1995), Pagano and Röell (1996), Rindi (2008), and Röell (1991).

3.1 Benchmark Model (B)

We consider a four-period ($t = t_1, t_2, t_3, t_4$) trading protocol that features a LOB for a security which pays v at the end of the trading game. The LOB is characterized by a set of four prices and associated quantities, denoted by $\{p_i^z \& q_i^z\}$, where $z = \{A, B\}$ indicates the ask or bid side of the market, and $i = \{1, 2\}$ the level on the price grid. Therefore, prices are defined relative to the common value of the asset, v:

$$p_2^A = v + \frac{3}{2}\tau \quad (1)$$

$$p_1^A = v + \frac{1}{2}\tau \quad (2)$$

$$p_1^B = v - \frac{1}{2}\tau \quad (3)$$

$$p_2^B = v - \frac{3}{2}\tau \ , \quad (4)$$

where we assume that the minimum price increment that traders are allowed to quote over the existing prices is equal to a constant τ. Hence τ is the minimum spread that can prevail on the LOB. The associated quantities denote the number of shares that are available at each price level. Following Parlour (1998) and Seppi (1997), we assume that a trading crowd absorbs whatever amount of the asset is demanded or offered at the highest ask and lowest bid on the price grid, which in our model are p_2^A and p_2^B. Therefore the book depth is unlimited at the second level, whereas the number of shares available at p_1^A (p_1^B) forms the state of the book at each time t and is defined as $b_t = [q_1^A q_1^B]$.

In our model all agents are fully rational and trade because they wish to trade. In each period t a new risk neutral trader joins the market. With equal probability the trader arrives with a 2 share (large) or a 1 share (small) order. Traders arriving with a large order can trade up to 2 shares, $j = \{0, 1, 2\}$, and we call them large traders (LT), whereas traders arriving with a small order can only trade 1 share or refrain from trading and we call them small traders (ST). Upon arrival, the trader selects an order type and his optimal trading strategy cannot be modified thereafter. The trader's personal valuation of the asset is represented by a multiplicative parameter, β, drawn from a uniform distribution with support $[0, 2]$: traders with a high value of β are impatient to buy the asset, while traders with a low β are impatient to sell it; traders with a β next to 1 are patient as their valuation of the asset is close to the common value.[10]

Traders observe the state of the LOB but not the identity of market participants. To select the optimal order type the incoming trader compares the expected profits from each of the different orders strategies, $\varphi(.)$, presented in Table 1. Both large and small traders can submit market orders to the first two levels of the price grid, $\varphi_M(j, p_i^z)$; they can post limit orders to the first level, $\varphi_L(j, p_1^z)$, and they can choose not to trade, $\varphi(0)$. The large trader can also split his order and combine the two different strategies, $\varphi_{ML}(1, p_i^z; 1, p_1^z)$. Marketable orders are large market orders that walk up or down the book in search of execution, and we label them $\varphi_M(2, p^z)$.[11] The profitability of the orders depends on the state of the book, b_t, and on the personal valuation of the trader, β.

[**Insert Table 1 and Figure 2 here**]

Figure 2 shows the extensive form of the trading game, in which the market opens at t_1 with an empty LOB, $b_{t_1} = [00]$. To keep the extensive form as simple as possible, we present only the equilibrium strategies. We refer to this extensive form to discuss the strategies available to both large and small traders conditional on the state of the LOB, as well as their payoffs and the effects of different orders on the state of the LOB. Assume first that an impatient large trader arrives, observes $b_{t_1} = [00]$ and opts for a 2-share market buy order that hits the trading crowd standing on the second level of the ask side, $\varphi_M(2, p_2^A)$. This order pays the spread and executes with certainty with the following payoff:

$$\pi_{t_1}[\varphi_M(2, p_2^A)] = (\beta_{t_1} v - p_2^A)\ 2\ . \tag{5}$$

After this order is executed, at t_2 the book still opens with no shares on the first level of the book, $b_{t_2} = [00]$. The same book opens at t_2 if a large traders submits at t_1 a market sell order, $\varphi_M(2, p_2^B)$, or if a small trader arrives and opts for a market buy or a market sell order, $\varphi_M(1, p_2^A)$ or $\varphi_M(1, p_2^B)$.

Considering again t_1, assume now that a moderately impatient large trader combines a market buy and a limit buy order, taking advantage of the better execution probability of a 1-unit limit

[10] Differently from Parlour (1998), we do not assume that traders arrive at the market with an exogenous probability of being a buyer or a seller, but let the individual β determine their trade direction. This way, agents are not forced to refrain from trading when they have a high or a low evaluation of the asset and nature selects them as sellers or buyers respectively.

[11] We omit the subscript i for the level of the book since the order will be executed at different prices.

order.[12] Part of his profits depends on the probability of the order being executed in the following trading rounds:

$$\begin{aligned}\pi^e_{t_1}[\varphi_{ML}(1, p^A_2; 1, p^B_1)] \;&=\; (\beta_{t_1}v - p^A_2) + (\beta_{t_1}v - p^B_1)\{\Pr_{w_{t_2}=1}(p^B_1|b_{t_2}) + \Pr_{w_{t_2}=0}(p^B_1|b_{t_2}) \\ &\quad [\Pr_{w_{t_3}=1,2}(p^B_1|b_{t_3}) + \Pr_{w_{t_3}=0}(p^B_1|b_{t_3}) \Pr_{w_{t_4}=1,2}(p^B_1|b_{t_4})]\}\ ,\end{aligned} \tag{6}$$

where $\Pr_{w_t}(p^B_1|b_t)$ is the probability that w_t shares get executed at t. It follows that the book opens at t_2 as $b_{t_2} = [01]$. The same book is obtained if at t_1 a patient small trader opts for a limit buy order, $\varphi_L(1, p^B_1)$. Analogously, the book will open at t_2 as $b_{t_2} = [10]$ if a large traders opts for a combination of market and limit orders to sell, or if a small trader chooses a limit sell order.

Finally, if a more patient large trader arrives at t_1 and chooses a limit order to buy, $\varphi_L(2, p^B_1)$, or to sell, $\varphi_L(2, p^A_1)$, at p^B_1 and p^A_1 respectively, the book will open at t_2 with 2 shares at the best bid or ask price, $b_{t_2} = [02]$ or $b_{t_2} = [20]$. As an example, we consider this last strategy of a large limit order to sell, $\varphi_L(2, p^A_1)$, and move to the next periods. The expected profits of the large trader opting for a limit sell order depend on the probability of the order being executed in the following trading rounds, t_2, t_3 and t_4:

$$\begin{aligned}\pi^e_{t_1}[\varphi_L(2, p^A_1)] \;&=\; (p^A_1 - \beta_{t_1}v)\{\sum_{w_{t_2}=1,2} w_{t_2} \Pr_{w_{t_2}}(p^A_1|b_{t_2}) + \sum_{w_{t_2}=0,1} \Pr_{w_{t_2}}(p^A_1|b_{t_2}) \\ &\quad [\sum_{w_{t_3}=1,2}^{2-w_{t_2}} w_{t_3} \Pr_{w_{t_3}}(p^A_1|b_{t_3}) + \sum_{w_{t_3}=0,1} \Pr_{w_{t_3}}(p^A_1|b_{t_3}) \sum_{w_{t_4}=1,2}^{2-w_{t_2}-w_{t_3}} w_{t_4} \Pr_{w_{t_4}}(p^A_1|b_{t_4})]\}\ .\end{aligned} \tag{7}$$

When the book opens with 2 shares at the best ask price, $b_{t_2} = [20]$, large and small traders arriving at t_2 can choose among the strategies shown in Figure 2.

Still as an example, we consider a small trader who arrives at t_2, observes the book $b_{t_2} = [20]$, and submits a limit buy order of 1 share so that the book will open at t_3 as $b_{t_3} = [21]$. Assume next that a large impatient seller arrives at t_3, and decides to submit a market order. Because there is only 1 share standing at the best bid price, he has the choice of either submitting a 1-unit market sell order, $\varphi_M(1, p^B_1)$, or a marketable sell order, $\varphi_M(2, p^B)$, that walks down the LOB hitting p^B_1 and p^B_2 to complete execution, with the following payoff:

$$\pi_{t_3}[\varphi_M(2, p^B)] = (p^B_1 + p^B_2) - 2\beta_{t_3}v\ . \tag{8}$$

[12] Because small traders trade only 1 unit, the execution probability of the first unit of a limit order is higher.

For both strategies, the book opens at t_4 as $b_{t_4} = [20]$. In this last period, traders do not submit limit orders because the market closes and their execution probability is zero. Therefore both large and small traders submit either market buy orders to p_1^A, or market sell orders to p_2^B, or refrain from trade and get no profits, $\pi_{t_4}[\varphi(0)] = 0$.

To summarize, at each trading round, the arriving risk-neutral trader selects the optimal order submission strategy which maximizes his expected profits, conditional on the state of the LOB, b_t, and on his type captured by his personal valuation of the asset, β. The large trader chooses:

$$\max_{\varphi} \pi_t^e[\varphi_M(j, p_i^z), \varphi_M(2, p^z), \varphi_{ML}(1, p_i^z; 1, p_1^z), \varphi_L(j, p_1^z), \varphi(0) \; |\beta_t, b_t] \ , \tag{9}$$

and the small trader:

$$\max_{\varphi} \pi_t^e[\varphi_M(1, p_i^z), \varphi_L(1, p_1^z), \varphi(0)|\beta_t, b_t] \ . \tag{10}$$

Note that in this model the standard trade-off between non-execution costs and price opportunity costs applies. Impatient traders generally minimize non-execution costs by choosing market orders, whereas patient traders minimize the costs of trading at an unfavorable price by choosing limit orders.

We find the solution of this game by backward induction, and from now on for simplicity we assume without loss of generality that $\tau = 0.08$ and $v = 1$. We start from the end-nodes at time t_4 and for all the possible states of the book we compare trading profits from both the large and the small traders' optimal strategies. This allows us to determine the probability of the equilibrium trading strategies at t_4, which can be market orders on the buy or sell side, as well as no trading. We can hence compute the execution probabilities of limit orders placed at t_3, which in turn allows us to derive the equilibrium order submission strategies for period t_3. Given the probability of market orders submitted at t_3, we can finally compute the equilibrium order submission strategies at t_2. The same procedure is then re-iterated to obtain the equilibrium order submission strategies at t_1.

In this model, traders are indifferent between orders with zero execution probability and therefore a unique equilibrium always exists due to the recursive structure of the game:

Definition 1 *An equilibrium of the trading game is a set of $n \in N_t$ order submission decisions, $\{\varphi_a^n\}$, where $a = \{LT, ST\}$, such that at each period the large and the small trader maximize* *the expected payoff π_t^e according to their Bayesian updated beliefs over the execution probabilities,* $\Pr_{w_t}(p_1^z|b_t)$.

3.2 Limit Order Book and Dark Pools

We now extend the model to include a dark pool that operates alongside our benchmark LOB. As mentioned, we consider two different types of dark pools: the PDP and the CDP. The PDP has periodic execution and resembles the Independent/Agency dark pools. The CDP captures the most relevant microstructure features of the Bank/Broker and Exchange-Based dark pools.

In the dark pool modeled herein traders are unable to observe the orders previously submitted by the other market participants. It follows that they can only infer the state of the dark pool by monitoring the LOB and by Bayesian updating their expectations. We assume that at t_1 the dark pool opens empty, $PDP_{t_1} = CDP_{t_1} = 0$.

3.2.1 Periodic Dark Pool Framework - *L&P*

A PDP is organized like an opaque crossing network where time priority is enforced. In this trading venue, orders are crossed at the end of the trading game only if enough orders on the opposite side have been submitted to the dark pool prior to the cross. The execution price is the spread midquote prevailing on the LOB at the end of period t_4 which we indicate with $\widetilde{p}_{Mid}$. Hence, in a PDP not only the execution probability is uncertain but also the execution price.

In this framework the large traders' action space includes, in addition to the orders presented for the benchmark model, the possibility to submit orders to buy or to sell the asset on the PDP, as shown in Table 1.[13] Therefore, each trader decides not only his optimal order type, as in the benchmark framework, but also his preferred trading venue. In particular, large traders can split their orders between the LOB and the PDP, by combining a 1-unit dark pool order with either a market order, $\varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid})$, or a limit order, $\varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid})$. Alternatively, they can submit their large order to the PDP, $\varphi_D(\pm j, \widetilde{p}_{Mid})$. As for the benchmark model, both small and large traders still compare the expected profits from the different order types but now the feasibility and profitability of these orders depend also on the expected state of the PDP at the time of the order submission, $\widetilde{PDP}_t$.

[13] In this model dark pools are designed for large traders. For this reason, we do not allow small traders to post their orders to the dark pool.

[**Insert Figure 3 here**]

In Figure 3 we present the extensive form of the game. At t_1 the $L\&P$ is similar to the benchmark case, because traders do not find it optimal to submit orders to the PDP. However, as liquidity starts to build up in the LOB, they switch to the dark market. Assume for example the at t_1 a large trader arrives and submits a limit sell order at p_1^A, $\varphi_L(2, p_1^A)$, so that the book opens at t_2 as $b_{t_2} = [20]$. A large incoming trader willing to sell may now, in addition to the benchmark strategies, submit a 2-unit order to the PDP, $\varphi_D(-2, \widetilde{p}_{Mid})$, that has the following expected payoff:

$$\pi_{t_2}^e[\varphi_D(-2, \widetilde{p}_{Mid})] = E[(\widetilde{p}_{Mid} \ - \beta_{t_2} v) \Pr_{-2}(\widetilde{p}_{Mid} \ |\Omega_{t_2})] \ 2 \ , \tag{11}$$

where $\Omega_{t_2} = \{b_{t_2}, PDP_{t_2}\}$ is the information set of the trader and $\Pr_{-2}(\widetilde{p}_{Mid} \ |\Omega_{t_2})$ is the probability that 2 shares to sell will be executed in the PDP at the end of the game. Clearly, this order submitted to the dark venue changes the state of the dark pool from $PDP_{t_2} = 0$ to $PDP_{t_3} = -2$: because neither no-trading nor other dark pool orders are equilibrium strategies when the book opens as $b_{t_2} = [20]$, if traders observe no changes in the state of the LOB at the opening of t_3, i.e., $b_{t_3} = [20]$, they infer that a 2-unit dark pool order to sell was submitted at t_2.

Alternatively, assume now that a small trader arrives at t_2 and submits a limit order to buy, so that the book opens in the following period as $b_{t_3} = [21]$. In this case, large buyers can submit a combination of limit and dark pool order, with the following expected pay-off:

$$\pi_{t_3}^e[\varphi_{LD}(1, p_1^B; +1, \widetilde{p}_{Mid})] = (\beta_{t_3} v - p_1^B) \Pr_1(p_1^B|\Omega_{t_3}) + E[(\beta_{t_3} v - \widetilde{p}_{Mid}) \Pr_{+1}(\widetilde{p}_{Mid} \ |\Omega_{t_3})] \ . \tag{12}$$

The incoming trader at t_4 observes the resulting opening book, $b_{t_4} = [22]$, and Bayesian updates his expectations on the state of the PDP, taking into account that this state of the LOB could be the result of either a limit and dark pool order by a large trader, or a limit order by a small trader:

$$\widetilde{PDP}_{t_4} = \begin{cases} +1 \ \ with\ prob = \frac{\Pr_{t_3} \varphi_{LD}(1, p_1^B; +1, \widetilde{p}_{Mid})}{\Pr_{t_3} \varphi_{LD}(1, p_1^B; +1, \widetilde{p}_{Mid}) + \Pr_{t_3} \varphi_L(1, p_1^B)} \\ 0 \ \ with\ prob = \frac{\Pr_{t_3} \varphi_L(1, p_1^B)}{\Pr_{t_3} \varphi_{LD}(1, p_1^B; +1, \widetilde{p}_{Mid}) + \Pr_{t_3} \varphi_L(1, p_1^B)} \end{cases} . \tag{13}$$

Large sellers, instead, may submit a combination of a market and a dark pool order to sell, with the following expected pay-off:

$$\pi^e_{t_3}[\varphi_{MD}(1, p^B_1; -1, \widetilde{p}_{Mid})] = (p^B_1 - \beta_{t_3} v) + E[(\widetilde{p}_{Mid} - \beta_{t_3} v) \Pr_{-1}(\widetilde{p}_{Mid} \,|\Omega_{t_3})] \; . \tag{14}$$

In this case, the incoming trader at t_4 observes $b_{t_4} = [20]$, and Bayesian updates his expectations on $\widetilde{PDP}_{t_4}$ as follows:

$$\widetilde{PDP}_{t_4} = \begin{cases} -1 \;\; with \; prob = \frac{\Pr_{t_3} \varphi_{MD}(1, p^B_1; -1, \widetilde{p}_{Mid})}{\Pr_{t_3} \varphi_{MD}(1, p^B_1; -1, \widetilde{p}_{Mid}) + \Pr_{t_3} \varphi_M(1, p^B_1)} \\ \;\; 0 \;\; with \; prob = \frac{\Pr_{t_3} \varphi_M(1, p^B_1)}{\Pr_{t_3} \varphi_{MD}(1, p^B_1; -1, \widetilde{p}_{Mid}) + \Pr_{t_3} \varphi_M(1, p^B_1)} \end{cases} . \tag{15}$$

If he wants to buy the asset, he can choose depending on his personal valuation β_{t_4} among submitting a regular market order, $\varphi_M(2, p^A_1)$, a 1-unit dark pool order, $\varphi_D(+1, \widetilde{p}_{Mid})$, or combining a market and a dark pool order, $\varphi_{MD}(1, p^A_1; +1, \widetilde{p}_{Mid})$. Depending on his β_{t_4} the trader can also market sell or choose not to trade.

To summarize, the existence of the dark pool option influences market participants' estimate of the state of the dark pool, and therefore their estimate of the execution probability of future dark orders as well as the order submission decisions of incoming traders.

More generally, at each trading round the fully rational risk-neutral large trader takes all these effects into account and chooses the optimal order submission strategy which maximizes his expected profits, conditional on his valuation of the asset, β_t, and his information set, Ω_t, respectively:

$$\begin{aligned} \max_{\varphi} \; & \pi^e_t[\varphi_M(j, p^z_i), \varphi_M(2, p^z), \varphi_{ML}(1, p^z_i; j, p^z_1), \varphi_{MD}(1, p^z_i; \pm 1, \widetilde{p}_{Mid}), \\ & \varphi_D(\pm j, \widetilde{p}_{Mid}), \varphi_{LD}(1, p^z_1; \pm 1, \widetilde{p}_{Mid}), \varphi_L(j, p^z_1), \varphi(0) \; |\beta_t, \Omega_t] \; . \end{aligned} \tag{16}$$

Small traders still solve problem (10), however they now condition their strategies not only on their own β_t and on the state of the LOB but also on the inferred state of the PDP. The game is solved as before by backward induction starting from t_4.

3.2.2 Continuous Dark Pool Framework - *L&C*

We now consider the dark pool that offers continuous execution. In our discrete model, this means that the CDP crosses orders at each trading round at the spread midquote prevailing on the LOB in that period, $\widetilde{p}_{Mid,t}$. In addition to the orders discussed so far, the CDP offers traders a more sophisticated strategy that allows large investors to simultaneously send orders to the LOB and to the dark venue (Table 1). Considering the sell side, a large impatient trader may send a IOC sell order to the CDP. If the order does not execute immediately, it is automatically routed to the LOB as a market order, $\varphi_D(-j, p_{Mid,t}, p_i^B)$. This strategy provides the following payoff:

$$\pi_t^e[\varphi_D(-j, p_{Mid,t}, p_i^B)] = \Pr_{-j,t}(p_{Mid,t} \, |\Omega_t)(p_{Mid,t} - \beta_t v) \, j + [1 - \Pr_{-j,t}(p_{Mid,t} \, |\Omega_t)](p_i^B - \beta_t v) \, j \; , \quad (17)$$

where $\Pr_{-j,t}(p_{Mid,t} \, |\Omega_t)$ is the probability that j shares to sell are executed in the CDP at t.[14]

Figure 4 illustrates the extensive form of the game. Consider again the book that opens as $b_{t_3} = [21]$, after a 2-unit limit sell order and a 1-unit limit buy order have been submitted at t_1 and t_2, respectively. If a large seller arrives at t_3 and submits a combination of market and dark pool order, as in the $L\&P$ framework presented before, the incoming trader at t_4 will update his expectation on the state of the CDP according to (15). Note that in this case by submitting a IOC, $\varphi_D(+2, p_{Mid,t_4}, p_1^A)$, the potential buyer can search execution first on the CDP and then on the LOB.

[**Insert Figure 4 here**]

Under this new trading protocol, at each trading round the risk-neutral large trader chooses the optimal order submission strategy which maximizes his expected profits, depending on his evaluation of the asset, β_t and on his information set, Ω_t:

$$\max_{\varphi} \; \pi_t^e[\varphi_M(j, p_i^z), \varphi_M(2, p^z), \varphi_{ML}(1, p_i^z; j, p_1^z), \varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid,t}), \quad (18)$$
$$\varphi_D(\pm j, \widetilde{p}_{Mid,t}, p_i^z), \varphi_D(\pm j, \widetilde{p}_{Mid,t}), \varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid,t}), \varphi_L(j, p_1^z), \varphi(0) \; |\beta_t, \Omega_t] \; .$$

As before, small traders solve problem (10), and shape their strategies depending on the expected state of the CDP.

4 What's Driving Volume into the Dark?

We solve both the benchmark model (B) and the two models with a periodic ($L\&P$) and a continuous ($L\&C$) dark pool alongside a LOB numerically. This allows us to answer a number of questions,

[14] When the opposite side of the LOB has only one share at the first level, the IOC bounces back as a marketable order, see Eq. (8).

related respectively to order migration, trade creation and volume creation, which we believe are of particular interest both to market participants and even more so to exchange officials.

When a dark pool is added alongside a LOB, should we expect orders to migrate to the dark venue? And if the dark pool generates order migration, should we expect orders simply to move from one trading platform to the other, or the aggregate execution rate of orders to increase, thus generating more trades? If orders migrate to the dark venue, are these the orders that in the LOB would have been submitted as limit orders or market orders, or both? Also, considering that orders may differ in size, should we expect this variation in the fill rate to lead to volume creation? Our model allows us to discuss these issues and also to investigate which factors attract order flow away from the lit market and into the dark pool. By comparing the $L\&P$ with the $L\&C$, we can also discuss how the design of the dark market affects the dynamics of such order flow. To emphasize the dynamics of our model, we solve the model both at t_1 with four periods remaining and at t_2 when we condition on the different equilibrium states of the opening book. When we report results at t_1, these are averages across future periods (t_1, t_2, t_3, and t_4) and include all nodes of the tree. The results at t_2 instead report the average across future periods (t_2, t_3, and t_4) and nodes of the remaining tree, conditional on the three possible order book outcomes after t_1, $b_{t_2} = [00]$, [10], and [20].

We define order migration (OM) as the average probability that in equilibrium an order migrates to the dark pool. The average is computed over a number of periods, T, of the game and over all of the equilibrium states of the book and of the dark pool:

$$OM = \frac{1}{T} \sum_{t} \Pr(LT)\ E_{\Omega_t} \left[\int_0^2 \varphi_{LT}^{n} \cdot f\left(\beta_t\right) d\beta_t \right] \ , \tag{19}$$

where for the $L\&P$ framework $\varphi_{LT}^n = \{\varphi_D(\pm j, \widetilde{p}_{Mid}), \varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid}), \varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid})\}$, whereas for the $L\&C$ framework $\varphi_{LT}^n = \{\varphi_D(\pm j, \widetilde{p}_{Mid,t}), \varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid,t}), \varphi_D(\pm j, \widetilde{p}_{Mid,t}, p_i^z),$ $\varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid,t})\}$.[15]

We define trade creation (TC) as the average difference between the sum of the fill rates on

[15] When we solve the four-period model starting at t_1, we average on four periods and therefore $T = 4$. When instead we solve the three-period models by conditioning on the different equilibrium states of the opening book at t_2, $T = 3$.

the LOB and the dark pool, and the fill rate in the benchmark model:

$$TC = \frac{1}{T} \sum_t (FR_t^Z - FR_t^B) \ , \tag{20}$$

where $Z = \{L\&P, L\&C\}$ and

$$FR_t^{Z,B} = \sum_{a=ST,LT} \Pr(a) \ E_{\Omega_t} \left[\int_0^2 \varphi_a^n \cdot f(\beta_t) \, d\beta_t \right] \ . \tag{21}$$

The equilibrium strategies (φ_a^n) considered in Eq. (21) include all -large and small- market orders for the B framework, and both market orders and executed dark pool orders for the $L\&P$ and $L\&C$ frameworks.

Finally, we define volume creation (VC) as the average LOB plus dark pool volume, V_t^Z, in excess of the total LOB volume in the benchmark framework, V_t^B:

$$VC = \frac{1}{T} \sum_t (V_t^Z - V_t^B) \ , \tag{22}$$

where we measure volume in each period t by weighting the fill rate, FR_t, by the order size, $q_t = \{1, 2\}$:

$$V_t^{Z,B} = \sum_{a=ST,LT} \Pr(a) E_{\Omega_t} \left[\int_0^2 q_t \cdot \varphi_a^n \cdot f(\beta_t) \, d\beta_t \right] \ . \tag{23}$$

Proposition 1 *In equilibrium, when a dark pool is introduced in a market with a limit order book, we observe:*

- *order migration, trade creation and volume creation.*

As order book depth builds,

- *order migration increases but trade creation and volume creation decrease.*

The effects of the introduction of a dark pool on order migration, trade creation and volume creation are stronger when traders use the continuous rather than the periodic dark pool, and weaker with a smaller tick size.

Figure 5 (Panel A) reports results for the model evaluated at t_1 with an empty book and four periods remaining, and at t_2 respectively with an empty book, one share on the first level of the ask side, or two shares on the same level of the book and three periods remaining. The results evaluated both at t_1 and at t_2 show that when traders who are active on a LOB are offered the additional option to trade in the dark at a better price but with execution uncertainty, orders migrate to the dark pool. Interestingly, the migration of orders for the four-period model is stronger that that of the three-period one, conditional on the same initial state of the book. This is a general property of the model which holds for all our results, and suggests that when we increase the number of periods, all the effects are magnified.

Orders migrate more intensively when the dark pool executes continuously ($L\&C$) than when it executes only at the end of the trading game ($L\&P$). In the $L\&C$ framework, dark pool orders become more attractive because they may be executed at each trading round (lower execution uncertainty). On the contrary, orders migrate less intensively when the tick size gets smaller as traders' incentive to use limit orders decreases (Figure 5, Panel B).[16]

[**Insert Figure 5 here**]

Results evaluated at t_2 further show that migration is more intense when the book becomes more liquid in terms of depth and spread: as liquidity increases, some traders find dark pool orders more attractive than limit and market orders (Table II). The reason is that when competition for the provision of liquidity increases, the queue becomes longer due to time priority and there is less room on the LOB; so, dark orders become attractive for the most aggressive of the patient traders. At the same time, as the liquidity of the book increases, the execution uncertainty of the dark pool decreases and dark orders become more attractive for the impatient traders. Because both the LOB and the dark pool open empty, traders start moving to the dark pool only at t_2 when some depth builds up in the book; it is important to notice, though, that traders go dark even if liquidity builds up only on one side of the LOB.[17] Clearly because the incentive to go dark is small, even the migration is small in size. However, Table II shows that by moving from the benchmark to the market with a dark pool in the $b_{t_2} = [20]$ book, both limit orders and market orders migrate to the dark pool.

[**Insert Table II here**]

[16]Note that when we solve the models with a smaller tick size we hold the number of price levels constant. As Buti, Consonni, Rindi, Wen and Werner (2014a) show, all else equal this new model specification results in a reduction of traders' incentive to supply liquidity.

[17]More precisely, investors choose to trade in the dark pool at t_2 only when there are at least 2 shares at the top of the ask side to the book.

On average dark orders become more attractive not only when the depth increases, but also when the spread becomes smaller. All else equal, a smaller spread increases competition for the provision of liquidity and makes the dark pool option more attractive for limit orders. A smaller spread also decreases the price improvement that the dark pool may offer to market orders but this effect is compensated by the increased execution probability generated in the dark pool by the migration of the limit orders.

This prediction is consistent with Ready (2013), who shows that the size of the spread on the primary market influences volumes on Liquidnet and ITG POSIT, two Independent/Agency dark pools with periodic crossing. According to Ready, the larger the percentage spread of a stock, the lower is the share of institutional volumes traded on these two dark pools. Our prediction is also consistent with Buti, Rindi and Werner (2011) who find evidence that stocks with narrower quoted spreads have greater dark pool volume, suggesting that dark pools are more attractive when the degree of competition on the LOB is high. Interestingly, the results from Buti, Rindi and Werner (2011) also hold overtime, thus supporting the predictions of our model which has a time series flavor.

Having discussed the migration of orders away from the lit market into the dark pool, we now consider the model's results on trade creation. TC measures the overall increase in the execution rate following the introduction of the dark pool and hence it is the sum of the orders executed in the dark and on the LOB in excess of the benchmark framework. First of all we study executions in the dark pools and notice that consistently with the pattern of OM, the dark orders' fill rate (FR) increases with the liquidity of the book at t_2 and it is greater in the $L\&C$ than in the $L\&P$ framework. When instead we consider the total fill rate (TC) that also includes the net fill rate of the LOB, we find that TC decreases with the liquidity of the book for both the $L\&P$ and the $L\&C$ (Figures 6, Panel A). This result is driven by the different effect that the migration of limit and market orders has on executions. When limit orders migrate from the LOB into the dark, executions overall increase, whereas when market orders migrate to the dark pool executions decrease as the execution probability of dark orders is larger than that of limit orders and smaller than that of market orders. When the book becomes deeper, traders use more market than limit orders and the second effect is stronger, so that total executions relative to the benchmark model decrease. This effect is evident by looking at the fill rate of the LOB relative to the benchmark that becomes substantially negative in the [20] book and counterbalances the positive increase of the fill rate in the dark pools. Indeed the dark pool fill rate is tiny in this model as both the LOB and the dark pools open empty, so both the incentive to go dark and the execution probability in the dark pools are minimized. However, the effect of the LOB liquidity is present both in the $L\&P$ and in the $L\&C$ frameworks, and because in the $L\&C$ traders use the dark pool more extensively, the effect is stronger in the $L\&C$ than in the $L\&P$.[18]

[**Insert Figure 6 here**]

Because we measure volume by weighing the fill rates by the size of the orders executed, a similar pattern characterizes the dynamics of the dark pool volume as well as the LOB volume relative to the benchmark and the VC (Figure 7, Panel A). Moreover, because the average size of the orders that migrate from the LOB to the dark pool is larger than the average size of the orders executed on the LOB, all the effects generated by the migration of orders to the dark pool are magnified when measured by volume rather than by fill rate. Hence the increased dark pool volume that we observe when the LOB gets deeper is greater than the increased dark pool fill rate and so is the reduction in VC compared to the reduction in TC.

[**Insert Figure 7 here**]

As for OM, we show that all the effects of the introduction of a dark pool on both fill rate and volume, as well as on TC and VC, are smaller when we solve the models with a smaller tick size (Figure 6 and 7, Panel B). In equilibrium the smaller the tick size, the smaller the proportion of limit to market orders (as limit orders are less profitable), so that when the dark market is introduced fewer limit orders switch to the dark pool. Furthermore, the smaller the tick size, the smaller the inside spread and the less expensive market orders are compared to dark pool orders. Hence all the effects previously described are smaller when the tick size decreases.

In DVW (2009), the introduction of a CN alongside a DM leads to the creation of new orders, and it generates order migration only as a secondary effect. This is due to the fact that in a DM traders cannot post limit orders, and the introduction of a CN has the main effect of attracting

[18] Note that TC reflects the impact that the introduction of the dark pool has on the LOB at t_1. In fact, even though orders do not migrate to the dark in that very first period as both the book and the dark pool open empty, traders switch from limit to market orders anticipating that in the future periods liquidity will build up in the dark pool, and with the dark pool attracting market orders, the execution probability of limit orders posted on the LOB will decrease. It is precisely because traders move from limit to market orders at t_1 that both the average FR of the LOB relative to the benchmark and the average TC are greater when the model is evaluated at t_1 than evaluated at t_2 with an empty opening book.

patient investors who otherwise would refrain from trading. By contrast, in our benchmark LOB model traders can post limit orders and therefore when competition for the provision of liquidity becomes strong the introduction of a dark pool attracts limit orders away from the LOB into the dark pool. The migration of limit orders turns out to be the main factor driving our results. In our model the effect of the creation of new orders that DVW obtain takes place almost exclusively in the last period of the trading game, when the market resembles a DM because limit orders have zero execution probability.

However, our model shares with DVW (2009), as well as with Hendershott and Mendelson (2000), a feedback effect generated by traders' perception of dark pool liquidity which influences traders' estimate of the execution probability of dark pool orders and hence their use. When traders perceive that liquidity is building in the dark pool, they update their estimate of the dark pool depth and assign a higher probability of execution to dark orders, the result being that they are more likely to opt for dark trading. This positive liquidity-externality effect intensifies when traders perceive that dark volume is growing. This prediction is consistent with the empirical results by Buti, Rindi and Werner (2011) that show the existence of a positive auto-correlation between contemporaneous and lagged dark activity.

5 Who Benefits from a Dark Pool?

Even though dark trading has existed for several decades, it is only recently that dark pool volume relative to consolidated equity volume has increased to more than 14% in the U.S. and almost 5% in Europe (Figure 1).

It is therefore understandable that regulators are concerned about the effects on market quality and traders' welfare of the widespread use of dark pools. Is market quality affected by the overall reduction in transparency that the growing use of dark trading entails? Should regulators be concerned about the welfare implications of dark trading? We address these issues by first investigating how the introduction of a dark pool affects the quality of the primary market. Because changes in market quality influence agents' gains from trade, we then study how total welfare and the distribution of welfare across market participants change after the introduction of a dark pool.

5.1 Market Quality

To evaluate the effect of dark trading on the quality of the LOB, we consider two standard measures of market quality, i.e., inside spread (S) and market depth (D). We compute expected spread and depth in period t_{i+1} by weighing the realized values in the equilibrium states of the book with the corresponding order submission probabilities in the previous periods:

$$y_{t_{i+1}} = \sum_{a=ST,LT} \Pr(a)\ E_{\Omega_{t_i}} \left[\int_0^2 y_{t_{i+1}} \cdot \varphi_a^n \cdot f\left(\beta_{t_i}\right) d\beta_{t_i} \right] \ , \tag{24}$$

where $y_{t_{i+1}} = \left\{S_{t_{i+1}}, D_{t_{i+1}}\right\}$. We then compute the percentage difference between these indicators of market quality for the $L\&P$ (and $L\&C$) and the B framework, and average them across periods:[19]

$$\Delta y = \tfrac{1}{T} \sum_t (y_t^Z - y_t^B)/y_t^B \ , \tag{25}$$

where $y = \{S, D\}$. The following Proposition summarizes our results.

Proposition 2 *In equilibrium, when a dark pool is introduced in a market with a limit order book:*

- *the inside spread and the inside order book depth worsen.*

As order book depth builds:

- *the negative effect on the inside spread weakens;*
- *there is a non-monotonic effect on inside order book depth.*

The effects of the introduction of a dark pool on spread and order book depth are stronger when traders use the continuous rather than the periodic dark pool, and weaker with a smaller tick size.

Our results show that the introduction of a dark pool has a negative effect on the liquidity of an empty LOB as both the inside spread and the depth at the best bid and offer worsen (Figure 8, Panel A). However, while the negative effect on spread decreases with the liquidity of the book, the effect on depth first improves and then, as traders heavily move to the dark pool, further deteriorates. We can explain the dynamic of spread and depth by considering once more how

[19] As in the initial period spread and depth are exogenous, $T = 3$ for the four-period model and $T = 2$ for the three-period models.

traders' order submission strategies react to the introduction of the dark pool and how the reaction changes as liquidity builds up both in the LOB and in the dark pool.

[**Insert Figure 8 here**]

At the beginning of the trading game when both the book and the dark pool open empty, the introduction of a dark pool makes traders switch from limit to market orders anticipating a future reduction in the execution probability of limit orders.[20] The resulting increased liquidity demand and reduced liquidity supply widen spread and reduce depth at the inside quotes. Subsequently, when liquidity builds up in the LOB two effects take place. On the one hand, when the book becomes deeper traders generally tend to use more market than limit orders and so fewer traders switch from limit to market orders when a dark pool is introduced. This first effect attenuates the negative impact on liquidity of the switch from limit to market orders. On the other hand, however, as liquidity builds up and the queue at the top of the LOB becomes longer, more limit orders are attracted to the dark pool and this effect reduces liquidity of the LOB, especially depth.

When the book moves from being empty to having 1 share at the best ask price, the first effect dominates and both spread and depth improve (the impact of the dark pool introduction is less negative). When instead depth builds up substantially and the book opens with 2 shares at the best ask price, the second effect dominates and while spread improves, depth drops due to the migration of limit orders. The reason why the spread improves but depth deteriorates when the book opens with 2 shares at the best ask price is that a minimum of 2 shares at the best ask is required for the introduction of the dark pool to generate a migration of limit orders -not only a switch from market to limit orders. So the intense migration of the 2-unit limit orders from the ask side does not impact spread heavily as the book has already 2 shares at p_1^A, whereas it does impact depth substantially.

Clearly and consistent with the dynamic of OM, all these effects are stronger for the $L\&C$ market in which traders use the dark pool more intensively.

When instead the tick size gets smaller, all the effects diminish for two reasons (Figure 8, Panel B). First, the tick size reduction makes traders switch from limit to market orders which attenuates all the effects at work. Second, a smaller tick size results in a narrower inside spread and therefore

[20] More precisely, traders switch from 2-unit limit orders to 2-unit market orders, and from 2-unit limit orders to 1-unit limit and 1-unit market orders.

in a reduced benefit from the dark pool execution at the midquote of the inside spread. Hence a smaller tick size results in fewer market orders migrating to the dark pool.

5.2 Welfare Analysis

Traders in our model have a private motive to trade. Hence we can fully characterize welfare and further differentiate between the effects of introducing a dark venue on small and large traders' welfare. In light of our results on OM, TC and VC, and on market quality, we can assess to what extent dark pools enable traders to realize welfare gains. Finally, we can address the policy question of whether in a competitive setting the dark trading option enhances total welfare.

Following Goettler, Parlour and Rajan (2005) and DVW (2009), we measure welfare for a large or a small trader as:

$$W_{a,t} = \int_0^2 \pi_t^e(\varphi_a^n) d\beta_t \ . \tag{26}$$

Total welfare at period t is equal to the sum of the gains from trade for both large and small traders:[21]

$$W_t = \sum_{a=ST,LT} \Pr(a) W_{a,t} \ . \tag{27}$$

We then compute the percentage difference between the $L\&P$ (and $L\&C$) and the B framework for each trader's type and in total in each period, and average them out across periods. The following Proposition summarizes our results.

Proposition 3 *The introduction of a dark pool to a market with a limit order book changes traders' welfare as follows.*

- *Both large and small traders are worse off, and total welfare decreases.*

As order book depth builds:

- *the negative welfare effects of the dark pool decline, and when liquidity is sufficiently large (2 shares at the best ask or at the best bid), the welfare gains of large traders outweigh the welfare losses of small traders, and total welfare increases as a result.*

The effects of the introduction of a dark pool are stronger when traders use the continuous dark pool, and weaker with a smaller tick size.

[21] We assume that the trading crowd acts as a group of traders who are indifferent between trading and not trading.

Welfare of large traders. Large traders do not benefit from the introduction of a dark pool when both the book and the dark pool open empty (Figure 9, Panel A). In this case the introduction of the dark pool has the initial effect of inducing traders to anticipate the future lower limit order execution probability and hence to switch from limit to market orders. As a result both spread and depth deteriorate and so does welfare. However, as the book becomes deeper and at the same time some liquidity builds up in the dark pool, large traders start using the dark pool intensively and benefit from dark trading.

[**Insert Figure 9 here**]

Welfare of small traders. The effect of the introduction of a dark pool on the welfare of small traders is mainly driven by the variation in the spread: because these agents only trade one unit, depth only marginally affects their profits. Spread deteriorates due to the introduction of the dark pool, which explains why small traders are worse off (Figure 9, Panel A). However, because spread improves with the liquidity of the LOB, so does small traders' welfare as their losses diminish.

Total welfare. When both the book and the dark pool open empty, the introduction of a dark pool deteriorates total welfare. When liquidity subsequently increases and large traders can access the dark pool intensively, total welfare increases. Note that small traders generally loose from the introduction of a dark pool, just because they cannot access the new trading venue, not because they may only trade orders of small size. We believe that small traders would benefit from the introduction of a dark pool if they were allowed to access the dark platform, even though with smaller profits than large traders.

When traders may access the dark pool continuously and manage to execute dark orders at each trading round, all effects are magnified; whereas, when the tick size decreases and traders' incentive to use limit orders also decreases, fewer limit orders migrate to the dark pool and all the welfare effects are attenuated (Figure 9, Panel B).

Interestingly, our results show that the welfare effects improve when liquidity of the book builds. The more liquid the LOB is, the more the market resembles a DM in which dealers provide infinite liquidity on both sides of the book. Therefore our result is reminiscent of DVW (2009) who find that when a dark pool is introduced along side a DM welfare increases because the dark pool attracts some traders who would otherwise not trade.

6 Empirical Implications

Strictly speaking, our model compares a market with only a LOB to that of a market with a LOB and a dark pool that opens empty and either crosses continuously or at the end of the trading game. However, in practice what we observe in data are stocks with varying amounts of dark pool activity, not the introduction of dark pool trading. Therefore, we reinterpret our theoretical predictions so that they are useful for empirical testing. The first step we take is to associate the model's introduction of a dark pool as an increase in dark pool activity. This is of course consistent with the equilibrium outcome of the model.

In addition to providing predictions starting from the very beginning, with both the LOB and the dark pool opening empty (t_1), our model also generates predictions for the subsequent period, when liquidity has had a chance to build (t_2). The second step we take is to examine the results for nodes of the tree that differ in the amount of liquidity that is available in the LOB. We interpret these differences in liquidity as a feature of the underlying stock. This allows us to discuss our predictions about the effect of dark pool activity. Finally, note that our model has a flavor of the time series and therefore our predictions can also be tested over time.

6.1 Predictions on fill rates and volume

General:

- Stocks with more dark pool activity have higher LOB fill rates and volume, and higher consolidated fill rates and trading volume. Dark pool fill rates and volume are higher for stocks with liquid books than for stocks with illiquid books.
- LOB fill rates and volume decline (increase) in dark pool activity for stocks with liquid (illiquid) books.
- Consolidated fill rates and volume decline (increase) in dark pool activity for stocks with liquid (illiquid) books.

To our knowledge no attempt has been made in the literature so far to test predictions regarding how fill rates are affected by dark pool trading. To test these predictions, order level data from both the dark pools and the lit market are required. To date, we are not aware of any researchers that have access to this type of granular data for both dark pools and lit markets.

The prediction that dark pool activity is higher for stocks with liquid books is in line both with Buti, Rindi and Werner (2011) and with Ready (2013). Buti, Rindi and Werner (2011) examine a unique dataset on self-reported dark pool activity for a sample covering a large cross section of U.S. securities. They find that liquid stocks are characterized by more intense dark pool activity. Specifically, stocks with higher market capitalization, higher volume, lower spread, higher depth, and lower volatility have significantly higher dark pool activity. Ready (2013) studies monthly dark pool volume stock-by-stock for two dark pools from June 2005 to September 2007. He shows that the size of the spread on the primary market influences volumes on the two dark pools: Liquidnet and ITG POSIT, both Independent/Agency dark pools with periodic crossing. According to Ready (2013), the larger the percentage spread of a stock, the lower is the share of institutional volumes traded on these two dark pools. He finds that dark pools execute most of their volume in liquid stocks (low spreads, high share volume), but they execute the smallest fraction of share of volume in those same stocks.

Besides liquidity, our model also suggests there is another factor that influences dark pool trading, namely the tick size.

Tick size:

- Dark pool activity is generally increasing in the tick size.
- The effects of dark pool activity on LOB and consolidated fill rates and volumes are generally stronger when the tick size is larger.

Because in the U.S. the tick size is one penny for all stocks priced above $1, this empirical prediction can be tested by considering changes in the price of the stock which affect the tick-to-price ratio. Indeed, O'Hara, Saar and Zhong (2014) use order level data from the NYSE to study how the tick-to-price ratio (relative tick size) affects liquidity. While they do not have access to information on external dark pool trading, they do study hidden orders within the NYSE's trading systems. Consistent with our prediction, they find that stocks with larger relative tick size are more likely to have hidden orders.

Note that the empirical predictions regarding the effect of the tick size should be tested with caution, as our model does not include sub-penny trading which may take place in some dark markets and is particularly sensitive to tick size variations (Buti, Consonni, Rindi, Wen, and

Werner, 2014b). We also note that to study the effect of the tick size on dark trading, empiricists should control for the average order size, which is generally smaller in dark venues that engage in sub-penny trading.

Finally, our model makes predictions on the differences in the effects of a continuous and periodic dark pool on trading patterns:

Continuous vs. Period Dark Pool:

- Dark pool activity is higher for a continuous dark pool than for a periodic dark pool.
- The effects of dark pool activity on LOB and consolidated fill rates and volumes are generally stronger for a continuous dark pool than for a periodic dark pool.

Unfortunately, data limitations make it difficult to test the predictions on the effects of a periodic vs. a continuous dark pool. However, the recently released Financial Industry Regulatory Authority (FINRA) data may be considered a good starting point as they allow researchers to study trading and share volume from 40 different ATSs on a weekly basis.

6.2 Predictions on market quality

- Stocks with more dark pool activity have wider LOB spreads and lower depths.
- The effect of dark pool activity on LOB spreads is smaller for liquid than for illiquid stocks.
- The effect of dark pool activity on LOB depth is larger for liquid than for illiquid stocks.

To empirically isolate the effect of dark trading on market quality is challenging because, as our model shows, dark pool activity is in itself determined by liquidity. Another challenge is that data on dark pool activity has not until recently been generally available.[22] Various data sets and empirical approaches have been used in the literature, and it is perhaps not surprising that the evidence is mixed. O'Hara and Ye (2011) find that the overall effect of fragmentation on NASDAQ and NYSE market quality is positive. As a proxy of volume on off-exchange venues they use trades reported to the Trade Reporting Facilities (TRFs). Unfortunately, the TRF data does not distinguish between dark markets, internalization by broker-dealers and fully transparent LOBs

[22] The SEC now mandates disclosure of aggregate volume by stock per day for U.S. dark pools.

like BATS or Direct Edge. Buti, Rindi and Werner (2011) find that both more dark internalization and more dark pool activity are on average associated with improved market quality. However, they also find that the beneficial effects on market quality are weaker for less liquid stocks than for liquid stocks.

Degryse, de Jong and van Kervel (2014) consider a sample of 52 Dutch stocks and analyze both lit fragmentation, (dark) internalized trades and trades sent to dark pools. They find that when the two sources of dark liquidity are combined, the overall effect on market quality is detrimental. By contrast, lit fragmentation is associated with improved aggregate liquidity (although it is associated with poorer liquidity at the listing venue). Gresse (2014) in her recent study on the effect of the Markets in Financial Instruments Directive (MiFID) on U.K. and Euronex listed stocks finds that lit fragmentation increases depth and narrow spreads, whereas (dark) internalized trades are associated with greater depth but wider spreads.[23] Unfortunately, Gresse does not have access to dark pool trading for her sample stocks.

The model also offers empirical predictions about how the tick size and the dark pool trading protocol influence the relationship between dark pool trading and market quality.

Tick size:

- The effects of dark pool activity on LOB spread and depth are generally stronger when the tick size is larger.

This hypothesis could be tested empirically by sub-sampling on stock price (as U.S. stocks above $1 have the same tick size) when studying the relationship between liquidity and dark pool activity. We are not aware of any empirical paper that has actually done this to date.

Continuous vs. Period Dark Pool:

- The effects of dark pool activity on LOB spread and depth are generally stronger for a continuous dark pool than for a periodic dark pool.

As mentioned above, data limitations unfortunately make it difficult to test the predictions on the effects of a periodic vs. a continuous dark pool.

[23] There are several additional papers on MiFID: Soltani, Mai, and Jerbi (2011) on Euronext stocks, Kohler and Wyss (2012) on Swiss stocks, Aitken, deB-Harris, Senenbrenner III (2012) on U.K. stocks. See also Riordan, Storkenmaier and Wagener (2011), and Gentile and Fioravanti (2012).

7 Conclusions and Policy Implications

Regulators, exchange officials, media, and even some market participants are voicing concerns about the growing level of dark trading in U.S. and European equity markets. They worry that the presence of dark venues reduces the incentives for liquidity provision in the lit market, potentially reducing the depth at the best bid offer, widening the displayed spread, impairing price discovery, and discouraging traders from participating in the market. In addition, exchange officials see their franchise threatened as more trading moves off-exchange. In this paper, we develop a theoretical dynamic equilibrium model to address these concerns. The model attempts to capture the salient features of the real market and permits traders to use a rich set of order submission strategies and venues. Overall, our results illustrate that there is a trade-off between trade and volume creation on the one hand, and displayed liquidity such as depth and spread on the other hand.

Specifically, we model a multi-period LOB market to study the consequences of introducing a dark pool. Our order book has two price levels on each side of the market and is fully transparent. All traders can execute market orders against available liquidity or post limit order for possible future execution. The trading protocol in the order book respects strict price/time priority, so a liquidity-supplying trader needs to improve price by at least one tick to gain priority over existing orders. The dark pool is available to traders that arrive to the market with a desire to trade a large order. It can either be a periodic dark pool which gathers orders and executes buy orders against sell orders at the end of the trading game, or a continuous dark pool which traders can use as a complement to the LOB to demand and supply liquidity at each trading round. While small traders are restricted to using market or limit orders on the LOB, large traders are able to optimally design their trading strategies by selecting a mix of order types, and may also split their orders between the LOB and the dark pool. In the continuous dark pool framework, large traders can even submit orders to the dark pool that resemble immediate or cancel orders combined with routing instructions. These orders are automatically routed to the order book if they are not filled in the dark pool.

Our results show that when a dark pool is introduced in an illiquid market, the expected fill rate for limit orders declines so that limit orders either switch to market orders or migrate to the dark pool, and fill rates and volume increase. In other words, the presence of a dark pool boosts overall trading activity and this is beneficial since each trader rationally decides that she is better off trading than not trading. However, traders using the order book reduce their reliance on limit orders and as a result the order book spread widens and depth decreases. So, the increase in trading takes place at on average worse prices, and welfare declines.

When liquidity builds up in the order book, limit order queues are long which reduces the execution probability of new limit orders. Traders therefore substitute away from limit orders, sending these trading interests to the dark pool. As the execution probability of dark orders increases, also market orders substantially migrate to the dark pool thus reducing the consumption of order book liquidity. Since a significant portion of the market orders left the lit market, order book fill rates and volume decline significantly and out-weigh the increase in dark pool trading activity, so that consolidated fill rates and volume decline. With fewer market orders remaining in the order book, there is less pressure on the displayed liquidity and spreads widen less for liquid than for illiquid books. However, the probability of execution for limit orders in liquid books declines sufficiently for inside depth to decline significantly more than is the case for illiquid books. As the execution probability of dark orders increases, the traders with access to the dark pool are better off.

We study two additional variations in market structure and trading protocols. First, we alter the trading protocol of the dark pool from crossing orders only in the last period of the trading game to executing orders continuously. Second, we investigate the consequences of adding a dark pool to order books with different tick size. We find that the effects of introducing a dark pool are amplified when the dark pool crosses orders continuously. Similarly, the effects of introducing a dark pool are amplified if the order book trades with a larger minimum price increment.

Our results suggest that the regulatory objective to preserve retail traders' welfare could clash with the objective of dark pool operators to maximize trade and volume-related revenues. The reason is that when large traders have access to a dark pool the lit market spread tends to widen, and small traders face higher trading costs as a result. Further, since fill rates and share volume in the public LOB increase when a dark pool is available, the operator of the dark pool has an incentive to boost dark trading even when the operator is the exchange which also runs the lit market. Our model also shows that managers of dark pools seeking to maximize revenue would prefer continuous executions to periodic crossings as this further enhances executions and share volume, but comes at an even higher cost to small traders in terms of a wider lit market spread. Finally, our model shows that regulators should be wary of widening the tick size for less liquid stocks. While a wider tick size in our model results in more order book and consolidated trading activity, this increase in trading activity comes at the price of wider spreads, less depth and a reduction in welfare.

Our model allows us to discuss a wide range of policy issues which are currently on the agenda of financial regulators. However, there are several caveats that should be kept in mind when deriving policy conclusions from our results. First, the model does not include asymmetric information, so we cannot say anything about whether dark markets are likely to affect price discovery. However, this topic is addressed in complementary theoretical work by Ye (2011) and Zhu (2014). Unfortunately, their models reach opposite conclusions: Ye (2011) finds that informed traders are attracted to the dark pool while Zhu (2014) finds that informed traders avoid the dark pool.

Second, we do not discuss price manipulation. While smart traders could in principle trade on the lit market in advance to manipulate the execution price in the dark, we conjecture that this would primarily be an issue for illiquid stocks. Therefore, the possibility of manipulation provides a further incentive for the regulator to limit dark pool volumes for illiquid stocks.

Third, our model does not embed sub-penny trading as our dark pool trades execute at the midpoint of the lit market spread. Buti, Consonni, Rindi, Wen, and Werner (2014b) show, however, that sub-penny trading also harms illiquid rather than liquid stocks. Therefore, our main policy implications are supported even for market structures where dark pools offer sub-penny trading.

Finally, our model focuses on the competition between a transparent LOB and a dark market. However, some exchanges also allow traders to use hidden orders, thus offering an alternative to dark pool trading. Among the wide range of existing undisclosed orders, the closest competitors to dark pool orders are Hidden Mid-Point Peg orders which are totally invisible and are submitted at the spread mid-point. Compared to dark pool orders, Hidden Mid-Point Peg execute against the LOB order flow and therefore have a higher execution probability than dark pool orders. Tackling the issue of competition for the provision of dark venues between exchanges and ATSs is therefore an extremely interesting issue that we leave for future research.

Appendix

Proof of Proposition 1

Consider first the benchmark case. The model is solved by backward induction, starting from $t = t_4$. The t_4-trader solves a simplified version of Eq. (9), if large, or Eq. (10), if small:

$$\max_{\varphi} \pi^e_{t_4} \left\{ \varphi_M(j, p_i^B), \varphi_M(2, p^B), \varphi(0), \varphi_M(2, p^A), \varphi_M(j, p_i^A) | \beta_{t_4}, b_{t_4} \right\} \tag{9'}$$

$$\max_{\varphi} \pi^e_{t_4} \left\{ \varphi_M(1, p_i^B), \varphi(0), \varphi_M(1, p_i^A) \,|\, \beta_{t_4}, b_{t_4} \right\} \ . \tag{10'}$$

Without loss of generality, assume that depending on β_{t_4} and the state of the book b_{t_4} the trader selects one of the equilibrium strategy φ^n_a, with $a = \{ST, LT\}$ and $n \in N_{t_4}$, being N_{t_4} the number of the equilibrium strategies at t_4. The β-thresholds between two different strategies are determined as follows:

$$\beta^{\varphi_a^{n-1}, \varphi_a^n}_{t_4} \therefore \pi^e_{t_4}(\varphi_a^{n-1} \,|\, b_{t_4}) - \pi^e_{t_4}(\varphi_a^n \,|\, b_{t_4}) = 0 \ . \tag{28}$$

These strategies are ordered in such a way that the β-thresholds are increasing, $\beta^{\varphi_a^{n-1}, \varphi_a^n}_{t_4} < \beta^{\varphi_a^n, \varphi_a^{n+1}}_{t_4}$. Hence, the ex-ante probability that a trader submits a certain order type at t_4 is determined as follows:

$$\Pr_{t_4}(\varphi_a^n \,|\, b_{t_4}) = F(\beta^{\varphi_a^n, \varphi_a^{n+1}}_{t_4} \,|\, b_{t_4}) - F(\beta^{\varphi_a^{n-1}, \varphi_a^n}_{t_4} \,|\, b_{t_4}) \ . \tag{29}$$

Consider now period t_3. The incoming trader solves Eq. (9) or (10) if large or small respectively, and uses $\Pr_{t_4}(\varphi_a^n \,|\, b_{t_4})$ to compute the execution probabilities of his limit orders. Given the optimal strategies at t_4, the β-thresholds and the order type probabilities at t_3 are derived using the same procedure as for period t_4, which is then reiterated for periods t_2 and t_1. When a trader is indifferent between strategies φ_a^{n-1} and φ_a^n, i.e., $\beta_t = \beta_t^{\varphi_a^{n-1}, \varphi_a^n}$, we assume without loss of generality that he chooses φ_a^{n-1}.

We provide an example of how the model is solved and from now onwards we assume that for large traders the optimal order size is $j^* = \max_j [\varphi \,|\, b_t]$, since $\partial \pi^e_t(\varphi)/\partial j \geq 0$ due to agents' risk neutrality.

To solve the model we consider all the possible opening LOBs at t_4, and following Figure 2 we present as an example $b_{t_4} = [20]$. From now onwards to ease the notation we omit that all profits are conditional to the state of the book. We start with the small trader's profits:

$$\pi_{t_4}[\varphi_M(1, p_2^B)] = (p_2^B - \beta_{t_4} v) = (1 - \tfrac{3\tau}{2} - \beta_{t_4}) \tag{30}$$

$$\pi_{t_4}[\varphi_M(1, p_1^A)] = (\beta_{t_4} v - p_1^A) = (\beta_{t_4} - 1 - \tfrac{\tau}{2}) \tag{31}$$

$$\pi_{t_4}[\varphi(0)] = 0 \ . \tag{32}$$

By solving Eq. (9') for this case, it is straightforward to show that all strategies are optimal in equilibrium ($N_{t_4} = 3$) and that for the small trader: $\varphi^1_{ST, b_{t_4}} = \varphi_M(1, p_2^B)$, $\varphi^2_{ST, b_{t_4}} = \varphi(0)$, and $\varphi^3_{ST, b_{t_4}} = \varphi_M(1, p_1^A)$. As an example we compute the probability of $\varphi^1_{ST, b_{t_4}}$ and to ease the notation in the following formula we omit the subscript "ST, b_{t_4}":

$$\beta^{\varphi^1, \varphi^2}_{t_4} \quad \therefore \quad \pi_{t_4}[\varphi^1] - \pi_{t_4}[\varphi^2] = 0, \text{ and therefore } \beta^{\varphi^1, \varphi^2}_{t_4} = 1 - \tfrac{3\tau}{2} \tag{33}$$

$$\Pr_{t_4} \varphi^1 = F(\beta^{\varphi^1, \varphi^2}_{t_4}) = \tfrac{1}{2}(1 - \tfrac{3\tau}{2}) \ . \tag{34}$$

Similarly for the large trader, we have $\varphi^1_{LT, b_{t_4}} = \varphi_M(2, p_2^B)$, $\varphi^2_{LT, b_{t_4}} = \varphi(0)$, and $\varphi^3_{LT, b_{t_4}} = \varphi_M(2, p_1^A)$.

For the other periods we only specify the profit formulas, as the derivation of both the β-thresholds and order probabilities follows the same steps presented for period t_4.

Still as an example, and following Figure 2, we consider the opening LOB $b_{t_3} = [21]$. Small traders' profits are as follows:

$$\pi_{t_3}[\varphi_M(1, p_1^B)] = (p_1^B - \beta_{t_3} v) \tag{35}$$

$$\pi^e_{t_3}[\varphi_L(1, p_1^A)] = \pi_{t_3}[\varphi(0)] = 0 \tag{36}$$

$$\pi^e_{t_3}[\varphi_L(1,p^B_1)] = (\beta_{t_3}v - p^B_1)\tfrac{1}{2}\Pr_{t_4}(\varphi_M(2,p^B_1)\,|\,b_{t_4}= [22]) \tag{37}$$

$$\pi_{t_3}[\varphi_M(1,p^A_1)] = (\beta_{t_3}v - p^A_1)\ . \tag{38}$$

Large traders' strategies are similar, the only difference being that $j = 2$ for a market sell order, and that conditioning on $b_{t_3} = [21]$ now traders willing to buy can combine market and limit orders, and traders willing to sell can walk down the book via a marketable order:

$$\pi^e_{t_3}[\varphi_{ML}(1,p^A_1;1,p^B_1)] = (\beta_{t_3}v - p^A_1)+(\beta_{t_3}v - p^B_1)\tfrac{1}{2}\Pr_{t_4}(\varphi_M(2,p^B_1)\,|\,b_{t_4}= [12]) \tag{39}$$

$$\pi_{t_3}[\varphi_M(2,p^B)] = (p^B_1 - \beta_{t_3}v) + (p^B_2 - \beta_{t_3}v)\ . \tag{40}$$

At t_2, again following Figure 2, we present as an example the book $b_{t_2} = [20]$. We consider a small trader and refer to Eq. (30), (31) and (32) for the profits of a market order to sell, to buy, or no trading, respectively. Profits of the other possible strategies of a small trader are:

$$\begin{aligned}\pi^e_{t_2}[\varphi_L(1,p^A_1)] = \quad & (p^A_1 - \beta_{t_2}v)\{\tfrac{1}{2}[\Pr_{t_3}(\varphi_M(2,p^A_1)\,|\,b_{t_3}= [30])\Pr_{w_{t_4}=1}(p^A_1\,|\,b_{t_4}= [10]) \\ & +\Pr_{t_3}(\varphi_{ML}(1,p^A_1;1,p^B_1)\,|\,b_{t_3}= [30])\Pr_{w_{t_4}=2}(p^A_1\,|\,b_{t_4}= [21])] \\ & +\tfrac{1}{2}\Pr_{t_3}(\varphi_M(1,p^A_1)\,|\,b_{t_3}= [30])\Pr_{w_{t_4}=2}(p^A_1\,|\,b_{t_4}= [20])\}\end{aligned} \tag{41}$$

$$\begin{aligned}\pi^e_{t_2}[\varphi_L(1,p^B_1)] = \quad & (\beta_{t_2}v - p^B_1)\{\tfrac{1}{2}[\Pr_{t_3}(\varphi_M(2,p^B)\,|\,b_{t_3}= [21]) + \Pr_{t_3}(\varphi_M(1,p^B_1)\,|\,b_{t_3}= [21]) \\ & +\Pr_{t_3}(\varphi_L(1,p^B_1)\,|\,b_{t_3}= [21])\Pr_{w_{t_4}=1,2}(p^B_1\,|\,b_{t_4}= [22]) + \Pr_{t_3}(\varphi_{ML}(1,p^A_1;1,p^B_1)\,|\,b_{t_3}= [21]) \\ & \Pr_{w_{t_4}=1,2}(p^B_1\,|\,b_{t_4}= [12]) + \Pr_{t_3}(\varphi_M(2,p^A_1)\,|\,b_{t_3}= [21])\Pr_{w_{t_4}=1}(p^B_1\,|\,b_{t_4}= [01])] \\ & +\tfrac{1}{2}[\Pr_{t_3}(\varphi^s_M(1,p^B_1)\,|\,b_{t_3}= [21]) + \Pr_{t_3}(\varphi^s_L(1,p^B_1)\,|\,b_{t_3}= [21])\Pr_{w_{t_4}=1,2}(p^B_1\,|\,b_{t_4}= [22]) \\ & +\Pr_{t_3}(\varphi_M(1,p^A_1)\,|\,b_{t_3}= [21])\Pr_{w_{t_4}=1}(p^B_1\,|\,b_{t_4}= [11])\}\ .\end{aligned} \tag{42}$$

Notice that when small and large traders optimally choose the same equilibrium strategy, we add the superscript "s" to indicate the order submitted by small traders. Notice also that to economize space we do not explicit execution probabilites at t_4. To provide an example:

$$\begin{aligned}\Pr_{w_{t_4}=1}(p^A_1\,|\,b_{t_4}= [10]) = \quad & \tfrac{1}{2}\Pr_{t_4}(\varphi^s_M(1,p^A_1)\,|\,b_{t_4}= [10]) \\ & +\tfrac{1}{2}[\Pr_{t_4}(\varphi_M(1,p^A_1)\,|\,b_{t_4}= [10]) + \Pr_{t_4}(\varphi_M(1,p^A)\,|\,b_{t_4}= [10])]\ .\end{aligned} \tag{43}$$

Large traders' strategies are similar and omitted, the only difference being that $j = 2$ for market and limit orders, and that traders can combine the two order types. We do not present profit formulas for period t_1, in which the opening LOB is $b_{t_1} = [00]$, because they are similar to the ones presented for period t_2, but now limit orders have an additional period to get executed.

L&P . The solution of the *L&P* framework follows the same methodology, but now the large trader solves Eq. (16). For brevity, we provide examples only for the last two periods of the trading game. The remaining two periods are solved in a similar way. To ensure the uniqueness of the equilibrium, we assume that when traders are indifferent between trading on the LOB or on the dark pool, they choose the LOB. We define the information set of the trader at t as $\Omega_{t_i} = [b_{t_i}, y_{t_{i-3}}, y_{t_{i-2}}, y_{t_{i-1}}]$, where y_{t_i} is the observed strategy on the LOB in period t_i.

Following Figure 3, at t_4 we consider again the book $b_{t_4} = [20]$ with the following information set, $\Omega_{t_4} = [20, \varphi_L(2,p^A_1), \varphi_L(1,p^B_1), \varphi_M(1,p^B_1)]$. Notice that when at t_3 traders observe $\varphi_M(1,p^B_1)$, they don't know whether a small trader submitted a 1-unit market order to sell, or whether a large trader submitted a market order combined with a dark pool order. Similarly, at t_2 when traders observe $\varphi_L(1,p^B_1)$ it could be a 1-unit limit order to buy or a combination of a limit and a dark pool order. Traders coming at t_4 Bayesian update their expectations on the state of the dark pool as follows (we omit that all probabilities are conditional to Ω_{t_4}):

$$\widetilde{PDP}_{t_4} = \begin{cases} +1 \;\; with\ prob = \frac{\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid}) \Pr_{t_3} \varphi_M(1,p_1^B)}{[\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B)][\Pr_{t_3} \varphi_M(1,p_1^B)+\Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})]} \\ 0 \;\; with\ prob = \frac{\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid}) \Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B) \Pr_{t_3} \varphi_M(1,p_1^B)}{[\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B)][\Pr_{t_3} \varphi_M(1,p_1^B)+\Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})]} \\ -1 \;\; with\ prob = \frac{\Pr_{t_2} \varphi_L(1,p_1^B) \Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})}{[\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B)][\Pr_{t_3} \varphi_M(1,p_1^B)+\Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})]} \end{cases} . \tag{44}$$

We refer to the B framework for the profits of the small trader, and discuss the large trader's profits. We omit regular market orders that we have already presented in the B framework, Eq. (30)-(31), and focus only on orders that involve the use of the PDP:

$$\pi^e_{t_4}[\varphi_{MD}(1,p_2^B;-1,\widetilde{p}_{Mid})] = (p_2^B - \beta_{t_4}v) + (\tfrac{p_1^A+p_2^B}{2} - \beta_{t_4}v)\Pr(\widetilde{PDP}_{t_4} = +1) \tag{45}$$

$$\pi^e_{t_4}[\varphi_D(-1,\widetilde{p}_{Mid})] = (\tfrac{p_1^A+p_2^B}{2} - \beta_{t_4}v)\Pr(\widetilde{PDP}_{t_4} = +1) \tag{46}$$

$$\pi^e_{t_4}[\varphi_D(+1,\widetilde{p}_{Mid})] = (\beta_{t_4}v - \tfrac{p_1^A+p_2^B}{2})\Pr(\widetilde{PDP}_{t_4} = -1) \tag{47}$$

$$\pi^e_{t_4}[\varphi_{MD}(1,p_1^A;+1,\widetilde{p}_{Mid})] = (\beta_{t_4}v - p_1^A) + (\beta_{t_4}v - \tfrac{p_1^A+p_2^B}{2})\Pr(\widetilde{PDP}_{t_4} = -1) . \tag{48}$$

To determine the equilibrium strategies $\varphi^n_{a,\Omega_{t_4}}$ at t_4 for $n \in N_{t_4}$, the model has to be solved up to period t_1. We anticipate that because in equilibrium $\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid}) = 0$ and $\Pr_{t_3} \varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid}) > 0$, $N_{t_4} = 5$ and the strategies of the large trader are as follows: $\varphi^1_{LT,\Omega_{t_4}} = \varphi_M(2,p_2^B)$, $\varphi^2_{LT,\Omega_{t_4}} = \varphi(0)$, $\varphi^3_{LT,\Omega_{t_4}} = \varphi_D(+1,\widetilde{p}_{Mid})$, $\varphi^4_{LT,\Omega_{t_4}} = \varphi_{MD}(1,p_1^A;+1,\widetilde{p}_{Mid})$, and $\varphi^5_{LT,\Omega_{t_4}} = \varphi_M(2,p_1^A)$.

As for the B framework, we now consider the case $b_{t_3} = [21]$ with the following information set, $\Omega_{t_3} = [21,\varphi_L(2,p_1^A),\varphi_L(1,p_1^B)]$. The expected state of the PDP is:

$$\widetilde{PDP}_{t_3} = \begin{cases} +1 \;\; with\ prob = \frac{\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})}{\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B)} \\ 0 \;\; with\ prob = \frac{\Pr_{t_2} \varphi_L(1,p_1^B)}{\Pr_{t_2} \varphi_{LD}(1,p_1^B;+1,\widetilde{p}_{Mid})+\Pr_{t_2} \varphi_L(1,p_1^B)} \end{cases} . \tag{49}$$

Compared to the B framework, the large trader has now additional strategies. He can combine market and dark orders:

$$\begin{aligned} \pi^e_{t_3}[\varphi_{MD}(1,p_1^A;+1,\widetilde{p}_{Mid})] = \;& (\beta_{t_3}v - p_1^A) + \Pr(\widetilde{PDP}_{t_3} = +1)(\beta_{t_3}v - \tfrac{p_1^A+p_1^B}{2}) \\ & \tfrac{1}{2}\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid}) \,|\, \Omega_{t_4}) + \Pr(\widetilde{PDP}_{t_3}{=}0)\tfrac{1}{2}[(\beta_{t_3}v - \tfrac{p_1^A+p_1^B}{2}) \\ & \Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4}) + (\beta_{t_3}v - \tfrac{p_1^A+p_2^B}{2})\Pr_{t_4}(\varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})|\,\Omega_{t_4})] , \end{aligned} \tag{50}$$

where $\Omega_{t_4} = [11,\varphi_L(2,p_1^A),\varphi_L(1,p_1^B),\varphi_M(1,p_1^A)]$;

$$\begin{aligned} \pi^e_{t_3}[\varphi_{MD}(1,p_1^B;-1,\widetilde{p}_{Mid})] = \;& (p_1^B - \beta_{t_3}v) + \Pr(\widetilde{PDP}_{t_3} = +1)\tfrac{1}{2}\{(\tfrac{p_1^A+p_2^B}{2} - \beta_{t_3}v) \\ & [1 + (1 - \Pr_{t_4}(\varphi_M(2,p_1^A)|\,\Omega_{t_4})] + (\tfrac{p_2^A+p_2^B}{2} - \beta_{t_3}v)\Pr_{t_4}(\varphi_M(2,p_1^A)|\,\Omega_{t_4})\} \\ & + \Pr(\widetilde{PDP}_{t_3}{=}0)\tfrac{1}{2}(\tfrac{p_1^A+p_2^B}{2} - \beta_{t_3}v) \\ & [\Pr_{t_4}(\varphi_D(+1,\widetilde{p}_{Mid}) \,|\, \Omega_{t_4}) + \Pr_{t_4}(\varphi_{MD}(1,p_1^A;+1,\widetilde{p}_{Mid}) \,|\, \Omega_{t_4})] , \end{aligned} \tag{51}$$

where $\Omega_{t_4} = [20,\varphi_L(2,p_1^A),\varphi_L(1,p_1^B),\varphi_M(1,p_1^B)]$.

Alternatively, he can combine a limit and dark order on the bid side of the market:

$$\begin{aligned}\pi^e_{t_3}[\varphi_{LD}(1,p^B_1;+1,\widetilde{p}_{Mid})]= \quad & (\beta_{t_3}v-p^B_1)\tfrac{1}{2}\Pr_{t_4}(\varphi_M(2,p^B_1)\,|\,\Omega_{t_4})+(\beta_{t_3}v-\tfrac{p^A_1+p^B_1}{2})\tfrac{1}{2}\{\Pr(\widetilde{PDP}_{t_3}{=}0)\\ & [\Pr_{t_4}(\varphi_{MD}(1,p^B_1;-1,\widetilde{p}_{Mid})|\,\Omega_{t_4})+\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4})]\\ & +\Pr(\widetilde{PDP}_{t_3}=+1)\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4})\}\ ,\end{aligned} \tag{52}$$

where $\Omega_{t_4}=[22,\varphi_L(2,p^A_1),\varphi_L(1,p^B_1),\varphi_L(1,p^B_1)]$.

Finally, they can submit a pure dark pool order to sell or buy:

$$\begin{aligned}\pi^e_{t_3}[\varphi_D(-2,\widetilde{p}_{Mid})]= \quad & \Pr(\widetilde{PDP}_{t_3}{=}+1)\tfrac{1}{2}\{(\tfrac{p^A_1+p^B_1}{2}-\beta_{t_3}v)[2(\Pr_{t_4}(\varphi_{MD}(1,p^A_1;+1,\widetilde{p}_{Mid})|\,\Omega_{t_4})\\ & +\Pr_{t_4}(\varphi_D(+2,\widetilde{p}_{Mid})|\,\Omega_{t_4}))+\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4})+\Pr_{t_4}(\varphi_M(1,p^A_1)|\,\Omega_{t_4})\\ & +\Pr_{t_4}(\varphi(0)|\,\Omega_{t_4})+\Pr_{t_4}(\varphi^s(0)|\,\Omega_{t_4})]+(\tfrac{p^A_1+p^B_2}{2}-\beta_{t_3}v)[\Pr_{t_4}(\varphi_M(1,p^B_1)|\,\Omega_{t_4})\\ & +\Pr_{t_4}(\varphi_M(1,p^B)|\,\Omega_{t_4})+\Pr_{t_4}(\varphi_{MD}(1,p^B_1;-1,\widetilde{p}_{Mid})|\,\Omega_{t_4})]\\ & +(\tfrac{p^A_2+p^B_1}{2}-\beta_{t_3}v)\Pr_{t_4}(\varphi_M(2,p^A_1)|\,\Omega_{t_4})\}+\Pr(\widetilde{PDP}_{t_3}{=}0)\tfrac{1}{2}(\tfrac{p^A_1+p^B_1}{2}-\beta_{t_3}v)\\ & [\Pr_{t_4}(\varphi_{MD}(1,p^A_1;+1,\widetilde{p}_{Mid})|\,\Omega_{t_4})+2\Pr_{t_4}(\varphi_D(+2,\widetilde{p}_{Mid})|\,\Omega_{t_4})]\end{aligned} \tag{53}$$

$$\begin{aligned}\pi^e_{t_3}[\varphi_D(+2,\widetilde{p}_{Mid})]= \quad & \Pr(\widetilde{PDP}_{t_3}{=}+1)\tfrac{1}{2}(\beta_{t_3}v-\tfrac{p^A_1+p^B_1}{2})\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4})\\ & +\Pr(\widetilde{PDP}_{t_3}{=}0)\tfrac{1}{2}[(\beta_{t_3}v-\tfrac{p^A_1+p^B_2}{2})\Pr_{t_4}(\varphi_{MD}(1,p^B_1;-1,\widetilde{p}_{Mid})|\,\Omega_{t_4})\\ & +(\beta_{t_3}v-\tfrac{p^A_1+p^B_1}{2})2\Pr_{t_4}(\varphi_D(-2,\widetilde{p}_{Mid})|\,\Omega_{t_4})]\ ,\end{aligned} \tag{54}$$

where in both cases $\Omega_{t_4}=[21,\varphi_L(2,p^A_1),\varphi_L(1,p^B_1),\varphi(0)]$.

L&C . We now consider the L&C framework, where the dark pool executes at each trading round. By comparing Eq. (30) and (17), we observe that market orders are always dominated by IOC dark pool orders, unless the probability that the order executes on the CDP is zero:

$$\pi^e_t[\varphi_D(\pm j,p_{Mid,t},p^z_i)]\geq\pi_t[\varphi_M(j,p^z_i)]\ . \tag{55}$$

As an example, we consider again the book $b_{t_4}=[20]$ with the following information set, $\Omega_{t_4}=[20,\varphi_L(2,p^A_1),\varphi_L(1,p^B_1),\varphi_M(1,p^B_1)]$. In this case when at t_3 a trader observes $\varphi_M(1,p^B_1)$, he doesn't know whether a small trader submitted a 1-unit market order, or whether a large trader submitted a market order combined with a dark pool order or a 2-unit IOC dark pool order that was partially executed on the CDP. Therefore, traders coming at t_4 Bayesian update their expectations on the state of the dark pool by using an extended version of Eq. (44) which takes into account that $\widetilde{CDP}_{t_4}=0$ with the additional probability $\Pr_{t_2}\varphi_{LD}(1,p^B_1;+1,\widetilde{p}_{Mid,t})\Pr_{t_3}\varphi_D(-2,p_{Mid,t_3},p^B_1)$. Traders coming at t_3 instead face the same uncertainty as for the $L\&P$ framework.

Compared to the $L\&P$ framework, the large trader can now submit a IOC dark pool order with the following profits:

$$\pi^e_{t_4}[\varphi_D(+2,p_{Mid,t_4},p^A_1)]=2\beta_{t_4}v-p^A_1[1+\Pr(\widetilde{CDP}_{t_4}=+1,0)]-\tfrac{p^A_1+p^B_2}{2}\Pr(\widetilde{CDP}_{t_4}=-1) \tag{56}$$

$$\pi^e_{t_4}[\varphi_D(-2,p_{Mid,t_4},p^B_2)]=p^B_2[1+\Pr(\widetilde{CDP}_{t_4}=-1,0)]+\tfrac{p^A_1+p^B_2}{2}\Pr(\widetilde{CDP}_{t_4}=+1)-2\beta_{t_4}v \tag{57}$$

To determine the equilibrium strategies $\varphi^n_{a,\Omega_{t_4}}$ at t_4 for $n\in N_{t_4}$, the model has again to be solved up to period t_1. We anticipate that because in equilibrium $\Pr_{t_2}\varphi_{LD}(1,p^B_1;+1,\widetilde{p}_{Mid,t})=0$ and $\Pr_{t_3}\varphi_{MD}(1,p^B_1;-1,\widetilde{p}_{Mid,t})>0$,

$N_{t_4} = 4$ and the strategies of the large trader are as follows: $\varphi^1_{LT,\Omega_{t_4}} = \varphi_M(2, p^B_2)$, $\varphi^2_{LT,\Omega_{t_4}} = \varphi(0)$, $\varphi^3_{LT,\Omega_{t_4}} = \varphi_D(+1, p_{Mid,t_4})$, $\varphi^4_{LT,\Omega_{t_4}} = \varphi_D(+2, p_{Mid,t_4}, p^A_1)$.

As for the other two frameworks, we now consider the case $b_{t_3} = [21]$ with the following information set, $\underset{\sim}{\Omega_{t_3}} = [20, \varphi_L(2, p^A_1), \varphi_L(1, p^B_1)]$. We refer to Eq. (49) for the probabilities of the expected state of the CDP, $\widetilde{CDP}_{t_3} = \{0, +1\}$. The large trader can now submit a IOC dark pool order to sell (the execution probability on the CDP of a IOC dark order to buy is zero, so traders never select this strategy):

$$\pi^e_{t_3}[\varphi_D(-2, p_{Mid,t_3}, p^B)] = p^B_1 + p^B_2 \Pr(\widetilde{CDP}_{t_3} = 0) + \frac{p^A_1 + p^B_1}{2} \Pr(\widetilde{CDP}_{t_3} = +1) - 2\beta_{t_3} v \tag{58}$$

Notice that the profits of combined market and dark orders, limit and dark orders and pure dark pool orders are similar to the ones presented for the $L\&P$ framework, the main differences being that the order is executed as soon as there is available liquidity on the CDP and not at the end of the trading game, and the presence of IOC dark pool orders. We provide one example:

$$\begin{aligned}\pi^e_{t_3}[\varphi_D(-2, \widetilde{p}_{Mid,t})] = \; & \Pr(\widetilde{CDP}_{t_3} = +1)(\tfrac{p^A_1 + p^B_1}{2} - \beta_{t_3} v)\{1 + \tfrac{1}{2}[\Pr_{t_4}(\varphi_D(+2, p_{Mid,t_4}, p^A_1)|\,\Omega_{t_4}) \\ & + \Pr_{t_4}(\varphi_D(+2, p_{Mid,t_4})|\,\Omega_{t_4})]\} + \Pr(\widetilde{CDP}_{t_3} = 0)(\tfrac{p^A_1 + p^B_1}{2} - \beta_{t_3} v) \\ & \tfrac{1}{2}[2 \Pr_{t_4}(\varphi_D(+2, p_{Mid,t_4}, p^A_1)|\,\Omega_{t_4}) + 2 \Pr_{t_4}(\varphi_D(+2, p_{Mid,t_4})|\,\Omega_{t_4})]\} \; ,\end{aligned} \tag{59}$$

where, as for the $L\&P$, $\Omega_{t_4} = [21, \varphi_L(2, p^A_1), \varphi_L(1, p^B_1), \varphi(0)]$.

OM . Results for OM presented in Figure 5 are derived by straightforward comparison of the equilibrium strategies for the three frameworks: B, $L\&P$ and $L\&C$. In Figures A1-A6 we provide plots at t_2 for the large trader's profits as a function of β, for both the $L\&P$ and $L\&C$ frameworks. Following the exposition in the main text, we focus on selling strategies. Each figure provides a graphical representation of the traders' optimization problem. Figure A1 shows how the introduction of a PDP changes the optimal order submission strategies of large traders by crowding out both market and limit orders, and generating OM. Consider first OM in the $L\&P$: compare Figures A1 and A3 for the effect of market depth, A1 and A5 for the effect of spread, and A1 and A7 for the tick size (to economize space we only report results for the book $b_{t_2} = [20]$). For the $L\&C$, compare instead Figures A2 and A4, A2 and A6, and A2 and A8 respectively.

TC and VC. Results for TC and VC presented in Figure 6 and 7, respectively, are obtained by comparing fill rates and volumes for the B, $L\&P$ and $L\&C$ frameworks, as shown in Eq. (20) and (22) respectively. As an example, we consider period t_1 of the B model and specify formulas for the estimated fill rate and volume in this period. Equilibrium strategies at t_1 for a large trader are as follows: $\varphi^1_{LT} = \varphi_M(2,p^B_2)$, $\varphi^2_{LT} = \varphi_{ML}(1, p^B_2; 1, p^A_1)$, $\varphi^3_{LT} = \varphi_L(2, p^A_1)$, $\varphi^4_{LT} = \varphi_L(2, p^B_1)$, $\varphi^5_{LT} = \varphi_{ML}(1, p^A_2; 1, p^B_1)$ and $\varphi^6_{LT} = \varphi_M(2,p^A_1)$. The ones for a small trader are: $\varphi^1_{ST} = \varphi_M(1,p^B_1)$, $\varphi^2_{ST} = \varphi_L(1, p^A_1)$, $\varphi^3_{ST} = \varphi_L(1, p^B_1)$ and $\varphi^4_{ST} = \varphi_M(1,p^A_1)$.

$$FR^B_{t_1,[22]} = \tfrac{1}{2}(\Pr_{t_1} \varphi^1_{ST} + \Pr_{t_1} \varphi^4_{ST}) + \tfrac{1}{2}(\Pr_{t_1} \varphi^1_{LT} + \Pr_{t_1} \varphi^2_{LT} + \Pr_{t_1} \varphi^5_{LT} + \Pr_{t_1} \varphi^6_{LT}) \tag{60}$$

$$V^B_{t_1,[22]} = \tfrac{1}{2}(\Pr_{t_1} \varphi^1_{ST} + \Pr_{t_1} \varphi^4_{ST}) + \tfrac{1}{2}(2 \Pr_{t_1} \varphi^1_{LT} + \Pr_{t_1} \varphi^2_{LT} + \Pr_{t_1} \varphi^5_{LT} + 2 \Pr_{t_1} \varphi^6_{LT}) \; . \tag{61}$$

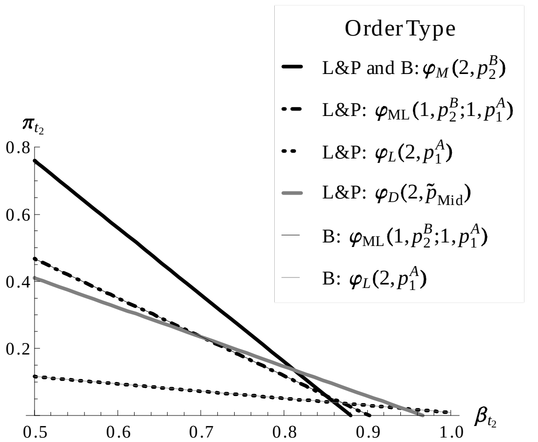

Figure A1. Order Migration on the L&P - b_{t_2}=[20]

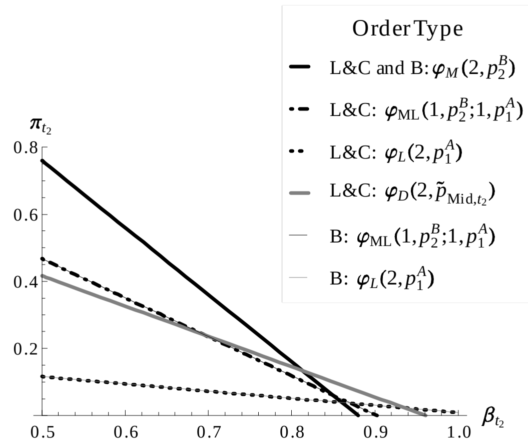

Figure A2. Order Migration on the L&C - b_{t_2}=[20]

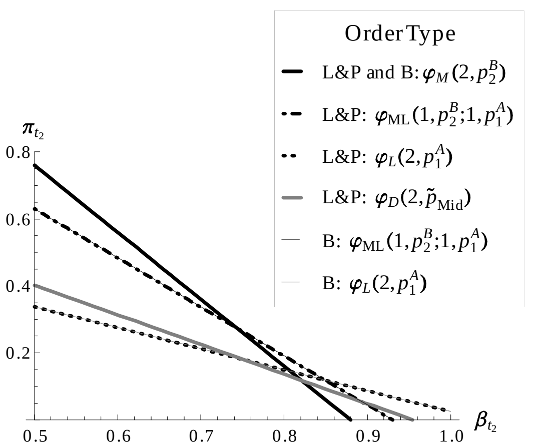

Figure A3. Order Migration on the L&P - b_{t_2}=[10]

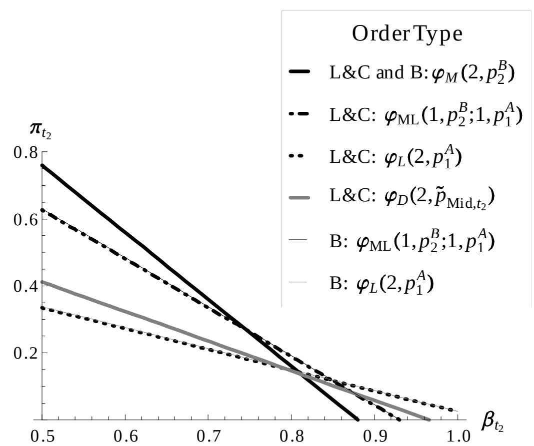

Figure A4. Order Migration on the L&C - b_{t_2}=[10]

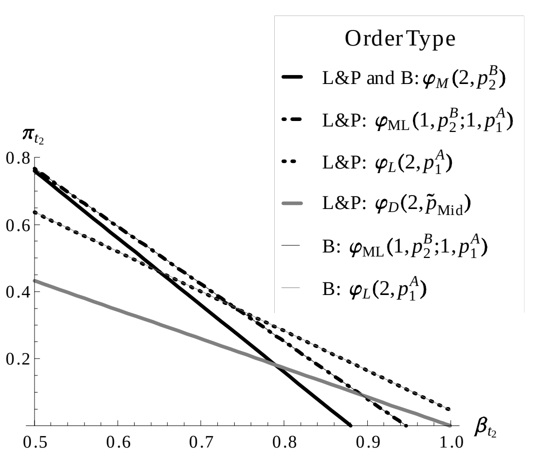

Figure A5. Order Migration on the L&P - b_{t_2}=[00]

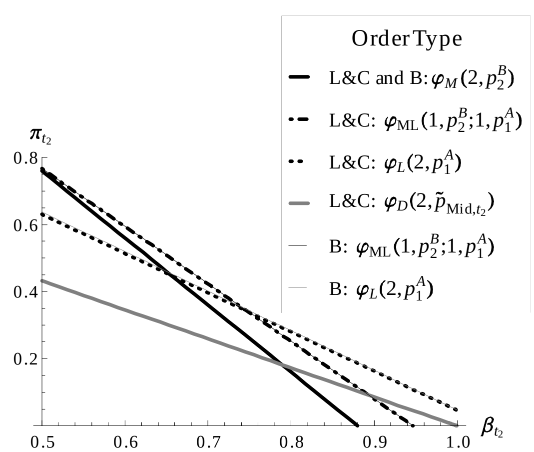

Figure A6. Order Migration on the L&C - b_{t_2}=[00]

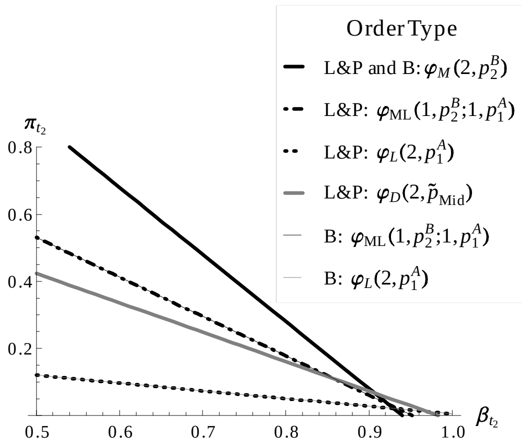

Figure A7. Order Migration on the L&P - b_{t_2}=[20] Small Tick

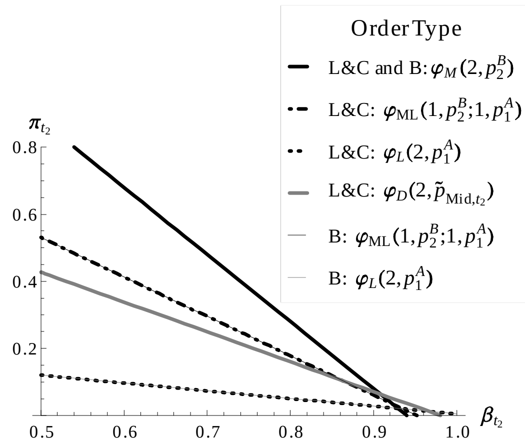

Figure A8. Order Migration on the L&C - b_{t_2}=[20] Small Tick

Proof of Proposition 2

Results for spread and depth presented in Figure 8 are obtained by comparing the two market quality measures for the B, $L\&P$ and $L\&C$ protocol. As an example, we consider again the B model and specify formulas for the estimated spread and depth at t_1. We refer to the proof of Proposition 1 for a list of the equilibrium strategies in this case.

$$\begin{aligned} S_{t_1}^B \quad = \quad & \tfrac{1}{2}[(p_2^A - p_2^B)(\Pr_{t_1} \varphi_{LT}^1 + \Pr_{t_1} \varphi_{LT}^6) + (p_1^A - p_2^B)(\Pr_{t_1} \varphi_{LT}^2 + \Pr_{t_1} \varphi_{LT}^3) \\ & + (p_2^A - p_1^B)(\Pr_{t_1} \varphi_{LT}^4 + \Pr_{t_1} \varphi_{LT}^5)] + \tfrac{1}{2}[(p_2^A - p_2^B)(\Pr_{t_1} \varphi_{ST}^1 + \Pr_{t_1} \varphi_{ST}^4) \\ & + (p_1^A - p_2^B) \Pr_{t_1} \varphi_{ST}^2 + (p_2^A - p_1^B) \Pr_{t_1} \varphi_{ST}^3] \end{aligned} \tag{62}$$

$$D_{t_1}^B = \tfrac{1}{2}[\Pr_{t_1} \varphi_{LT}^2 + \Pr_{t_1} \varphi_{LT}^5 + 2(\Pr_{t_1} \varphi_{LT}^3 + \Pr_{t_1} \varphi_{LT}^4)] + \tfrac{1}{2}(\Pr_{t_1} \varphi_{ST}^2 + \Pr_{t_1} \varphi_{ST}^3) \ . \tag{63}$$

Similar computations make it possible to derive the market quality measures for all the other cases.

Proof of Proposition 3

Results for welfare presented in Figure 9 are obtained by comparing welfare values for the large trader, the small trader and on average in the B, $L\&P$ and $L\&C$ protocol. To provide an example, we consider again the B model and specify the welfare formula at t_1 for the large trader. We refer again to the proof of Proposition 1 for a list of the equilibrium strategies in this case.

$$\begin{aligned} W_{LT,t_1}^B = & \int_0^{\beta_{t_1}^{\varphi_{LT}^1,\varphi_{LT}^2}} \pi_{t_1}(\varphi_{LT}^1) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi_{LT}^1,\varphi_{LT}^2}}^{\beta_{t_1}^{\varphi_{LT}^2,\varphi_{LT}^3}} \pi_{t_1}(\varphi_{LT}^2) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi_{LT}^2,\varphi_{LT}^3}}^{\beta_{t_1}^{\varphi_{LT}^3,\varphi_{LT}^4}} \pi_{t_1}(\varphi_{LT}^3) d\beta_{t_1} \\ & + \int_{\beta_{t_1}^{\varphi_{LT}^3,\varphi_{LT}^4}}^{\beta_{t_1}^{\varphi_{LT}^4,\varphi_{LT}^5}} \pi_{t_1}(\varphi_{LT}^4) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi_{LT}^4,\varphi_{LT}^5}}^{\beta_{t_1}^{\varphi_{LT}^5,\varphi_{LT}^6}} \pi_{t_1}(\varphi_{LT}^4) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi_{LT}^5,\varphi_{LT}^6}}^{2} \pi_{t_1}(\varphi_{LT}^6) d\beta_{t_1} \ . \end{aligned} \tag{64}$$

Similarly, we can derive welfare values for the small trader, and for the other protocols.

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Figure 1 - Dark Pools Volume: Percentage of Consolidated U.S., European and Canadian Equity Volume, December 2012. Data source: Rosenblatt Securities Inc.

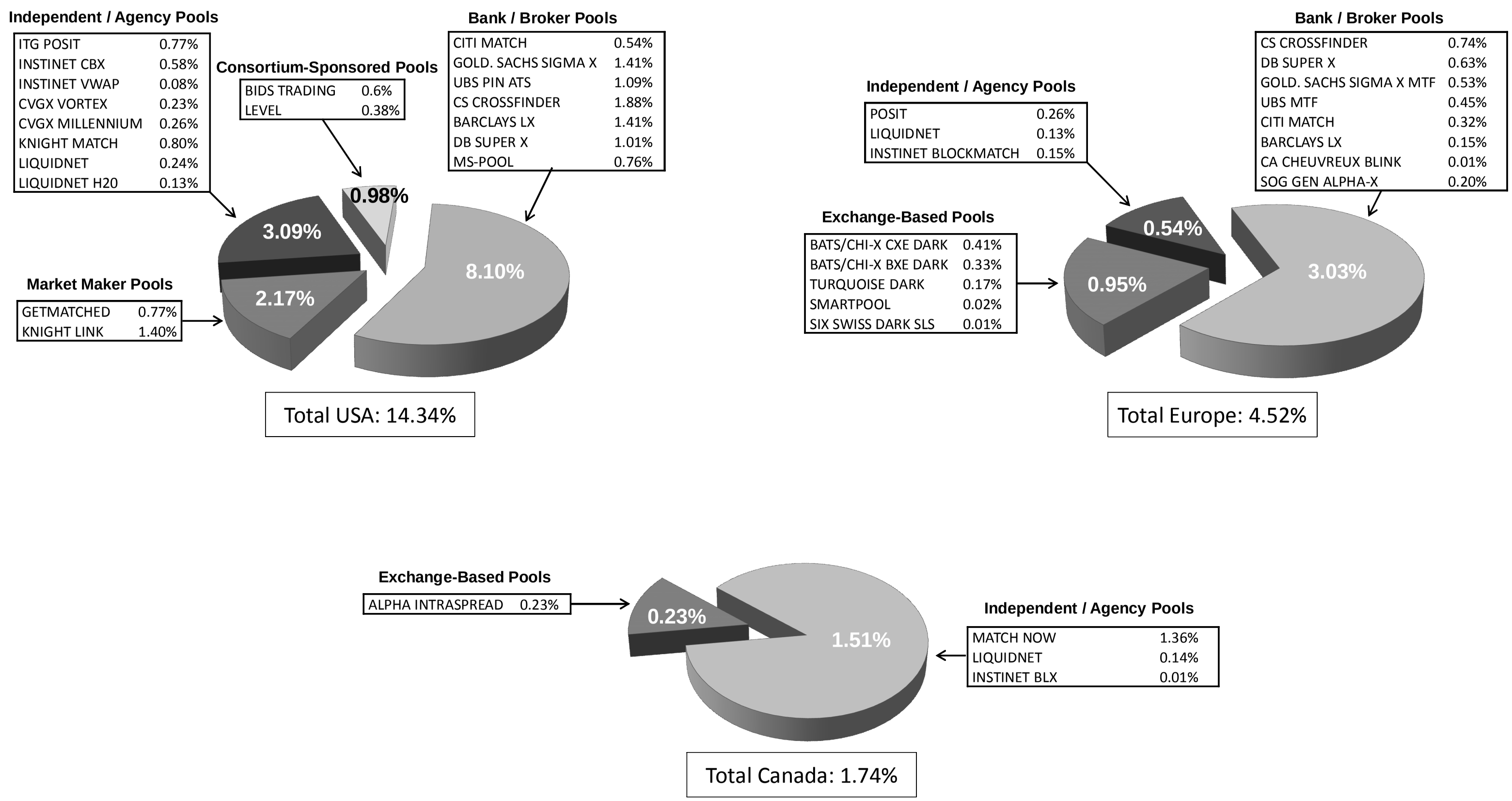


Figure 2 - Benchmark Model of Limit Order Book (B). Example of the extensive form of the game for the benchmark model when the opening book at t_1 is $b_{t_1} = [00]$, where j indicates the number of shares traded by the large trader. Only equilibrium strategies are presented.

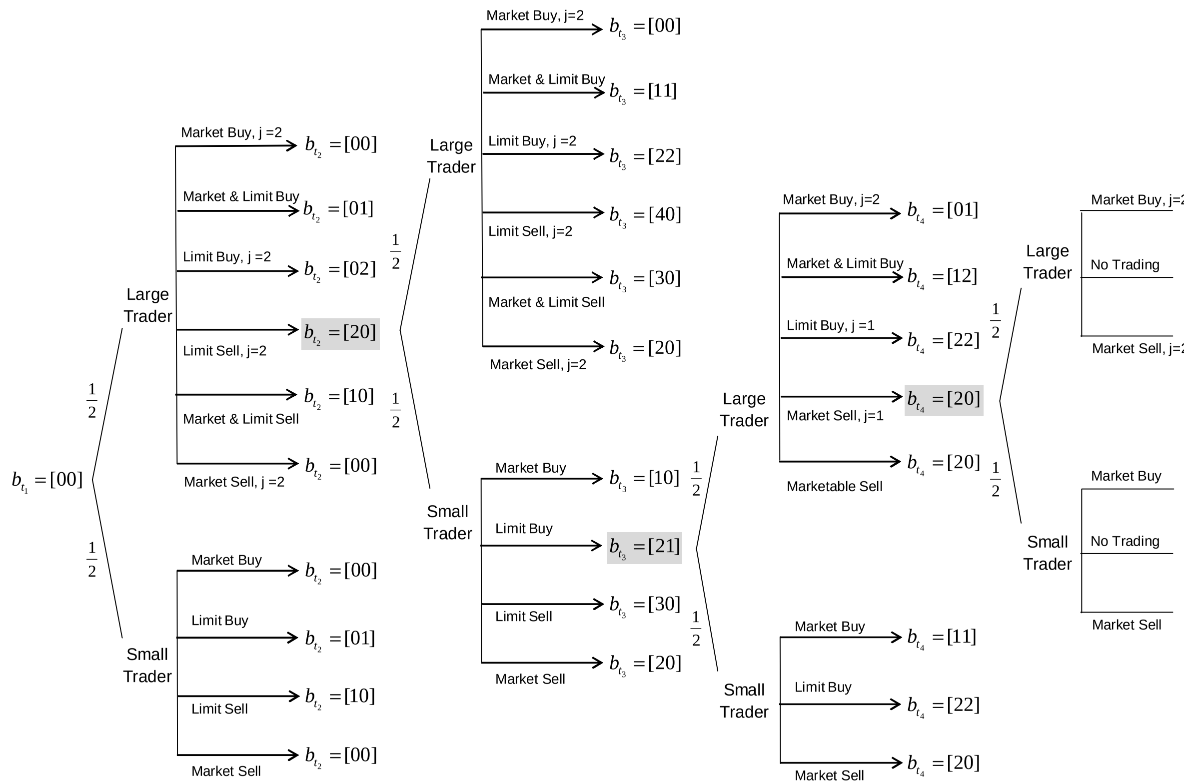


Figure 3 - Limit Order Book and Periodic Dark Pool (L&P). Example of the extensive form of the game for the model with a periodic dark pool when the opening book at t_1 is $b_{t_1} = [00]$, where j indicates the number of shares traded by the large trader. Books that belong to the same information set, and hence are undistinguishable, are inside a squared box and have the same line format. For example, $b_{t_4} = [22]$ can be observed when at t_3 either a small trader arrives and submits a limit buy, or a large trader arrives and submits a combined limit and dark order to buy. Only equilibrium strategies are presented.

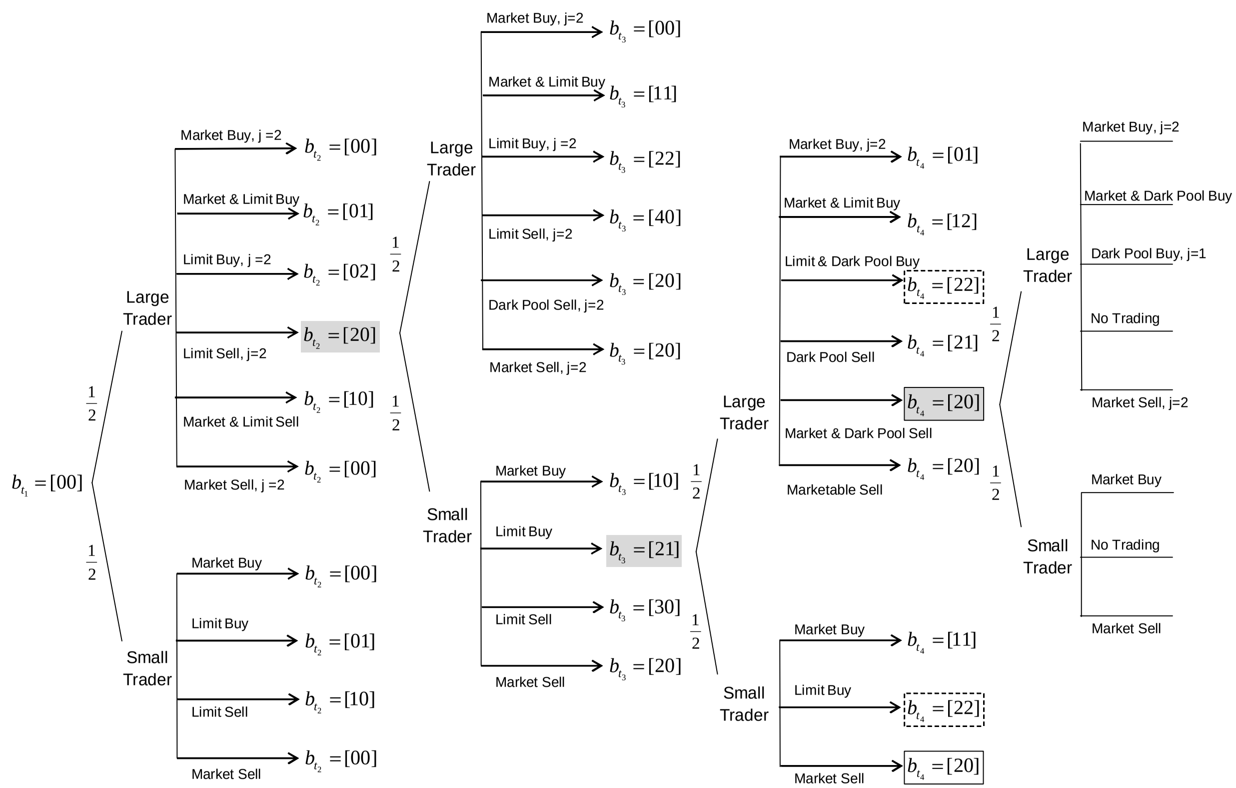


Figure 4 - Limit Order Book and Continuous Dark Pool (L&C). Example of the extensive form of the game for the model with a continuous dark pool when the opening book at t_1 is $b_{t_1} = [00]$, where j indicates the number of shares traded by the large trader. Books that belong to the same information set, and hence are undistinguishable, are inside a squared box and have the same line format. For example, $b_{t_4} = [21]$ can be observed when a large trader arrives at t_3 and submits either a dark pool order to sell or to buy. Only equilibrium strategies are presented.

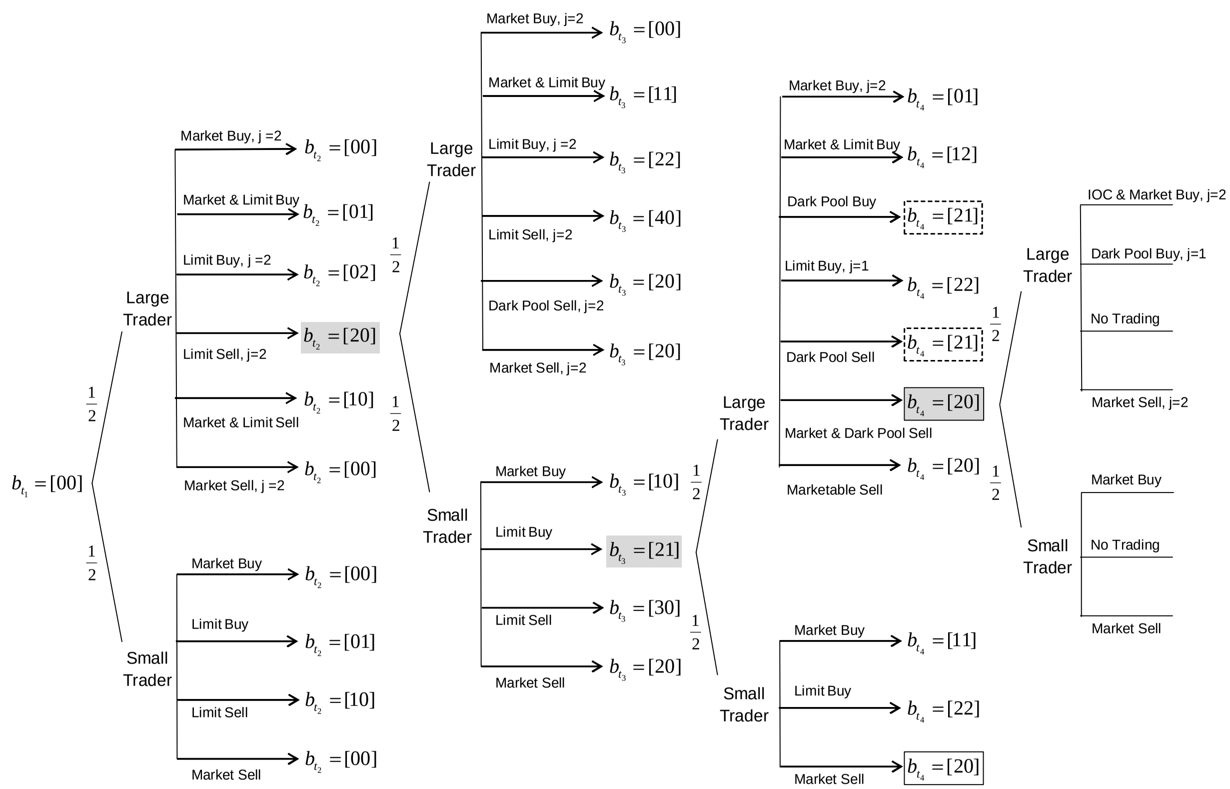
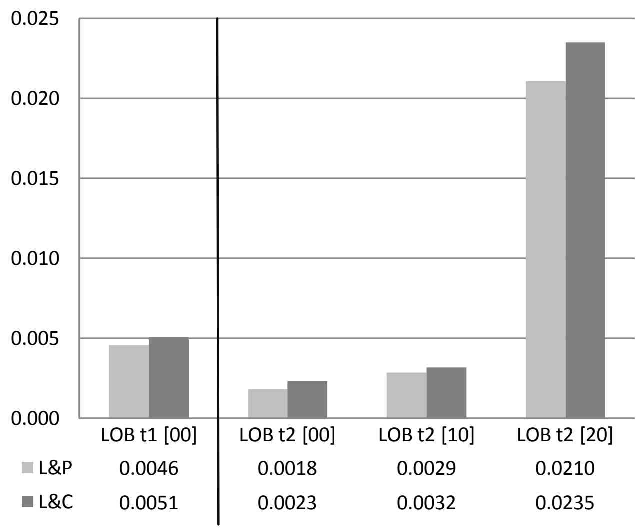


Figure 5 - Order Migration. This Figure presents results for the two frameworks, $L\&P$ and $L\&C$. The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second combines a LOB and a continuous dark pool (CDP). For each framework we report order migration (OM) which is the average probability that an order migrates to the dark pool. We report results for both the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and for the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$ (Panel A) and to $\tau = 0.04$ (Panel B).

PANEL A: $\tau = 0.08$



PANEL B: $\tau = 0.04$

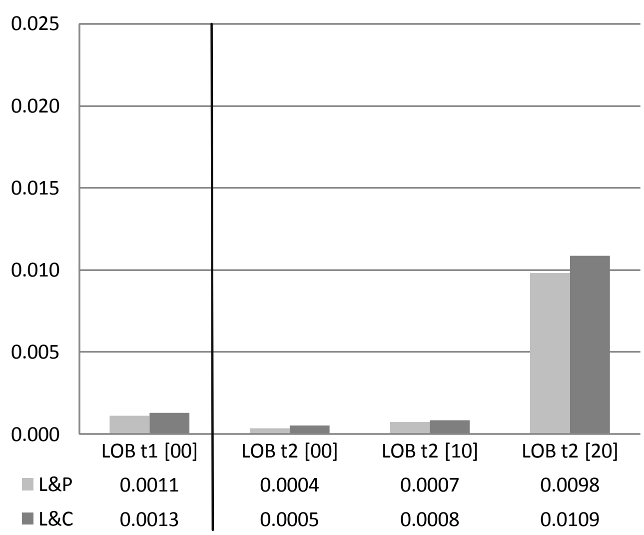
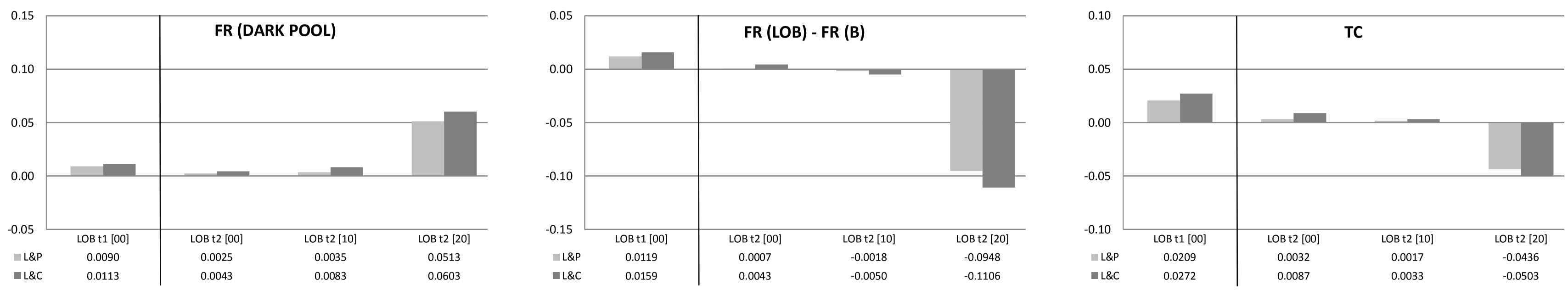


Figure 6 - Trade Creation. This Figure presents results for the two frameworks, $L\&P$ and $L\&C$. The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second combines a LOB and a continuous dark pool (CDP). For each framework we report trade creation (TC) that is the sum of two components. The first one is the average fill rate in the dark pool, $FR(DARK\ POOL)$. The second one is the difference between the average LOB fill rate in the $L\&P$ or $L\&C$, $FR(LOB)$, and the average LOB fill rate in the benchmark, $FR(B)$. We report results for both the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and for the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$ (Panel A) and to $\tau = 0.04$ (Panel B).

PANEL A: $\tau = 0.08$



PANEL B: $\tau = 0.04$

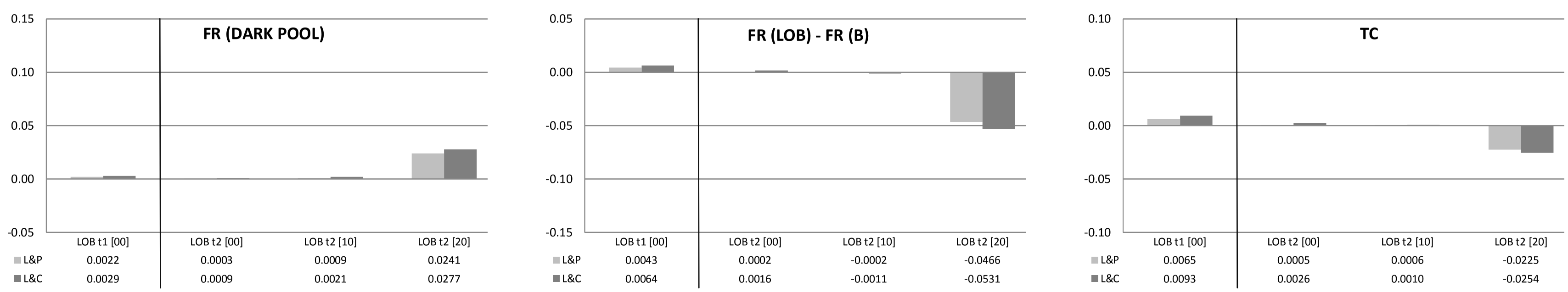
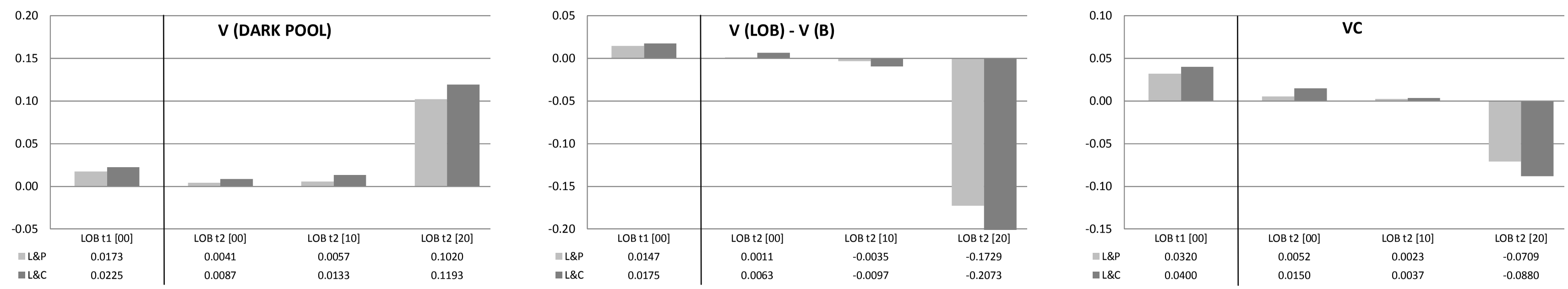


Figure 7 - Volume Creation . This Figure presents results for the two frameworks, $L\&P$ and $L\&C$. The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second combines a LOB and a continuous dark pool (CDP). For each framework we report volume creation (VC) that is the sum of two components. The first one is the average volume in the dark pool, $V(DARK\ POOL)$. The second one is the difference between the average LOB volume in the $L\&P$ or $L\&C$, $V(LOB)$, and the average LOB volume in the benchmark, $V(B)$. We report results for both the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and for the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$ (Panel A) and to $\tau = 0.04$ (Panel B).

PANEL A: $\tau = 0.08$



PANEL B: $\tau = 0.04$

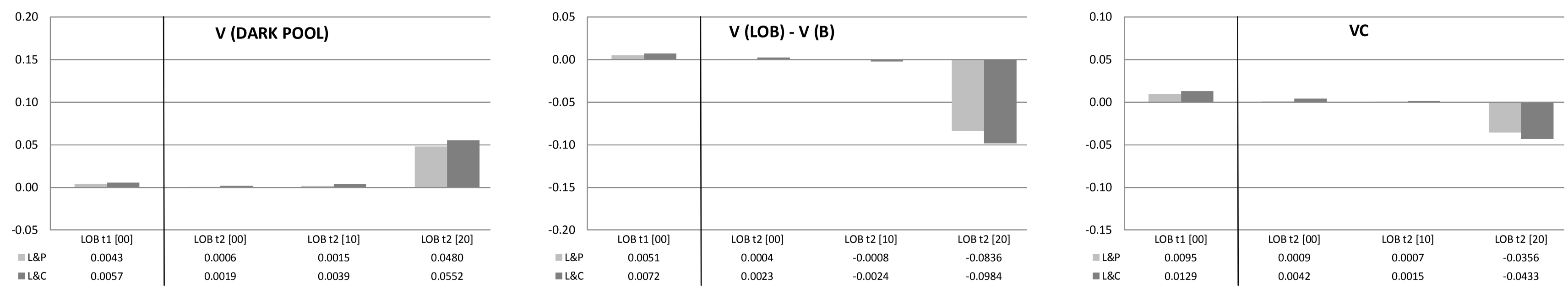
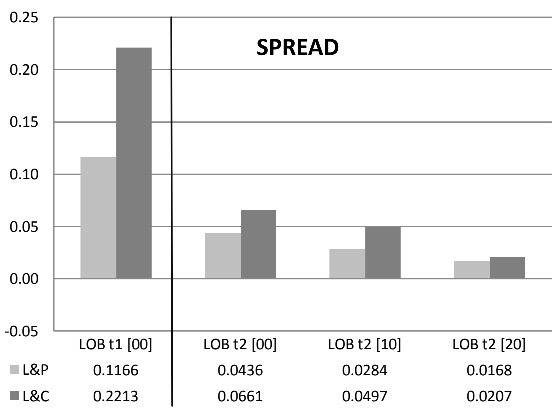
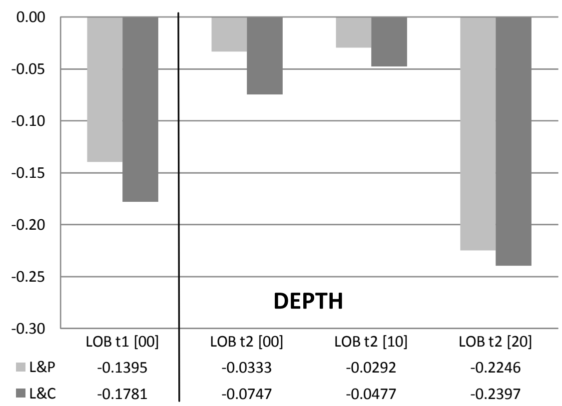


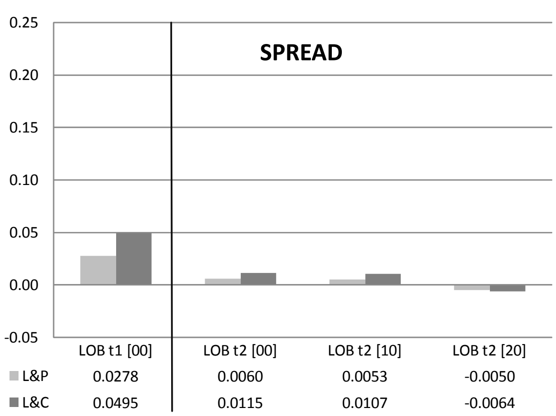
Figure 8 - Market Quality: Spread and Depth. This Figure presents results for spread and depth in the two frameworks, $L\&P$ and $L\&C$. $L\&P$ combines a limit order book (LOB) and a periodic dark pool (PDP); $L\&C$ combines a LOB and a continuous dark pool (CDP). The market quality measures are computed as the average percentage difference between their value for the $L\&P$ or $L\&C$ framework and the benchmark framework. As spread and depth are exogenous in the initial period we do not include it in the average. We report results for both the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and for the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$ (Panel A) and $\tau = 0.04$ (Panel B).

PANEL A: $\tau = 0.08$





PANEL B: $\tau = 0.04$



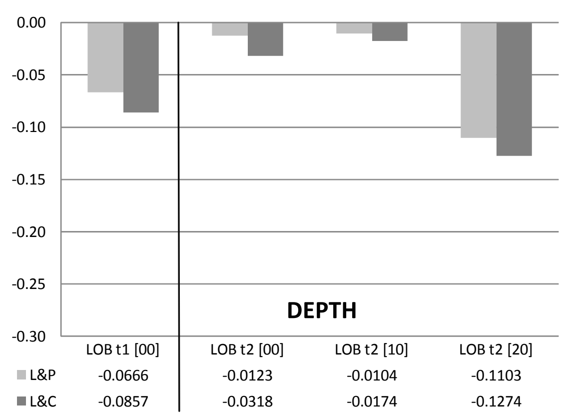
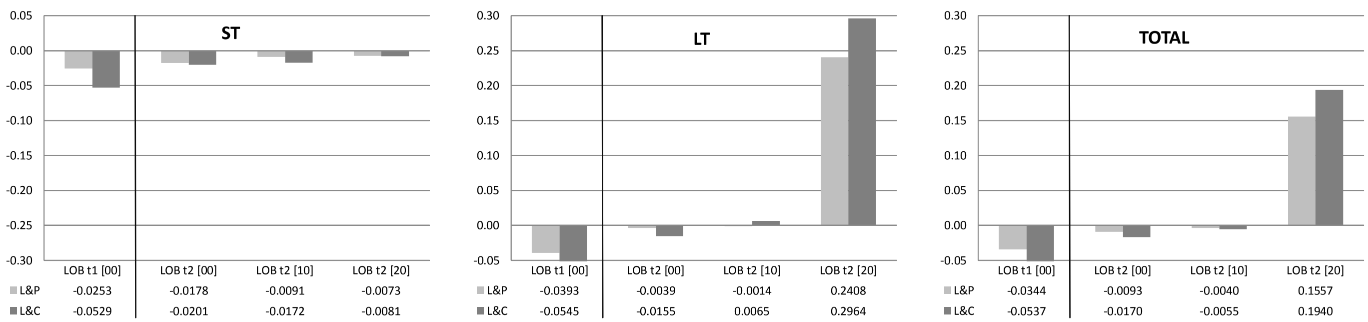


Figure 9 - Welfare. This Figure presents results for the two frameworks, $L\&P$ and $L\&C$. The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second combines a LOB and a continuous dark pool (CDP). For each framework three measures of welfare are computed as the average percentage difference across periods between their value in the $L\&P$ or $L\&C$ framework and in the benchmark framework. The three measures are: the welfare of a small trader (ST), the welfare of a large trader (LT), and total welfare. We report results for both the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and for the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$ (Panel A) and $\tau = 0.04$ (Panel B).

PANEL A: $\tau = 0.08$



PANEL B: $\tau = 0.04$

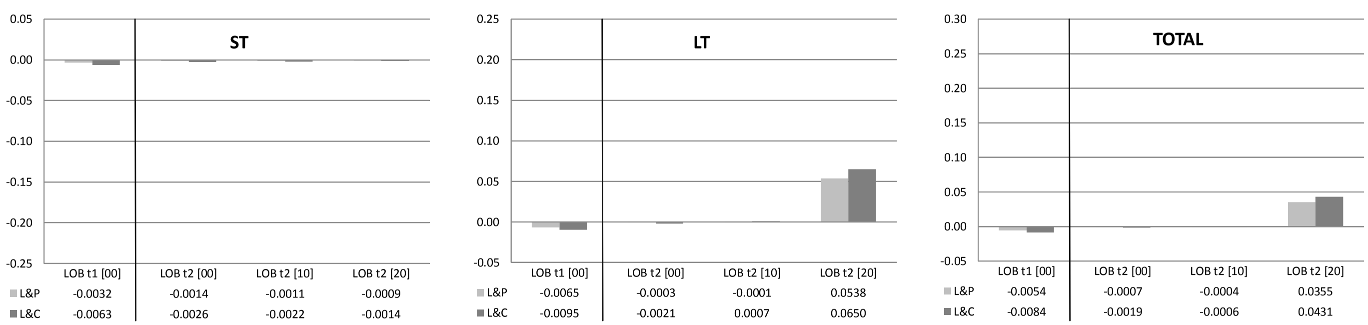


Table I: Order Submission Strategies. This Table reports the trading strategies, φ, available to large (LT) and small traders (ST) for the three different frameworks considered: a benchmark model (B) with a limit order book (LOB), and either a periodic ($L\&P$) or a continuous dark pool ($L\&C$) competing with a LOB. The LOB is characterized by a set of four prices, denoted by p_i^z, where $z = \{A, B\}$ indicates the ask or bid side of the market, and $i = \{1, 2\}$ the level on the price grid. In the $L\&P$, $\widetilde{p}_{Mid}$ indicates the spread midquote on the LOB prevailing at the end of period t_4, when the dark pool crosses orders. In the $L\&C$, $\widetilde{p}_{Mid,t}$ indicates the spread midquote on the LOB prevailing in period t. IOC indicates Immediate-or-Cancel orders. The LT trades up to 2 shares, $j = \{0, 1, 2\}$, while the ST trades up to 1 share.

Strategies LT (B)	**Notation**	**Strategies ST (B – $L\&P$ – $L\&C$)**	**Notation**
Market order of j shares	$\varphi_M(j, p_i^z)$	Market order of 1 share	$\varphi_M(1, p_i^z)$
Limit order of j shares	$\varphi_L(j, p_1^z)$	Limit order of 1 share	$\varphi_L(1, p_1^z)$
No trading	$\varphi(0)$	No trading	$\varphi(0)$
Market & limit order of 1 share each	$\varphi_{ML}(1, p_i^z; 1, p_1^z)$		
Marketable order of 2 shares	$\varphi_M(2, p^z)$		
Additional Strategies LT ($L\&P$)			
Market & dark pool order of 1 share each	$\varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid})$		
Limit & dark pool order of 1 share each	$\varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid})$		
Dark pool order of j shares	$\varphi_D(\pm j, \widetilde{p}_{Mid})$		
Additional Strategies LT ($L\&C$)			
Market & dark pool order of 1 share each	$\varphi_{MD}(1, p_i^z; \pm 1, \widetilde{p}_{Mid,t})$		
Limit & dark pool order of 1 share each	$\varphi_{LD}(1, p_1^z; \pm 1, \widetilde{p}_{Mid,t})$		
Dark pool order of j shares	$\varphi_D(\pm j, \widetilde{p}_{Mid,t})$		
IOC on dark pool or market order of j shares	$\varphi_{DM}(\pm j, \widetilde{p}_{Mid,t}, p_i^z)$		

Table II: Order Submission Probabilities at t_1. This Table reports the submission probabilities of large (LT) and small traders (ST) for the orders listed in column 1 for the benchmark framework (B), for the model with a limit order book and a periodic dark pool ($L\&P$) and for the model with a limit order book and a continuous dark pool ($L\&C$). We consider the t_1 equilibrium strategies from the four-period model that opens with an empty book at t_1, $b_{t_1} = [00]$, and the t_2 equilibrium strategies from the three-period models that open at t_2 according to the three equilibrium opening books (at t_2) from the four-period model, and that differ for the number of shares at the first level of the ask side of the book: $b_{t_2} = [00]$, $b_{t_2} = [10]$, and $b_{t_2} = [20]$. Results are computed assuming that the tick size is equal to $\tau = 0.08$.

PANEL A - ST	$b_{t_1} = [00]$			$b_{t_2} = [00]$			$b_{t_2} = [10]$			$b_{t_2} = [20]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M(1, p_2^B)$	0.0399	0.0410	0.0429	0.2376	0.2384	0.2385	0.3706	0.3709	0.3724	0.4245	0.4245	0.4245
$\varphi_L(1, p_1^A)$	0.4601	0.4590	0.4571	0.2624	0.2616	0.2615	0.1250	0.1247	0.1231	0.0628	0.0628	0.0628
$\varphi_L(1, p_1^B)$	0.4601	0.4590	0.4571	0.2624	0.2616	0.2615	0.1306	0.1306	0.1305	0.1406	0.1377	0.1386
$\varphi_M(1, p_1^A)$							0.3738	0.3738	0.3741	0.3721	0.3750	0.3741
$\varphi_M(1, p_2^A)$	0.0399	0.0410	0.0429	0.2376	0.2384	0.2385						

PANEL B - LT	$b_{t_1} = [00]$			$b_{t_2} = [00]$			$b_{t_2} = [10]$			$b_{t_2} = [20]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M(2, p_2^B)$	0.0399	0.0410	0.0429	0.2376	0.2384	0.2385	0.3706	0.3709	0.3724	0.4245	0.4061	0.4067
$\varphi_{ML}(1, p_2^B; 1, p_1^A)$	0.2686	0.2778	0.2780	0.1330	0.1324	0.1354	0.0539	0.0536	0.0521	0.0110		
$\varphi_D(-2, \widetilde{p}_{Mid})$											0.0652	
$\varphi_D(-2, \widetilde{p}_{Mid,t_2})$												0.0614
$\varphi_L(2, p_1^A)$	0.1915	0.1812	0.1791	0.1294	0.1291	0.1261	0.0692	0.0693	0.0692	0.0505	0.0147	0.0181
$\varphi_L(2, p_1^B)$	0.1915	0.1812	0.1791	0.1294	0.1291	0.1261	0.0639	0.0623	0.0624	0.0714	0.0713	0.0698
$\varphi_{ML}(1, p_1^A; 1, p_1^B)$							0.2048	0.2054	0.2054	0.0687	0.0689	0.0700
$\varphi_{ML}(1, p_2^A; 1, p_1^B)$	0.2686	0.2778	0.2780	0.1330	0.1324	0.1354						
$\varphi_M(2, p_1^A)$										0.3738	0.3738	0.3741
$\varphi_M(2, p^A)$							0.2376	0.2384	0.2385			
$\varphi_M(2, p_2^A)$	0.0399	0.0410	0.0429	0.2376	0.2384	0.2385						