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Orthogonal Decompositions in Hilbert A-Modules*

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Abstract

Pre-Hilbert A-modules are a natural generalization of inner product spaces in which the scalars are allowed to be from an arbitrary algebra. In this perspective, submodules are the generalization of vector subspaces. The notion of orthogonality generalizes in an obvious way too. In this paper, we provide necessary and sufficient topological conditions for a submodule to be orthogonally complemented. We present four applications of our results. The most important ones are Doob's and Kunita-Watanabe's decompositions for conditionally square-integrable processes. They are obtained as orthogonal decomposition results carried out in an opportune pre-Hilbert A-module. Second, we show that a version of Stricker's Lemma can be also derived as a corollary of our results. Finally, we provide a version of the Koopman-von-Neumann decomposition theorem for a specific pre-Hilbert module which is useful in Ergodic Theory.

1 Introduction

Pre-Hilbert A-modules are to algebras as inner product spaces are to the real/complex field. In fact, they can be defined by simply replacing in the definition of inner product space the real/complex field with an algebra A (for example, of functions). In this paper, compared to the vast majority of the literature, we focus on the case A is a *real* algebra. We assume that A is an Archimedean f-algebra with multiplicative unit and we provide topological conditions that guarantee that a submodule is (orthogonally) complemented.[1] In this work, the most important examples of Archimedean f-algebras with unit will be the following three: $\mathcal{L}^{\infty}(\Omega, \mathcal{G}, P)$, $\mathcal{L}^{0}(\Omega, \mathcal{G}, P)$, and the space of predictable processes.

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[1]Since we will only study *orthogonal* complementation, we will just refer to it as complementation.

More formally, given a pre-Hilbert A-module $(H,+,\cdot,\langle\ ,\ \rangle_H)$ and a submodule $M \subseteq H$, we define

$$M^{\perp} = \{y \in H : \langle x, y\rangle_H = 0 \quad \forall x \in M\}.$$

In this paper, we provide conditions on A and topological conditions on M that guarantee that M is such that

$$H = M \oplus M^{\perp}.$$

As in the standard case of Hilbert spaces, we will see that the problem of M being complemented is strictly connected to the problem of M being self-dual and, in studying complementation, we will also provide a new topological condition which is equivalent to self-duality (see Theorem 7). We conclude the paper by providing four applications of our results. In particular, we show how versions of different famous decomposition results can be better understood once framed within the Hilbert module framework (see Proposition 4 which is a version of Stricker's Lemma, Theorem 4 which generalizes the Koopman-von-Neumann decomposition result to modules, and Corollaries 3 and 4 which correspond, respectively, to the Doob's and Kunita-Watanabe's decomposition).

Related literature The literature on complementation in pre-Hilbert A-modules (similarly to the literature on self-duality) can be roughly divided in two main streams. The first one introduced the notion of Hilbert A-modules and considers *complex* C^*-algebras A. The second one focuses on a particular algebra of functions, namely, $\mathcal{L}^0(\mathcal{G}) = \mathcal{L}^0(\Omega,\mathcal{G},P)$ (either complex or real). On the one hand, the notion of pre-Hilbert A-module was introduced by Kaplansky [21]. Kaplansky [21] considers Hilbert modules over commutative (complex) AW^*-algebras A with unit and shows that a self-dual submodule is always complemented [21, Theorem 3].[2] Frank [13, Theorem 2.8] shows that for a generic C^*-algebra self-duality of a submodule M implies M being complemented. As a consequence Frank obtains that: a) *complete* (hence, *closed*) finitely generated submodules are always complemented, b) that complete submodules are always complemented,[3] provided A is finite dimensional (see [13, Corollary 2.9] and the references therein). Finally, if A is a W^*-algebra, Frank and Troitsky [14] show that, given a subset $M \subseteq H$, $M^{\perp}$ and $M^{\perp\perp}$ are direct summands of H.[4] On the other hand, Guo [15] studies pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-modules H and shows that a submodule is complemented if and only if it is closed with respect to a particular metrizable topology.

[2] In this context, self-duality is characterized in terms of algebraic properties, rather than topological ones.

[3] Completeness, in this case, is expressed in terms of the norm $\|\ \|_H$ below.

[4] See also [24, Proposition 2.5.4 and Lemma 3.6.1], for a textbook exposition.

Our contributions In this paper, we focus on *real* commutative algebras. We provide (topological) conditions on A and H that will allow us to conclude that a submodule of a pre-Hilbert A-module H is complemented. We start by considering A to be an Arens algebra of $\mathcal{L}^\infty$ type (Definition 2). In this case, H can be suitably topologized with several norm topologies as well as with a topology induced by an invariant metric d_H. In particular, two norms stand out: $\| \|_H$ and $\| \|_m$ (Subsection 2.2). We will discuss two results. Conceptually, the first provides topological conditions that guarantee the self-duality of the submodule M, while the second provides topological conditions that generalize the well known complementation result for standard Hilbert spaces.

When A is of $\mathcal{L}^\infty$ type and H is a self-dual pre-Hilbert A-module, in Theorem 1, we show that the following conditions are equivalent:

(i) M is "weakly" closed;

(ii) $M \cap B_H$ is "weakly" closed (where B_H is the unit ball induced by $\| \|_H$);

(iii) $M \cap B_H$ is "weakly" compact;

(iv) $M \cap B_H$ is $\| \|_m$ closed;

(v) $M \cap B_H$ is d_H closed;

(vi) $H = M \oplus M^\perp$;

(vii) $M = M^{\perp\perp}$.

Condition (v) builds on a new characterization of self-duality, which is contained in Theorem 7. This theorem is an interesting result in itself. Indeed, the topology induced by the metric d_H is the only topology that can be considered in both types of pre-Hilbert A-modules, that is, the one written over algebras A of either $\mathcal{L}^\infty$ type or $\mathcal{L}^0$ type (Definition 3). In the former case, Theorem 7 shows that self-duality amounts to d_H completeness of the unit ball B_H. In the latter case, [9, Theorem 5] shows that self-duality amounts to d_H completeness of the entire space H. Thus, Theorem 7 illustrates what is the common "topological trait" of these two classes of self-dual pre-Hilbert A-modules.

When A is of $\mathcal{L}^\infty$ type and H is a self-dual pre-Hilbert A-module, in Theorem 2, we show that the following conditions are equivalent:

(i) M is "weakly" closed;

(ii) M is d_H closed;

(iii) M is $\| \|_m$ closed;

(iv) $H = M \oplus M^{\perp}$.

Note that, when $A = \mathbb{R}$, it is easy to show that $\| \|_m$ coincides with the usual norm topology and d_H induces the same topology. Thus, in this case, properties (i)-(iv) are well known to be equivalent and we can conclude that our Theorem 2 is a natural generalization of the classical complementation theorem for Hilbert spaces. Indeed, note that, in that context, self-duality is equivalent to completeness in norm.

We then move to consider A to be an f-algebra of $\mathcal{L}^0$ type (Definition 3). In this case, H can be topologized with an invariant metric d_H. When A is of $\mathcal{L}^0$ type and H is a self-dual pre-Hilbert A-module, in Theorem 3, we show that the following conditions are equivalent:

(i) M is d_H closed;

(ii) $H = M \oplus M^{\perp}$.

We are thus able to obtain Guo's complementation result ([15, Theorem 47]). The contribution to the literature of our Theorem 3 is twofold: 1) it applies to a larger class of f-algebras (see Subsection 4.3), 2) paired with Theorem 2, it highlights the connection between pre-Hilbert $\mathcal{L}^0$-modules and pre-Hilbert $\mathcal{L}^\infty$-modules. Such a connection is again provided by the metric d_H.

Finally in both cases, we pay particular attention to finitely generated submodules and their complementation (see Corollaries 1 and 2). Indeed, finitely generated submodules and their closure play a key role in Finance where pre-Hilbert modules are useful in modelling asset pricing with conditional information (see Subsection 4.1 as well as [17] and [10]). Most notably, we show that finitely generated submodules are *always* closed and complemented in a self-dual pre-Hilbert $\mathcal{L}^0$-module.

Outline of the paper Section 2 introduces the two classes of algebras A we will consider in studying the orthogonal complementation problem and contains all the useful definitions and facts concerning pre-Hilbert A-modules.

Section 3 starts by studying few natural and useful properties that come with the orthogonal complementation procedure, paying, in Subsection 3.1, particular attention to finitely generated submodules. Subsection 3.2 contains our first set of results on complementation, while Subsection 3.3 contains the second one. Section 4 deals with the application of our results.

Since this paper is a paper about orthogonal complementation, we relegate all the self-duality results to Appendices A and B.

2 Mathematical preliminaries

2.1 Algebras

We are going to consider two classes of algebras which are strictly connected: Arens algebras of $\mathcal{L}^\infty$ type and f-algebras of $\mathcal{L}^0$ type. The reader, at a first read, might want to think of the former class as the class of standard $\mathcal{L}^\infty(\Omega, \mathcal{G}, P)$ spaces and of the latter as the class of standard $\mathcal{L}^0(\Omega, \mathcal{G}, P)$ spaces.[5,6]

Arens algebras of $\mathcal{L}^\infty$ type. Given a commutative *real* Banach algebra A with multiplicative unit e, we denote by $\| \, \|_A$ the norm of A.

Definition 1 *A commutative real Banach algebra A with unit e such that*

$$\|e\|_A = 1 \text{ and } \|a\|_A^2 \leq \left\|a^2 + b^2\right\|_A \quad \forall a, b \in A$$

is called an Arens algebra.

These algebras admit a concrete representation as a space of continuous functions over a compact Hausdorff topological space and were first studied by Arens [7] and Kelley and Vaught [22].[7] The cone generated by the squares of A induces a natural order relation on A: $a \geq b$ if and only if $a-b$ belongs to the norm closure of $\{c^2 : c \in A\}$. By using standard techniques, one can show that $(A, \geq)$ is a Riesz space with *strong order* unit e. Moreover, $\| \, \|_A$ is a lattice norm such that

$$\|a\|_A = \min\{\alpha \geq 0 : |a| \leq \alpha e\} \text{ and } \left\|a^2\right\|_A = \|a\|_A^2 \qquad \forall a \in A.$$

In light of these observations, note that for each $a \geq 0$, there exists a unique $b \geq 0$ such that $b^2 = a$. From now on, we will denote such an element by $a^{\frac{1}{2}}$ or $\sqrt{a}$. If A admits a strictly positive linear functional $\bar{\varphi} : A \to \mathbb{R}$ (wlog $\bar{\varphi}(e) = 1$), then we could also consider A endowed with the invariant metric $d : A \times A \to [0, \infty)$, defined by $d(a, b) = \bar{\varphi}(|b-a| \wedge e)$ for all $a, b \in A$. It is immediate to see that

$$d(a, b) \leq \|b - a\|_A \qquad \forall a, b \in A. \tag{1}$$

We conclude by defining a particular class of Arens algebras which are isomorphic to some space $\mathcal{L}^\infty(\Omega, \mathcal{G}, P)$ (see [1, Corollary 2.2]).

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[5] Subsection 4.3 is a notable exception to this statement.

[6] In this case, the functional $\bar{\varphi}$ we are going to encounter below is nothing but the expected value: $\bar{\varphi}(a) = \int a dP$ for all $a \in \mathcal{L}^\infty(\Omega, \mathcal{G}, P)$. The element e is the function that takes constant value 1.

[7] See also [3] for a modern treatment of the subject.

Definition 2 *Let A be an Arens algebra. We say that A is of $\mathcal{L}^\infty$ type if and only if A is* Dedekind complete *and admits a* strictly positive order continuous *linear functional $\bar{\varphi}$ on A.*

f-algebras of $\mathcal{L}^0$ type. Assume that A is an Archimedean f-algebra with unit $e \neq 0$ (see Aliprantis and Burkinshaw [5, Definition 2.53]). It is well known that e is a *weak order* unit. If A is Dedekind complete and $a \geq \frac{1}{n}e$ for some $n \in \mathbb{N}$, then there exists a unique $b \in A_+$ such that $ab = e$.[8] We denote this element by a^{-1}. If $a \geq 0$ is such that there exists a^{-1} and $b \in A$, then we alternatively denote ba^{-1} by b/a. By [18, Theorem 3.9], if A is also Dedekind complete, for each $a \geq 0$, there exists a unique $b \geq 0$ such that $b^2 = a$. Also in this case, we will denote such an element by $a^{\frac{1}{2}}$ or $\sqrt{a}$. The principal ideal generated by e is the set

$$A_e = \{a \in A : \exists \alpha > 0 \text{ s.t. } |a| \leq \alpha e\}.$$

It is immediate to see that A_e is a subalgebra of A with unit e. If A is an Arens algebra, then A is an Archimedean f-algebra with unit e and $A_e = A$. If there exists a strictly positive linear functional $\bar{\varphi} : A_e \to \mathbb{R}$ (wlog $\bar{\varphi}(e) = 1$), then we can define $d : A \times A \to [0, \infty)$ by

$$d(a, b) = \bar{\varphi}(|b - a| \wedge e) \qquad \forall a, b \in A.$$

As in the case of an Arens algebra, d is an invariant metric.

Definition 3 *Let A be an Archimedean f-algebra with unit e. We say that A is an f-algebra of $\mathcal{L}^0$ type if and only if A_e is an* Arens algebra of $\mathcal{L}^\infty$ type *and A is* Dedekind complete *and d* complete.

If $b \geq 0$ and $b \in A_e$, then there exist $c \geq 0$ and $0 \leq d \in A$ such that $cb = b$, $c^2 = c$, and $bd = c$. We refer to such an element c as the basic component of b and we denote it by c_b. Similarly, we denote the element d by d_b.[9] Moreover, if $\{a_n\}_{n \in \mathbb{N}} \subseteq A$ is such that $a_n = ca_n$ for all $n \in \mathbb{N}$ and $a_n b \xrightarrow{d} l$, then $a_n \xrightarrow{d} ld$.

Some common properties. Let A be either of $\mathcal{L}^\infty$ type or $\mathcal{L}^0$ type. In both cases, d is generated by the Riesz pseudonorm $c \mapsto \bar{\varphi}(|c| \wedge e)$. By [4, Theorems 2.28 and 4.7], it is easy to prove that the topology generated by d is linear, locally solid, and Fatou. Moreover, it can be shown that:

[8] Recall that $A_+ = \{a \in A : a \geq 0\}$.

[9] Clearly, in this context, d is an element of A and is not connected to the metric d, we previously discussed. Loosely speaking, if $A = \mathcal{L}^0(\Omega, \mathcal{G}, P)$ and $b \geq 0$, then $c = 1_C$ where $C = \{\omega \in \Omega : b(\omega) > 0\}$ and $d : \Omega \to \mathbb{R}$ is such that $d(\omega) = \frac{1}{b(\omega)}$ on C and zero otherwise.

1. If $a_n \downarrow 0$ and $b \geq 0$, then $a_n b \downarrow 0$ and $a_n b \xrightarrow{d} 0$;

2. If $b \in A$ and $a_n \xrightarrow{d} a$, then $ba_n \xrightarrow{d} ba$.

3. If $\lambda > 0$ and $\{a_n\}_{n\in\mathbb{N}} \subseteq [-\lambda e, \lambda e]$ and $\{a_n\}_{n\in\mathbb{N}}$ is a d Cauchy sequence, then there exists $a \in A$ such that $a_n \xrightarrow{d} a \in [-\lambda e, \lambda e]$.[10]

2.2 Pre-Hilbert A-modules

Let A be an Archimedean f-algebra with unit e. The 4-tuple $(H, +, \cdot, \langle\ ,\ \rangle_H)$ is a pre-Hilbert A-module if and only if $(H, +, \cdot)$ is a left A-module (see [2, p. 107]) and $\langle\ ,\ \rangle_H : H \times H \to A$ is such that for each $a \in A$ and for each $x, y, z \in H$:

1. $\langle x, x\rangle_H \geq 0$, with equality if and only if $x = 0$;

2. $\langle x, y\rangle_H = \langle y, x\rangle_H$;

3. $\langle x + y, z\rangle_H = \langle x, z\rangle_H + \langle y, z\rangle_H$;

4. $\langle a \cdot x, y\rangle_H = a \langle x, y\rangle_H$.

Observe that a pre-Hilbert A-module H is endowed with a natural scalar product. In fact, we can define $\cdot^e : \mathbb{R} \times H \to H$ to be such that $\alpha \cdot^e x = (\alpha e) \cdot x$. It is immediate to check that $(H, +, \cdot^e)$ is a vector space. For each $\alpha \in \mathbb{R}$ and $x \in H$, we denote $\alpha \cdot^e x = \alpha x$.

Since the Cauchy-Schwarz inequality holds for $\langle\ ,\ \rangle_H$, that is,

$$\langle x, y\rangle_H^2 \leq \langle x, x\rangle_H \langle y, y\rangle_H \qquad \forall x, y \in H, \tag{2}$$

we say that H is self-dual if and only if for each $f : H \to A$ that satisfies:

- **A-linearity**: $f(a \cdot x + b \cdot y) = af(x) + bf(y)$ for all $a, b \in A$ and for all $x, y \in H$

- **Boundedness**: There exists $c \in A_+$ such that $f^2(x) \leq c \langle x, x\rangle_H$ for all $x \in H$

there exists (a unique) $z \in H$ such that $f(x) = \langle x, z\rangle_H$ for all $x \in H$.

Arens algebras of $\mathcal{L}^\infty$ type. Define $N : H \to A_+$ by $N(x) = \langle x, x\rangle_H^{\frac{1}{2}}$ for all $x \in H$. The function N is a vector-valued norm. Moreover, we can endow H with several

[10] Here, $[-\lambda e, \lambda e] = \{a \in A : -\lambda e \leq a \leq \lambda e\}$ and the statement follows from Nakano's theorem (see [4, Theorem 4.28]).

topologies.[11] In this paper, we will consider two topologies generated by a norm, one generated by a metric, and a weak topology. The first two norms we will consider are:

1. $\|x\|_H = \sqrt{\|\langle x, x\rangle_H\|_A}$ for all $x \in H$;
2. $\|x\|_m = \sqrt{\bar{\varphi}\left(\langle x, x\rangle_H\right)}$ for all $x \in H$.

The metric we will consider is instead defined by

$$d_H(x, y) = \bar{\varphi}\left(N(x-y) \wedge e\right) = d\left(0, N(x-y)\right) \quad \forall x, y \in H.$$

Before defining the weak topology, note that we can define a standard real valued inner product by the formula

$$\langle x, y\rangle_m = \bar{\varphi}\left(\langle x, y\rangle_H\right) \qquad \forall x, y \in H.$$

It follows that $(H, +, \cdot^e, \langle\ ,\ \rangle_m)$ is a standard pre-Hilbert space. We can finally define the weak topology $\sigma(H, S(H))$, that is, given a net $\{x_i\}_{i\in I} \subseteq H$

$$x_i \overset{\sigma(H,S(H))}{\to} x \overset{def}{\iff} \langle x_i, y\rangle_m \to \langle x, y\rangle_m \qquad \forall y \in H.$$

The relations among these topologies is the following one

$$x_n \overset{\|\ \|_H}{\to} x \implies x_n \overset{\|\ \|_m}{\to} x \implies \begin{cases} x_n \overset{d_H}{\to} x \\ x_n \overset{\sigma(H,S(H))}{\to} x \end{cases}.$$

In characterizing self-duality, $\|\ \|_H$ plays only an ancillary role. Indeed, define $B_H = \{x \in H : \|x\|_H \leq 1\}$. By [9, Theorem 3] and Theorem 7, H is self-dual if and only if B_H is $\|\ \|_m$ complete if and only if B_H is d_H complete if and only if B_H is $\sigma(H, S(H))$ compact.[12]

***f*-algebras of $\mathcal{L}^0$ type**. As before, we can define $N : H \to A_+$ by $N(x) = \langle x, x\rangle_H^{\frac{1}{2}}$ for all $x \in H$. The function N is a vector-valued norm. In this case, we can endow H with only one natural topology: the one generated by the metric

$$d_H(x, y) = \bar{\varphi}\left(N(x-y) \wedge e\right) = d\left(0, N(x-y)\right) \quad \forall x, y \in H. \tag{3}$$

[11] We refer the interested reader to Cerreia-Vioglio, Maccheroni, and Marinacci [9] for a detailed study of these and other topologies as well as all the mathematical facts reported in this part of the paper.

[12] If H is self-dual, then H is $\|\ \|_H$ complete, but, typically, it is neither $\|\ \|_m$ nor d_H complete. At the same time, if H is a self-dual pre-Hilbert A-module, where A is an f-algebra of $\mathcal{L}^0$ type, then the norm $\|\ \|_H$ cannot be defined, yet H turns out to be d_H complete. Since in these two cases, topological completeness refers to different concepts, depending on which algebra A we use, in this paper, we refrain to formally talk about Hilbert A-modules.

By [9, Theorem 5], H is self-dual if and only if H is d_H complete.

Some common properties. Let A be either of $\mathcal{L}^\infty$ type or $\mathcal{L}^0$ type. We have that the map $x \mapsto \langle x, y \rangle_H$ is $d_H - d$ continuous. Finally, if $\{a_n\}_{n\in\mathbb{N}} \subseteq A$, $x \in H$, and $a_n \xrightarrow{d} a$, then $a_n \cdot x \xrightarrow{d_H} a \cdot x$.

3 Orthogonal decompositions and projections

In this section, we recall the notion of submodule and define the one of orthogonality. As a by-product, we also define orthogonal complements. This latter concept will allow us to naturally define (modular) orthogonal projections. The rest of the section will be devoted to provide necessary and sufficient conditions for the decomposition of a pre-Hilbert A-module H in a submodule and its orthogonal complement. We will pay particular attention to finitely generated submodules.

Definition 4 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. A nonempty subset M of H is a submodule if and only if $a \cdot x + b \cdot y \in M$ for all $a, b \in A$ and for all $x, y \in M$.*

Observe that each submodule M is a vector subspace of H, yielding that $0 \in M$.

Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. Given two elements $x, y \in H$, we say that x and y are *orthogonal* if and only if $\langle x, y \rangle_H = 0$. Given a nonempty subset $M \subseteq H$, we define

$$M^\perp = \{y \in H : \langle x, y \rangle_H = 0 \quad \forall x \in M\}.$$

It is immediate to verify that $M^\perp$ is a submodule and $M \cap M^\perp \subseteq \{0\}$. We will call $M^\perp$ the orthogonal complement of M. Define also $M^{\perp\perp} = \left(M^\perp\right)^\perp$. Given a submodule $M \subseteq H$, we will say that M is (orthogonally) *complemented* if and only if $H = M \oplus M^\perp$.[13]

If M is a complemented submodule, then it induces a natural pair of projections. Indeed, note that for each $x \in H$ there exist unique $y_1 \in M$ and $y_2 \in M^\perp$ such that $x = y_1 + y_2$. Define $P_M : H \to M$ and $P_{M^\perp} : H \to M^\perp$ to be such that $P_M(x) = y_1$ and $P_{M^\perp}(x) = y_2$ for all $x \in H$. By definition, we have that

$$P_M(x) = x \quad \forall x \in M \text{ and } P_{M^\perp}(x) = x \quad \forall x \in M^\perp.$$

[13] Since M and $M^\perp$ are submodules, they are vector subspaces too, therefore $M \oplus M^\perp$ means, as usual, $M + M^\perp$ and $M \cap M^\perp = \{0\}$.

Let us denote by P either P_M or $P_{M^\perp}$. It is immediate to verify that

$$P\left(a\cdot x+b\cdot y\right)=a\cdot P\left(x\right)+b\cdot P\left(y\right)\qquad\forall a,b\in A,\forall x,y\in H.$$

Finally, we have that $P_M\left(x\right)+P_{M^\perp}\left(x\right)=x$ as well as $\left\langle P_M\left(x\right),P_{M^\perp}\left(y\right)\right\rangle_H=0$ for all $x,y\in H$.

Lemma 1 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. If M_1 and M_2 are two submodules, then the following statements are true:*

1. $(M_1+M_2)^\perp=M_1^\perp\cap M_2^\perp$.
2. $M_1\subseteq M_1^{\perp\perp}$.
3. *If $M_1\subseteq M_2$, then $M_2^\perp\subseteq M_1^\perp$.*
4. *If M_1 is self-dual, then M_1 is complemented.*
5. *If M_1 is complemented, then $M_1=M_1^{\perp\perp}$.*

Proof. Given their importance in the sequel, we only prove points 4 and 5, since the other points are proven by replicating well known techniques in Hilbert space theory.

4. Clearly, $M_1\oplus M_1^\perp\subseteq H$. As for the opposite inclusion, consider $y\in H$. Since M_1 is a submodule of H, if we define $\langle\ ,\ \rangle_{M_1}$ as the restriction of $\langle\ ,\ \rangle_H$ to $M_1\times M_1$, then $\left(M_1,+,\cdot,\langle\ ,\ \rangle_{M_1}\right)$ is a pre-Hilbert A-module. The map defined on M_1 by $x\mapsto\langle x,y\rangle_H$ is A-linear and bounded. Since M_1 is self-dual, it follows that there exists a unique $y_1\in M_1$ such that

$$\langle x,y_1\rangle_H=\langle x,y_1\rangle_{M_1}=\langle x,y\rangle_H\qquad\forall x\in M_1.$$

Define $y_2=y-y_1$. We have that

$$\langle x,y_2\rangle_H=\langle x,y-y_1\rangle_H=0\qquad\forall x\in M_1,$$

that is, $y_2\in M_1^\perp$. It is also immediate to see that $y_1+y_2=y$. Since y was arbitrarily chosen, it follows that $H\subseteq M_1\oplus M_1^\perp$.

5. Since $M_1\subseteq M_1^{\perp\perp}$, we only need to prove the opposite inclusion. By assumption, if $x\in M_1^{\perp\perp}$, then there exists $x_{M_1}\in M_1$ and $x_{M_1^\perp}\in M_1^\perp$ such that $x=x_{M_1}+x_{M_1^\perp}$. Since $M_1\subseteq M_1^{\perp\perp}$, we have that $M_1^\perp\ni x_{M_1^\perp}=x-x_{M_1}\in M_1^{\perp\perp}$. Since $M_1^\perp\cap M_1^{\perp\perp}=\{0\}$, this implies that $x-x_{M_1}=0$, that is, $x=x_{M_1}\in M_1$, proving the opposite inclusion. ■

Remark 1 The sufficiency of self-duality for a submodule M to be complemented was already noted by Frank [13], when A is a C^*-algebra. His proof relies on a result of Paschke [26]. Here, instead, we prove this fact for a different class of algebras and we rely on a more direct argument, which replicates the techniques used for standard Hilbert spaces.

In the next result, we characterize when complementation is preserved by the Minkowski's sum. Inter alia, this result will help us in providing conditions that guarantee that finitely generated submodules are complemented.

Proposition 1 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. If M_1 and M_2 are two submodules such that M_1 is complemented, the following statements are equivalent:*

(i) $P_{M_1^\perp}(M_2)$ *is complemented;*

(ii) $M_1 + M_2$ *is complemented.*

Moreover, we have that

$$M_1 + M_2 = M_1 + P_{M_1^\perp}(M_2). \tag{4}$$

Proof. First, we prove (4). Consider $x \in M_1 + P_{M_1^\perp}(M_2)$. It follows that there exists $x_i \in M_i$ for $i \in \{1, 2\}$ such that

$$\begin{aligned} x &= x_1 + P_{M_1^\perp}(x_2) = x_1 - P_{M_1}(x_2) + P_{M_1}(x_2) + P_{M_1^\perp}(x_2) \\ &= (x_1 - P_{M_1}(x_2)) + x_2 \in M_1 + M_2. \end{aligned}$$

Viceversa, consider $x \in M_1 + M_2$. There exists $x_i \in M_i$ for $i \in \{1, 2\}$ such that $x = x_1 + x_2$. We can conclude that

$$x = x_1 + x_2 = x_1 + \Big(P_{M_1}(x_2) + P_{M_1^\perp}(x_2)\Big) = (x_1 + P_{M_1}(x_2)) + P_{M_1^\perp}(x_2)$$

belongs to $M_1 + P_{M_1^\perp}(M_2)$. Define $M_3 = P_{M_1^\perp}(M_2)$ and $M = M_1 + P_{M_1^\perp}(M_2) = M_1 + M_3$. Let $y \in M_1$. Since $M_3 = P_{M_1^\perp}(M_2) \subseteq M_1^\perp$, we observe that $\langle x, y\rangle_H = 0$ for all $x \in M_3$, proving that $y \in M_3^\perp$, that is, $M_1 \subseteq M_3^\perp$. By point 1 of Lemma 1 and since $P_{M_1^\perp}\left(M_3^\perp\right) \subseteq M_1^\perp$ and $P_{M_1^\perp}\left(M_3^\perp\right) \subseteq M_3^\perp$,[14] this implies that $M^\perp = M_1^\perp \cap M_3^\perp \supseteq P_{M_1^\perp}\left(M_3^\perp\right)$.

(i) implies (ii). Since clearly $M \oplus M^\perp \subseteq H$, we only need to prove the opposite inclusion. Consider $\bar{x} \in H$. Since $M_3 \subseteq M_1^\perp$, note that

$$\begin{aligned} \bar{x} &= P_{M_1}(\bar{x}) + P_{M_1^\perp}(\bar{x}) = P_{M_1}(\bar{x}) + P_{M_1^\perp}\Big(P_{M_3}(\bar{x}) + P_{M_3^\perp}(\bar{x})\Big) \\ &= P_{M_1}(\bar{x}) + P_{M_1^\perp}(P_{M_3}(\bar{x})) + P_{M_1^\perp}\Big(P_{M_3^\perp}(\bar{x})\Big) \\ &= (P_{M_1}(\bar{x}) + P_{M_3}(\bar{x})) + P_{M_1^\perp}\Big(P_{M_3^\perp}(\bar{x})\Big), \end{aligned}$$

[14] Consider $y \in P_{M_1^\perp}\left(M_3^\perp\right)$. Let $x \in M_3^\perp$ be such that $y = P_{M_1^\perp}(x)$. Clearly, $x = P_{M_1}(x) + P_{M_1^\perp}(x)$. Since $M_1 \subseteq M_3^\perp$, it follows that $y = P_{M_1^\perp}(x) = x - P_{M_1}(x) \in M_3^\perp$.

where $P_{M_1}(\bar{x}) \in M_1$, $P_{M_3}(\bar{x}) \in M_3$, and $P_{M_1^\perp}\left(P_{M_3^\perp}(\bar{x})\right) \in M^\perp$, proving that $P_{M_1}(\bar{x}) + P_{M_3}(\bar{x}) \in M_1 + M_3 = M$ and $P_{M_1^\perp}\left(P_{M_3^\perp}(\bar{x})\right) \in M^\perp$.

(ii) implies (i). Clearly, we have $M_3 \oplus M_3^\perp \subseteq H$. Viceversa, consider $\bar{x} \in H$. Since $M_1 \subseteq M$, note that

$$
\begin{aligned}
\bar{x} = P_M(\bar{x}) + P_{M^\perp}(\bar{x}) &= P_M\left(P_{M_1}(\bar{x}) + P_{M_1^\perp}(\bar{x})\right) + P_{M^\perp}(\bar{x}) \\
&= P_M(P_{M_1}(\bar{x})) + P_M\left(P_{M_1^\perp}(\bar{x})\right) + P_{M^\perp}(\bar{x}) \\
&= P_{M_1}(\bar{x}) + P_M\left(P_{M_1^\perp}(\bar{x})\right) + P_{M^\perp}(\bar{x}) \\
&= P_M\left(P_{M_1^\perp}(\bar{x})\right) + (P_{M_1}(\bar{x}) + P_{M^\perp}(\bar{x})).
\end{aligned}
$$

Observe that $y = P_{M_1^\perp}(\bar{x}) \in M_1^\perp$. By point 1 of Lemma 1 and since $y = P_M(y) + P_{M^\perp}(y)$, we have that $P_M(y) = y - P_{M^\perp}(y) \in M_1^\perp$. Since $P_M(y) \in M = M_1 + M_3$ and $M_1 \cap M_3 = \{0\}$, it follows that $P_M\left(P_{M_1^\perp}(\bar{x})\right) = P_M(y) \in M_3$. Finally, by point 1 of Lemma 1, note that since $M_1 \subseteq M_3^\perp$, $P_{M_1}(\bar{x}) \in M_1$, and $P_{M^\perp}(\bar{x}) \in M^\perp = M_1^\perp \cap M_3^\perp$, we have that $P_{M_1}(\bar{x}) + P_{M^\perp}(\bar{x}) \in M_3^\perp$. We can conclude that $\bar{x} = P_M\left(P_{M_1^\perp}(\bar{x})\right) + (P_{M_1}(\bar{x}) + P_{M^\perp}(\bar{x}))$ where $P_M\left(P_{M_1^\perp}(\bar{x})\right) \in M_3$ and $P_{M_1}(\bar{x}) + P_{M^\perp}(\bar{x}) \in M_3^\perp$. ■

We conclude by observing that, if A is either an f-algebra of $\mathcal{L}^0$ type or an Arens algebra of $\mathcal{L}^\infty$ type, then the orthogonal complement of a nonempty subset M is necessarily d_H closed. This provides a hint for our characterization of complemented submodules. Indeed, by point 5 of Lemma 1, if M is a complemented submodule, then necessarily $M = \left(M^\perp\right)^\perp$, thus necessarily it must be d_H closed. Later on in the paper, we will show that this is also a sufficient condition for complementation.

Lemma 2 *Let A be either an f-algebra of $\mathcal{L}^0$ type or an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If $\emptyset \neq M \subseteq H$, then $M^\perp$ is d_H closed.*

Proof. Recall that the map $z \mapsto \langle z, x\rangle_H$ is A-linear and $d_H - d$ continuous for all $x \in H$. In particular, $z \mapsto \langle z, x\rangle_H$ is linear. Fix $x \in H$ and define $\ker\{x\} = \{y \in H : \langle x, y\rangle_H = 0\}$. It follows immediately that $\ker\{x\}$ is d_H closed. Since $M^\perp = \bigcap_{x\in M} \ker\{x\}$, the statement follows. ■

3.1 Finitely generated submodules

In this subsection, we recall the notion of span for (pre-Hilbert) A-modules and we provide a sufficient condition for a finitely generated submodule to be complemented (see Proposition 2).

Definition 5 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. Given a finite set $\{x_i\}_{i=1}^n \subseteq H$, we define* $\operatorname{span}_A \{x_i\}_{i=1}^n$ *as the smallest submodule of H containing $\{x_i\}_{i=1}^n$.*

Similarly to what happens for standard vector spaces (see [6, p. 31]), $\operatorname{span}_A \{x_i\}_{i=1}^n$ is well defined and is characterized as the intersection of all submodules which contain $\{x_i\}_{i=1}^n$ as well as the set of all A-linear combinations of the elements in $\{x_i\}_{i=1}^n$, that is,

$$\operatorname{span}_A \{x_i\}_{i=1}^n = \left\{ x \in H : \exists \{a_i\}_{i=1}^n \subseteq A \text{ s.t. } x = \sum_{i=1}^n a_i \cdot x_i \right\}. \tag{5}$$

Proposition 2 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. If H is such that for each $x \in H$ the submodule* $\operatorname{span}_A \{x\}$ *is complemented, then for each finite collection $\{x_i\}_{i=1}^n \subseteq H$ the submodule* $\operatorname{span}_A \{x_i\}_{i=1}^n$ *is complemented.*

Proof. We proceed by induction.

Initial Step. $n = 1$. It follows by assumption.

Inductive Step. The statement is true for n. We next show it holds for $n+1$. By (5), it is immediate to see that

$$\operatorname{span}_A \{x_i\}_{i=1}^{n+1} = \operatorname{span}_A \{x_i\}_{i=1}^n + \operatorname{span}_A \{x_{n+1}\}. \tag{6}$$

By inductive assumption, $M_1 = \operatorname{span}_A \{x_i\}_{i=1}^n$ is complemented. If we define $M_2 = \operatorname{span}_A \{x_{n+1}\}$, then we have that $P_{M_1^\perp}(\operatorname{span}_A \{x_{n+1}\}) = \operatorname{span}_A \left\{ P_{M_1^\perp}(x_{n+1}) \right\}$, where the latter submodule is complemented by assumption. By Proposition 1, $M_1 + M_2$ is complemented. By (6), the inductive step follows.

By induction, the statement follows. ■

Since the span of one element plays a fundamental role, we next study the topological properties of this object. Later on in the paper, this will yield that, for self-dual pre-Hilbert A-modules over algebras of $\mathcal{L}^0$ type, finitely generated submodules are *always* complemented. While, for self-dual pre-Hilbert A-modules over algebras of $\mathcal{L}^\infty$ type, submodules generated by *one* element are always complemented, provided the norm of the generator is invertible.

Lemma 3 *Let A be either an f-algebra of $\mathcal{L}^0$ type or an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ and $\bar{x} \in H$ is such that $N(\bar{x}) \leq e$ then:*

1. *If A is an f-algebra of $\mathcal{L}^0$ type, by defining $\tilde{a}_n = c_{N(\bar{x})} a_n$ for all $n \in \mathbb{N}$, we have that $a_n \cdot \bar{x} = \tilde{a}_n \cdot \bar{x}$ for all $n \in \mathbb{N}$ and*

$$a_n \cdot \bar{x} \overset{d_H}{\to} y \iff \tilde{a}_n \overset{d}{\to} a \in A \text{ and } y = a \cdot \bar{x}.$$

2. *If A is an Arens algebra of $\mathcal{L}^{\infty}$ type, $N(\bar{x})$ is invertible,*[15] *and $\{a_n \cdot \bar{x}\}_{n\in\mathbb{N}} \subseteq B_H$, then*
$$a_n \cdot \bar{x} \xrightarrow{d_H} y \iff a_n \xrightarrow{d} a \in A \text{ and } y = a \cdot \bar{x} \in B_H.$$

Proof. 1. Since $0 \leq N(\bar{x}) \in A_e$, we can consider its basic component $c_{N(\bar{x})}$. For the sake of brevity, we will denote it by c. Recall that $c^2 = c$. Note that

$$\begin{aligned} N(a_n \cdot \bar{x} - \tilde{a}_n \cdot \bar{x}) &= N((a_n - \tilde{a}_n) \cdot \bar{x}) = |a_n - \tilde{a}_n| N(\bar{x}) = |a_n - \tilde{a}_n| cN(\bar{x}) \\ &= |a_n - \tilde{a}_n| |c| N(\bar{x}) = |ca_n - c\tilde{a}_n| N(\bar{x}) = \left|ca_n - c^2 a_n\right| N(\bar{x}) \\ &= |ca_n - ca_n| N(\bar{x}) = 0, \end{aligned}$$

proving that $a_n \cdot \bar{x} = \tilde{a}_n \cdot \bar{x}$ for all $n \in \mathbb{N}$. Next, assume that $a_n \cdot \bar{x} \xrightarrow{d_H} y$. It follows that $\tilde{a}_n \cdot \bar{x} \xrightarrow{d_H} y$. Observe that

$$\begin{aligned} |\tilde{a}_n N(\bar{x}) - \tilde{a}_m N(\bar{x})| &= |\tilde{a}_n - \tilde{a}_m| |N(\bar{x})| = |\tilde{a}_n - \tilde{a}_m| N(\bar{x}) = N((\tilde{a}_n - \tilde{a}_m) \cdot \bar{x}) \\ &= N(\tilde{a}_n \cdot \bar{x} - \tilde{a}_m \cdot \bar{x}) \qquad \forall m, n \in \mathbb{N}. \end{aligned}$$

This implies that

$$\begin{aligned} d(\tilde{a}_n N(\bar{x}), \tilde{a}_m N(\bar{x})) &= \bar{\varphi}(|\tilde{a}_n N(\bar{x}) - \tilde{a}_m N(\bar{x})| \wedge e) = \bar{\varphi}(N(\tilde{a}_n \cdot \bar{x} - \tilde{a}_m \cdot \bar{x}) \wedge e) \\ &= d_H(\tilde{a}_n \cdot \bar{x}, \tilde{a}_m \cdot \bar{x}) \qquad \forall m, n \in \mathbb{N}. \end{aligned}$$

Since $\{\tilde{a}_n \cdot \bar{x}\}_{n\in\mathbb{N}}$ is d_H convergent (in particular, it is d_H Cauchy), we can conclude that $\{\tilde{a}_n N(\bar{x})\}_{n\in\mathbb{N}} \subseteq A$ is d Cauchy. Since A is d complete, $\tilde{a}_n N(\bar{x}) \xrightarrow{d} l$. We can conclude that $\tilde{a}_n \xrightarrow{d} l d_{N(\bar{x})}$. Define $a = l d_{N(\bar{x})}$. It follows that $a_n \cdot \bar{x} = \tilde{a}_n \cdot \bar{x} \xrightarrow{d_H} a \cdot \bar{x}$. Since the limit is unique, we have that $y = a \cdot \bar{x}$, proving one implication. On the other hand, if $\tilde{a}_n \xrightarrow{d} a$, then $a_n \cdot \bar{x} = \tilde{a}_n \cdot \bar{x} \xrightarrow{d_H} a \cdot \bar{x}$, proving the opposite implication.

2. Assume that $a_n \cdot \bar{x} \xrightarrow{d_H} y$. Since $\{a_n \cdot \bar{x}\}_{n\in\mathbb{N}} \subseteq B_H$, we have that $|a_n N(\bar{x})| = |a_n| N(\bar{x}) = N(a_n \cdot \bar{x}) \leq e$ for all $n \in \mathbb{N}$. This implies that $\{a_n N(\bar{x})\}_{n\in\mathbb{N}} \subseteq [-e, e]$. At the same time, by the same arguments of before, we have that $d(a_n N(\bar{x}), a_m N(\bar{x})) = d_H(a_n \cdot \bar{x}, a_m \cdot \bar{x})$ for all $n, m \in \mathbb{N}$. Since $\{a_n \cdot \bar{x}\}_{n\in\mathbb{N}}$ is d_H convergent (in particular, it is d_H Cauchy), we can conclude that $\{a_n N(\bar{x})\}_{n\in\mathbb{N}} \subseteq A$ is d Cauchy. Since $\{a_n N(\bar{x})\}_{n\in\mathbb{N}} \subseteq A$ is d Cauchy and $[-e, e]$ is d complete, $a_n N(\bar{x}) \xrightarrow{d} l \in [-e, e]$. Since $N(\bar{x})$ is invertible, we can conclude that $a_n \xrightarrow{d} l(N(\bar{x}))^{-1}$. Define $a = l(N(\bar{x}))^{-1}$. It follows that $a_n \cdot \bar{x} \xrightarrow{d_H} a \cdot \bar{x}$. Since the limit is unique, we have that $y = a \cdot \bar{x}$. Since the topology induced by d is locally solid, we also have that $|a_n N(\bar{x})| \xrightarrow{d} |aN(\bar{x})| = N(a \cdot \bar{x}) = |l| \in [-e, e]$, proving that $a \cdot \bar{x} \in B_H$ and the implication. On the other hand, if $a_n \xrightarrow{d} a$, then $a_n \cdot \bar{x} \xrightarrow{d_H} a \cdot \bar{x}$, proving the opposite implication. ■

[15] That is, there exists $b \geq 0$, denoted by $N(\bar{x})^{-1}$, such that $bN(\bar{x}) = e$.

Lemma 4 *Let A be either an f-algebra of $\mathcal{L}^0$ type or an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If $x \in H$, then there exists $\bar{x} \in H$ such that $N(\bar{x}) \leq e$ and* $\operatorname{span}_A\{x\} = \operatorname{span}_A\{\bar{x}\}$. *Moreover,*

1. *If A is an f-algebra of $\mathcal{L}^0$ type, then* $\operatorname{span}_A\{x\}$ *is a d_H closed set.*

2. *If A is an Arens algebra of $\mathcal{L}^\infty$ type and $N(x)$ is invertible, then* $\operatorname{span}_A\{x\} \cap B_H$ *is a d_H closed set.*

Proof. Given $x \in H$, define $\bar{a} = N(x) + e$, $\bar{b} = \bar{a}^{-1}$, and $\bar{x} = \bar{b} \cdot x$. Since $\bar{b} \geq 0$, it follows that $N(\bar{x}) = \bar{b}N(x) \leq \bar{b}\bar{a} = e$. Next, note that if $y \in \operatorname{span}_A\{x\}$, then there exists $a \in A$ such that $y = a \cdot x$. It follows that $y = \left(a\left(\bar{a}\bar{b}\right)\right) \cdot x = (a\bar{a}) \cdot \left(\bar{b} \cdot x\right) = (a\bar{a}) \cdot \bar{x} \in \operatorname{span}_A\{\bar{x}\}$. Viceversa, if $y \in \operatorname{span}_A\{\bar{x}\}$, then there exists $a \in A$ such that $y = a \cdot \bar{x} = \left(a\bar{b}\right) \cdot x \in \operatorname{span}_A\{x\}$.

1. We next prove that $\operatorname{span}_A\{\bar{x}\} = \operatorname{span}_A\{x\}$ is d_H closed. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{span}_A\{\bar{x}\}$ be such that $x_n \overset{d_H}{\to} y$. It follows that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ such that $a_n \cdot \bar{x} = x_n \overset{d_H}{\to} y$. By point 1 of Lemma 3, we can conclude that there exists $a \in A$ such that $x_n = a_n \cdot \bar{x} \overset{d_H}{\to} a \cdot \bar{x} \in \operatorname{span}_A\{\bar{x}\}$, proving that $\operatorname{span}_A\{\bar{x}\}$ is d_H closed.

2. We next prove that $\operatorname{span}_A\{\bar{x}\} \cap B_H = \operatorname{span}_A\{x\} \cap B_H$ is d_H closed. Since $N(x)$ is invertible, so is $N(\bar{x})$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{span}_A\{\bar{x}\} \cap B_H$ be such that $x_n \overset{d_H}{\to} y$. It follows that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ such that $a_n \cdot \bar{x} = x_n \overset{d_H}{\to} y$ and $\{a_n \cdot \bar{x}\}_{n\in\mathbb{N}} \subseteq B_H$. By point 2 of Lemma 3, we can conclude that there exists $a \in A$ such that $x_n = a_n \cdot \bar{x} \overset{d_H}{\to} a \cdot \bar{x} \in \operatorname{span}_A\{\bar{x}\} \cap B_H$, proving that $\operatorname{span}_A\{\bar{x}\} \cap B_H$ is d_H closed. ∎

3.2 Arens algebras of $\mathcal{L}^\infty$ type

In a standard Hilbert space, it is well known that, given a vector subspace $M \subseteq H$, $H = M \oplus M^\perp$ if and only if M is norm closed. Since a vector subspace is convex, this is also equivalent to M being closed in the weak topology.

Given a pre-Hilbert A-module where A is an Arens algebra of $\mathcal{L}^\infty$ type, the generalization of the above facts leaves several options open. The first thing to observe is that the pre-Hilbert requirement needs to be strengthened. We will in fact assume, as a premise, that H is self-dual. Clearly, in the standard case of $A = \mathbb{R}$, self-duality is equivalent to completeness in norm. At the same time, in studying a submodule $M \subseteq H$, we will show that there are several "right" notions of closure of M. In Lemma 2, we saw already that closure with respect to the metric d_H could be a possible candidate. The next lemma shows that closure with respect to the topology $\sigma(H, S(H))$ could be another.

Lemma 5 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If $\emptyset \neq M \subseteq H$, then $M^\perp$ is $\sigma(H, S(H))$ closed.*

Proof. Fix $x \in H$ and define $\ker\{x\} = \{y \in H : \langle x, y\rangle_H = 0\}$. Consider a net $\{y_i\}_{i\in I} \subseteq \ker\{x\}$ such that $y_i \overset{\sigma(H,S(H))}{\rightarrow} y$. It follows that $0 = a\langle y_i, x\rangle_H = \langle y_i, a\cdot x\rangle_H$ for all $i \in I$ and for all $a \in A$. This implies that

$$0 = \bar{\varphi}(\langle y_i, a\cdot x\rangle_H) \to \bar{\varphi}(\langle y, a\cdot x\rangle_H) \qquad \forall a \in A.$$

We can conclude that $\bar{\varphi}(a\langle x, y\rangle_H) = \bar{\varphi}(\langle y, a\cdot x\rangle_H) = 0$ for all $a \in A$. If we define $\bar{a} = \langle x, y\rangle_H \in A$, this implies that $0 = \bar{\varphi}(\bar{a}\langle x, y\rangle_H) = \bar{\varphi}(\bar{a}^2)$. Since $\bar{\varphi}$ is strictly positive, this implies that $\bar{a}^2 = 0$, that is, $\bar{a} = 0$. By definition of $\bar{a}$, we can conclude that $y \in \ker\{x\}$, proving that $\ker\{x\}$ is $\sigma(H, S(H))$ closed. Since $M^\perp = \cap_{x\in M}\ker\{x\}$, the statement follows. ■

We are ready to prove our first two main results on complementation. On the one hand, the first set of topological conditions (Theorem 1), which characterize when a submodule M is complemented, consists of conditions which characterize the self-duality of M (see also point 4 of Lemma 1). On the other hand, the second set of topological conditions (Theorem 2) consists instead of "genuine" topological conditions on M, which, in particular, highlight the connection between pre-Hilbert A-modules over Arens algebras of $\mathcal{L}^\infty$ type and f-algebras of $\mathcal{L}^0$ type (cf. Condition (ii) in Theorem 2 and Condition (i) in Theorem 3).

Theorem 1 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If H is self-dual and M is a submodule of H, then the following statements are equivalent:*

(i) M *is* $\sigma(H, S(H))$ *closed;*

(ii) $M \cap B_H$ *is* $\sigma(H, S(H))$ *closed;*

(iii) $M \cap B_H$ *is* $\sigma(H, S(H))$ *compact;*

(iv) $M \cap B_H$ *is* $\|\ \|_m$ *closed;*

(v) $M \cap B_H$ *is* d_H *closed;*

(vi) $H = M \oplus M^\perp$*;*

(vii) $M = M^{\perp\perp}$.

Proof. By [9, Theorem 3] and since H is self-dual, we have that B_H is $\sigma(H, S(H))$ compact.

(i) implies (ii). Since $\sigma(H, S(H))$ is Hausdorff, B_H is closed. The implication trivially follows.

(ii) implies (iii). Since $M \cap B_H \subseteq B_H$ is $\sigma(H, S(H))$ closed and B_H is $\sigma(H, S(H))$ compact, it follows that $M \cap B_H$ is $\sigma(H, S(H))$ compact.

(iii) implies (iv). Since $M \cap B_H$ is $\sigma(H, S(H))$ compact, it is also $\sigma(H, S(H))$ closed. Consider now a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M \cap B_H$ such that $x_n \overset{\| \|_m}{\to} x$. Since $x_n \overset{\| \|_m}{\to} x$ implies $x_n \overset{\sigma(H,S(H))}{\to} x$, it follows that $x \in M \cap B_H$, proving the implication.

(iv) implies (vi). Since M is a submodule of H, if we define $\langle\ ,\ \rangle_M$ as the restriction of $\langle\ ,\ \rangle_H$ to $M \times M$, then $(M, +, \cdot, \langle\ ,\ \rangle_M)$ is a pre-Hilbert A-module. We also have that $M \cap B_H = B_M$ and the norm $\|x\|_{m'} = \sqrt{\bar{\varphi}(\langle x, x\rangle_M)}$ coincides with $\| \|_m$ on M. Since H is self-dual, B_H is $\| \|_m$ complete. By assumption, this implies that B_M is $\| \|_m$ complete, that is, B_M is $\| \|_{m'}$ complete. By [9, Theorem 3], we have that M is self-dual. By point 4 of Lemma 1, the implication follows.

(vi) implies (vii). By point 5 of Lemma 1, the implication follows.

(vii) implies (i). By Lemma 5 and since $M = M^{\perp\perp} = \left(M^{\perp}\right)^{\perp}$, it follows that M is $\sigma(H, S(H))$ closed.

It follows that conditions (i), (ii), (iii), (iv), (vi), and (vii) are equivalent. Finally observe that:

(v) implies (iv). Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M \cap B_H$ such that $x_n \overset{\| \|_m}{\to} x$. Since $M \cap B_H$ is d_H closed and $x_n \overset{\| \|_m}{\to} x$ implies that $x_n \overset{d_H}{\to} x$, we can conclude that $x \in M \cap B_H$, proving the implication.

(vii) implies (v). By Lemma 2 and since $M = M^{\perp\perp} = \left(M^{\perp}\right)^{\perp}$, M is d_H closed. By Theorem 7 and since H is self-dual, B_H is d_H complete. It follows that $M \cap B_H$ is d_H closed, proving the implication. ■

As a corollary, under the assumption that the norm of the generator is invertible, we obtain that submodules generated by one element are complemented, and, in particular, they are $\sigma(H, S(H))$ as well as $\| \|_H$ closed. Finally, the same holds for the sum of two complemented and *orthogonal* submodules.

Corollary 1 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If H is self-dual, then the following statements are true:*

1. *For each $x \in H$ such that $N(x)$ is invertible, the submodule* $\operatorname{span}_A\{x\}$ *is complemented. In particular,* $\operatorname{span}_A\{x\}$ *is $\sigma(H, S(H))$ and $\| \|_H$ closed.*

2. If M_1 and M_2 are complemented and orthogonal submodules, then their sum is complemented. In particular, $M_1 + M_2$ is $\sigma(H, S(H))$ and $\| \|_H$ closed.

Proof. 1. Let $x \in H$. By point 2 of Lemma 4, $\operatorname{span}_A \{x\} \cap B_H$ is d_H closed. By Theorem 1, we have that $\operatorname{span}_A \{x\}$ is complemented.

2. Since M_1 and M_2 are orthogonal, it follows that $M_2 \subseteq M_1^\perp$ and $P_{M_1^\perp}(M_2) = M_2$. By Proposition 1 and since M_1 and M_2 are complemented, we have that $M_1 + M_2$ is complemented.

In both cases, by Theorem 1, the property of being complemented yields that the new (sum) submodule is $\sigma(H, S(H))$ closed, which immediately implies that it is $\| \|_H$ closed too. ■

In the next result, we show that removing the hypothesis of invertibility in point 1 of Corollary 1 easily generates counterexamples.

Example 1 Let $(\Omega, \mathcal{F}, P) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), P)$ where $P(A) = \sum_{i \in A} \frac{1}{2^i}$ for all $A \in \mathcal{P}(\mathbb{N})$. Consider $A = H = \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$.The space A is an Arens algebra of $\mathcal{L}^\infty$ type. The space H is a pre-Hilbert A-module where $+$ is the usual sum. The outer product $\cdot$ is the usual product and $\langle x, y \rangle_H = xy$. In particular, $H = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ of Section 4 when $(\Omega, \mathcal{F}, P) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), P)$ and $\mathcal{G} = \mathcal{F}$. This yields that H is self-dual (see [9, Theorem 7]). Consider $x \in H$ to be such that $x(i) = \frac{1}{i}$ for all $i \in \mathbb{N}$. Note that $N(x) = x$ is not invertible in A. Consider $M = \operatorname{span}_A \{x\}$. Note that if $y \in M^\perp$, then $\langle x, y \rangle_H = xy = 0$, that is, $0 = x(i)\, y(i) = \frac{1}{i} y(i)$ for all $i \in \mathbb{N}$, that is, $y(i) = 0$ for all $i \in \mathbb{N}$. Thus, $M^\perp = \{0\}$. At the same time, consider $z \in H$ such that $z(i) = 1$ for all $i \in \mathbb{N}$. It is immediate to see that $z \notin M$. We can conclude that

$$z \in H \text{ and } z \notin M = M \oplus M^\perp.$$

The next result, which provides topological conditions for M being complemented, generalizes the usual known result for standard Hilbert spaces. Indeed, when $A = \mathbb{R}$, $\| \|_m$ is the norm induced by the inner product and $\sigma(H, S(H))$ is the weak topology induced by the norm dual of H.

Theorem 2 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If H is self-dual and M is a submodule of H, then the following statements are equivalent:*

(i) M is $\sigma(H, S(H))$ closed;

(ii) M is d_H closed;

(iii) M is $\| \|_m$ closed;

(iv) $H = M \oplus M^{\perp}$.

Proof. By Theorem 1, (i) is equivalent to (iv).

(ii) implies (iii). Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ such that $x_n \overset{\| \|_m}{\to} x$. Since $x_n \overset{\| \|_m}{\to} x$ implies $x_n \overset{d_H}{\to} x$ and M is d_H closed, it follows that $x \in M$, proving the implication.

(iii) implies (iv). By [9, Theorem 3] and since H is self-dual, it follows that B_H is $\| \|_m$ complete, in particular, it is $\| \|_m$ closed. This implies that $B_H \cap M$ is $\| \|_m$ closed. By Theorem 1, the implication follows.

(iv) implies (ii). By point 5 of Lemma 1, we have that $M = M^{\perp\perp}$. By Lemma 2 and since $M^{\perp\perp} = \left(M^{\perp}\right)^{\perp}$, it follows that M is d_H closed. ■

We conclude by observing that the orthogonal complement of a submodule M coincides with the orthogonal complement computed in a standard pre-Hilbert space. The result then should clarify the meaning of Lemma 5 and point (iii) of Theorem 2. Indeed, first recall that $\langle\ ,\ \rangle_m : H \times H \to \mathbb{R}$, defined by

$$\langle x, y\rangle_m = \bar{\varphi}\left(\langle x, y\rangle_H\right) \qquad \forall x, y \in H,$$

makes $(H, +, \cdot^e, \langle\ ,\ \rangle_m)$ a standard pre-Hilbert space.[16]

Proposition 3 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If $M \subseteq H$ is a submodule, then*

$$M^{\perp} = \left\{y \in H : \langle x, y\rangle_m = 0 \quad \forall x \in M\right\}.$$

Proof. If $y \in M^{\perp}$, then $\langle x, y\rangle_H = 0$ for all $x \in M$, yielding that $\langle x, y\rangle_m = 0$ for all $x \in M$. Viceversa, assume that $y \in H$ is such that $\langle x, y\rangle_m = 0$ for all $x \in M$. By contradiction, assume that $y \notin M^{\perp}$, that is, $a = \langle \bar{x}, y\rangle_H \neq 0$ for some $\bar{x} \in M$. It follows that either $0 \leq a^+ \neq 0$ or $0 \leq a^- \neq 0$ or both. In the first case, by [8, Lemma 3], we have that there exists $c \in A$ such that $c = c^2$ and $a^+ = ca$. Since M is a submodule, observe that $c \cdot \bar{x} \in M$. Since $a^+ > 0$ and $\bar{\varphi}$ is strictly positive, this implies that $0 < \bar{\varphi}(a^+) = \bar{\varphi}(c\langle \bar{x}, y\rangle_H) = \bar{\varphi}(\langle c\cdot\bar{x}, y\rangle_H) = \langle c\cdot\bar{x}, y\rangle_m = 0$, a contradiction. Similarly, in the second case, by [8, Lemma 3], we have that there exists $c \in A$ such that $c = c^2$ and $-a^- = ca$. Since M is a submodule, observe that $c \cdot \bar{x} \in M$. Since $a^- > 0$ and $\bar{\varphi}$ is strictly positive, this implies that $0 > \bar{\varphi}(-a^-) = \bar{\varphi}(c\langle \bar{x}, y\rangle_H) = \bar{\varphi}(\langle c\cdot\bar{x}, y\rangle_H) = \langle c\cdot\bar{x}, y\rangle_m = 0$, a contradiction. ■

[16] Note that H is typically *not* an Hilbert space, that is, it is not $\| \|_m$ complete, even if H is self-dual. See, for example, the self-dual pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ of Section 4.

3.3 f-algebras of $\mathcal{L}^0$ type

As we already observed before, in a standard Hilbert space, it is well known that, given a vector subspace $M \subseteq H$, $H = M \oplus M^{\perp}$ if and only if M is norm closed.

Given a pre-Hilbert A-module H where A is an Arens algebra of $\mathcal{L}^{\infty}$ type, we saw that the generalization of this result to pre-Hilbert modules was not completely obvious (see Theorems 1 and 2), given the several different natural topologies that one can consider on H. When A is an f-algebra of $\mathcal{L}^0$ type, the generalization is much more intuitive: M is complemented if and only if it is closed with respect to the metric d_H.

Theorem 3 *Let A be an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module. If H is self-dual and M is a submodule of H, then the following statements are equivalent:*

(i) M is d_H closed;

(ii) $H = M \oplus M^{\perp}$;

(iii) $M = M^{\perp\perp}$.

Proof. (i) implies (ii). Since M is a submodule of H, if we define $\langle\ ,\ \rangle_M$ as the restriction of $\langle\ ,\ \rangle_H$ to $M \times M$, then $(M, +, \cdot, \langle\ ,\ \rangle_M)$ is a pre-Hilbert A-module. It is immediate to see that $d_M = d_H$ once the latter is restricted to $M \times M$. By [9, Theorem 5] and since H is self-dual, H is d_H complete. By [9, Theorem 5] and since M is d_H closed, M is $d_M = d_H$ complete and it follows that M is self-dual. By point 4 of Lemma 1, the statement follows.

(ii) implies (iii). By point 5 of Lemma 1, the implication follows.

(iii) implies (i). By Lemma 2 and since $M = M^{\perp\perp} = \left(M^{\perp}\right)^{\perp}$, it follows that M is d_H closed. ∎

Remark 2 Guo [15] proves a similar result when $A = \mathcal{L}^0(\Omega, \mathcal{G}, P)$. His proof is different from ours since it relies on a version of the projection theorem for pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-modules. Here, instead, we prove it by relying on self-duality. Most importantly, our result holds for a larger class of algebras (see A in Subsection 4.3).

Corollary 2 *Let A be an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module. If H is self-dual, then the following statements are true:*

1. *For each $\{x_i\}_{i=1}^n \subseteq H$ the submodule $\operatorname{span}_A \{x_i\}_{i=1}^n$ is d_H closed and complemented.*

2. *For each* $\{x_i\}_{i=1}^n \subseteq H$ *and each* d_H *closed submodule* M*, we have that* $M + \operatorname{span}_A \{x_i\}_{i=1}^n$ *is* d_H *closed and complemented.*

3. *If* M_1 *and* M_2 *are* d_H *closed and orthogonal submodules, then their sum is* d_H *closed and complemented.*

Proof. 1. First, we prove the statement for $n = 1$. Let $x_1 \in H$. By point 1 of Lemma 4, $\operatorname{span}_A \{x_1\}$ is d_H closed. By Theorem 3, we have that $\operatorname{span}_A \{x_1\}$ is complemented. By Proposition 2, it follows that $\operatorname{span}_A \{x_i\}_{i=1}^n$ is complemented for any collection $\{x_i\}_{i=1}^n \subseteq H$.

2. Define $M_1 = M$ and $M_2 = \operatorname{span}_A \{x_i\}_{i=1}^n$. By Theorem 3, M_1 is complemented. Note also that $P_{M_1^\perp}(M_2) = \operatorname{span}_A\{P_{M_1^\perp}(x_i)\}_{i=1}^n$. By point 1, this implies that $P_{M_1^\perp}(M_2)$ is complemented as well. By Proposition 1, $M + \operatorname{span}_A \{x_i\}_{i=1}^n$ is complemented.

3. By Theorem 3 and since M_1 is d_H closed, M_1 is complemented. Since M_1 and M_2 are orthogonal, it follows that $M_2 \subseteq M_1^\perp$ and $P_{M_1^\perp}(M_2) = M_2$. By Theorem 3 and since M_2 is d_H closed, $P_{M_1^\perp}(M_2)$ is complemented. By Proposition 1, $M_1 + M_2$ is complemented.

In all three cases, by Theorem 3, the property of being complemented yields that the new (sum) submodule is d_H closed. ■

Note that for self-dual pre-Hilbert A-modules over f-algebras of $\mathcal{L}^0$ type, finitely generated submodules are *always* d_H closed: a key property in Finance applications.

4 Applications

We will first introduce two pre-Hilbert A-modules that will play a key role in our applications: $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$ and $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$. Consider a nonempty set Ω, a σ-algebra of subsets of Ω denoted by $\mathcal{F}$, a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, and a probability measure $P : \mathcal{F} \to [0, 1]$. Two $\mathcal{F}$-measurable random variables are defined to be equivalent if and only if they coincide almost surely. Define:

1. $A = \mathcal{L}^0(\mathcal{G})$, that is, A is the space (of equivalence classes) of real valued and $\mathcal{G}$-measurable functions;[17]

2. $b \geq a$ if and only if $b(\omega) \geq a(\omega)$ almost surely;

3. $e = 1_\Omega$, that is, e is the function that takes constant value 1;

[17] As usual, we view the equivalence classes as functions. This convention will apply throughout the rest of the paper.

4. It follows that $A_e = \mathcal{L}^\infty(\mathcal{G})$, that is, A_e is the space of all essentially bounded and $\mathcal{G}$-measurable functions;

5. $\bar{\varphi} : \mathcal{L}^\infty(\mathcal{G}) \to \mathbb{R}$ as

$$\bar{\varphi}(a) = \int a dP = \mathbb{E}a \qquad \forall a \in \mathcal{L}^\infty(\mathcal{G});$$

6. $d : \mathcal{L}^0(\mathcal{G}) \times \mathcal{L}^0(\mathcal{G}) \to \mathbb{R}$ as

$$d(a,b) = \bar{\varphi}(|b-a| \wedge e) = \int (|b-a| \wedge e)\, dP \qquad \forall a,b \in \mathcal{L}^0(\mathcal{G}).$$

   Note that the topology induced by d is the one of convergence in probability P.

It is immediate to verify that $(A =)\mathcal{L}^0(\mathcal{G})$ is an f-algebra of $\mathcal{L}^0$ type and $(A_e =)\mathcal{L}^\infty(\mathcal{G})$ is an Arens algebra of $\mathcal{L}^\infty$ type.[18]

We denote by $\mathcal{L}^0(\mathcal{F}) = \mathcal{L}^0(\Omega, \mathcal{F}, P)$ the space of real valued and $\mathcal{F}$-measurable functions. We call x, y, and z the elements of $\mathcal{L}^0(\mathcal{F})$. Given an $\mathcal{F}$-measurable function $x : \Omega \to \mathbb{R}$ such that $x \geq 0$, we denote by $\mathbb{E}(x||\mathcal{G})$ its conditional expected value with respect to P given $\mathcal{G}$ (see Loeve [23, Section 27] and Shiryaev [28, p. 213]) which exists and is unique P-a.s. Observe that $\mathbb{E}(x||\mathcal{G})$ might not be real valued. If $x \ngeq 0$, we define $\mathbb{E}(x||\mathcal{G}) = \mathbb{E}(x^+||\mathcal{G}) - \mathbb{E}(x^-||\mathcal{G})$, provided $\mathbb{E}(x^+||\mathcal{G}), \mathbb{E}(x^-||\mathcal{G}) \in \mathcal{L}^0(\mathcal{G})$. As for integrable random variables, one can show that if $x, y \in \mathcal{L}^0(\mathcal{F})$ and $\mathbb{E}(x||\mathcal{G}), \mathbb{E}(y||\mathcal{G})$ are well defined, then

1. $\mathbb{E}(ax + by||\mathcal{G}) = a\mathbb{E}(x||\mathcal{G}) + b\mathbb{E}(y||\mathcal{G})$ for all $a, b \in \mathcal{L}^0(\mathcal{G})$;

2. $\varphi(\mathbb{E}(x||\mathcal{G})) \leq \mathbb{E}(\varphi(x)||\mathcal{G})$, provided $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and the latter expectation is well defined (Conditional Jensen's inequality)

Denote by

$$H = \mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{ x \in \mathcal{L}^0(\mathcal{F}) : \sqrt{\mathbb{E}(x^2||\mathcal{G})} \in \mathcal{L}^0(\mathcal{G}) \right\}.$$

We endow H with two operations:

1. $+ : H \times H \to H$ which is the usual pointwise sum operation;

2. $\cdot : A \times H \to H$ such that $a \cdot x = ax$ where ax is the usual pointwise product.

[18] Both spaces are endowed with the usual operations of sum, scalar product, and multiplication. The norm on $\mathcal{L}^\infty(\mathcal{G})$ is the essential sup norm.

The space $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$ was introduced by Hansen and Richard [17]. Finally, we also define an inner product, namely, $\langle\ ,\ \rangle_H : H \times H \to \mathcal{L}^0(\mathcal{G})$ by

$$\langle x, y \rangle_H = \mathbb{E}(xy||\mathcal{G}) \qquad \forall x, y \in H.$$

Inter alia, Hansen and Richard [17, p. 592] show that this generalized inner product is well defined and, in particular, a conditional version of the Cauchy-Schwarz's inequality holds:

$$|\mathbb{E}(xy||\mathcal{G})| \leq \sqrt{\mathbb{E}(x^2||\mathcal{G})}\sqrt{\mathbb{E}(y^2||\mathcal{G})} \qquad \forall x, y \in H. \tag{7}$$

Note that $d_H : H \times H \to [0, \infty)$

$$d_H(x, y) = \int \left(\sqrt{\mathbb{E}\left((x-y)^2||\mathcal{G}\right)} \wedge 1_\Omega\right) dP \qquad \forall x, y \in H. \tag{8}$$

It is well known that H is a self-dual pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-module (see [17], [15], and [9, Theorem 6]). Thus, in particular, H is d_H complete.

We next consider another pre-Hilbert module. Denote by $\mathcal{L}^2(\mathcal{F}) = \mathcal{L}^2(\Omega, \mathcal{F}, P)$ the space of $\mathcal{F}$-measurable and square integrable functions. Denote also by

$$H = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{x \in \mathcal{L}^2(\mathcal{F}) : \sqrt{\mathbb{E}(x^2||\mathcal{G})} \in \mathcal{L}^\infty(\mathcal{G})\right\} \subseteq \mathcal{L}^2(\mathcal{F}).$$

If we restrict the two above operations, $+$ and $\cdot$, to $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ and $\mathcal{L}^\infty(\mathcal{G})$ and we also restrict $(x, y) \mapsto \mathbb{E}(xy||\mathcal{G})$ to $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$, then it is not hard to show that $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. This space was studied in Ergodic theory (in dealing with compact extensions) by Tao [29].[19]

Note that in this case $\|\ \|_H : H \to [0, \infty)$ is such that

$$\|x\|_H = \sqrt{\|\mathbb{E}(x^2||\mathcal{G})\|_{\mathcal{L}^\infty(\mathcal{G})}} \qquad \forall x \in H.$$

Similarly, we have that

$$\|x\|_m = \sqrt{\int \mathbb{E}(x^2||\mathcal{G})\, dP} = \sqrt{\int x^2 dP} = \|x\|_{\mathcal{L}^2(\mathcal{F})} \qquad \forall x \in H.$$

By [9, Theorem 7], we have that H is a self-dual pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module.

4.1 Stricker's Lemma

In Finance, it is common to assume that there are n primary assets that are traded at an initial date $t = 0$. Typically, each primary asset x_i is modelled to be a contingent payment or a stream of contingent payments. Mathematically, x_i is an $\mathcal{F}$-measurable

[19] Tao [29] focuses on the complex case. See also Zhao [31] for the real case.

function from Ω to either $\mathbb{R}$ (see, e.g., Hansen and Richard [17]) or a space of sequences (see, e.g., Hansen [16] and Cochrane [11]). At $t = 0$, it is possible to trade in the market these primary assets. In particular, a payoff vector of the form $\sum_{i=1}^{n} a_i \cdot x_i$ can be traded. From an economic point of view, the element a_i specifies the quantity to buy/sell of asset i at 0. Moreover, each a_i might be more than a number, it can be a function which depends on the information at time 0, that is, a_i is a $\mathcal{G}$-measurable function from Ω to $\mathbb{R}$. Intuitively, it follows that the space of marketed contingent claims is nothing but $\mathrm{span}_A \{x_i\}_{i=1}^n$. The closure of $\mathrm{span}_A \{x_i\}_{i=1}^n$ is then a fundamental condition for providing versions of the Fundamental Theorem of Asset Pricing (see [17, Assumption 2.1], [16], [27], [11, p. 17], and [10]). Depending on the space H the set $\{x_i\}_{i=1}^n$ is assumed to belong to, this amounts to prove a version of Stricker's Lemma. For example, if $\{x_i\}_{i=1}^n \subseteq \mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$, as in [17, Assumption 2.1] and [10], we have:

Proposition 4 *Let $H = \mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$. If $\{x_i\}_{i=1}^n \subseteq H$, then $\mathrm{span}_A \{x_i\}_{i=1}^n$ is d_H closed.*

Proof. By Corollary 2 and since H is a self-dual pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-module, the statement follows. ■

Remark 3 The original Stricker's Lemma (see [27, Lemma 2.3]) proves that, given $\{x_i\}_{i=1}^n \subseteq \mathcal{L}^0(\mathcal{F})$, the set

$$\left\{ x \in \mathcal{L}^0(\mathcal{F}) : \exists \{a_i\}_{i=1}^n \subseteq \mathcal{L}^0(\mathcal{G}) \text{ s.t. } x = \sum_{i=1}^{n} a_i x_i \right\}$$

is closed with respect to the topology of convergence in probability. The arguments contained in [27, Lemma 2.4] show that this is equivalent to prove the same result for $\{x_i\}_{i=1}^n \subseteq \mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$. Thus, despite being sufficient for financial applications (see [17, Assumption 2.1] and [10]), our result is weaker and differs from the original Stricker's Lemma in only one, but key dimension: We show that $\mathrm{span}_A \{x_i\}_{i=1}^n$ is d_H closed, rather than closed with respect to the topology of convergence in probability. Note that the latter is coarser than the former. At the same time, if $\mathcal{G} = \mathcal{F}$, then $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P) = \mathcal{L}^0(\mathcal{F})$ and d_H indeed metrizes the topology of convergence in probability. Thus, in this special case, our result *coincides* with Stricker's Lemma.

4.2 Ergodic theory

Consider the pre-Hilbert module $H = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$. Consider also a map $\tau : \Omega \to \Omega$ such that τ is $\mathcal{F}/\mathcal{F}$-measurable and $P(E) = P(\tau^{-1}(E))$ for all $E \in \mathcal{F}$. Let $\mathcal{G}$ be the sub-σ-algebra of invariant events. Define $T : H \to H$ to be such that $x \mapsto x \circ \tau$. An element $x \in H$ is said to be (conditionally) weak mixing if and only if

$$\frac{1}{N} \sum_{n=0}^{N-1} \left\| \langle T^n(x), x \rangle_H \right\|_{\mathcal{L}^2(\mathcal{F})} \to 0. \tag{9}$$

We denote by H_{wm} the set $\{x \in H : x \text{ is weak mixing}\}$.

Theorem 4 *Let $H = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$. If $\mathcal{G}$ is the sub-σ-algebra of invariant events, then $H_{wm} \cap B_H$ is $\| \ \|_{\mathcal{L}^2(\mathcal{F})}$ closed. In particular, we have that*

$$H = H_{wm} \oplus H_{wm}^{\perp}. \tag{10}$$

Proof. One can show that H_{wm} is a submodule.[20] Note that T is linear and such that $\|T(x)\|_H = \|x\|_H$ and $\|T(x)\|_{\mathcal{L}^2(\mathcal{F})} = \|x\|_{\mathcal{L}^2(\mathcal{F})}$ for all $x \in H$.[21] Moreover, observe that

$$\begin{aligned}
\sqrt{\mathbb{E}\left|\langle x, y\rangle_H\right|^2} &= \sqrt{\mathbb{E}\langle x, y\rangle_H^2} \leq \sqrt{\mathbb{E}\left(\langle x, x\rangle_H \langle y, y\rangle_H\right)} \leq \sqrt{\mathbb{E}\left(\left\|\langle x, x\rangle_H\right\|_{\mathcal{L}^\infty(\mathcal{G})} \langle y, y\rangle_H\right)} \\
&= \sqrt{\left\|\langle x, x\rangle_H\right\|_{\mathcal{L}^\infty(\mathcal{G})} \mathbb{E}\left(\langle y, y\rangle_H\right)} = \sqrt{\left\|\langle x, x\rangle_H\right\|_{\mathcal{L}^\infty(\mathcal{G})}} \sqrt{\mathbb{E}\left(\langle y, y\rangle_H\right)} \\
&= \|x\|_H \|y\|_{\mathcal{L}^2(\mathcal{F})}.
\end{aligned}$$

Let $\{x_k\}_{k\in\mathbb{N}} \subseteq H_{wm} \cap B_H$ to be such that $x_k \overset{\| \ \|_{\mathcal{L}^2(\mathcal{F})}}{\to} x \in H$. This implies that for each $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\begin{aligned}
\left\|\langle T^n(x), x\rangle_H - \langle T^n(x_k), x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} &\leq \left\|\langle T^n(x), x\rangle_H - \langle T^n(x_k), x\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&\quad + \left\|\langle T^n(x_k), x\rangle_H - \langle T^n(x_k), x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&= \left\|\langle T^n(x - x_k), x\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} + \left\|\langle T^n(x_k), x - x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&= \left\|\langle x, T^n(x - x_k)\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} + \left\|\langle T^n(x_k), x - x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&\leq \|x\|_H \left\|T^n(x - x_k)\right\|_{\mathcal{L}^2(\mathcal{F})} + \left\|T^n(x_k)\right\|_H \|x - x_k\|_{\mathcal{L}^2(\mathcal{F})} \\
&= \|x\|_H \|x - x_k\|_{\mathcal{L}^2(\mathcal{F})} + \|x_k\|_H \|x - x_k\|_{\mathcal{L}^2(\mathcal{F})} \\
&\leq \left(\|x\|_H + 1\right) \|x - x_k\|_{\mathcal{L}^2(\mathcal{F})}.
\end{aligned}$$

[20] A sketch of the proof is contained in [29, p. 206].

[21] Note that $\langle T(x), T(x)\rangle_H = T(\langle x, x\rangle_H) = \langle x, x\rangle_H$ for all $x \in H$. This yields that

$$\|T(x)\|_H = \sqrt{\left\|\langle T(x), T(x)\rangle_H\right\|_{\mathcal{L}^\infty(\mathcal{G})}} = \sqrt{\left\|\langle x, x\rangle_H\right\|_{\mathcal{L}^\infty(\mathcal{G})}} = \|x\|_H.$$

Similarly, we have that

$$\|T(x)\|_{\mathcal{L}^2(\mathcal{F})} = \sqrt{\mathbb{E}\left(\langle T(x), T(x)\rangle_H\right)} = \sqrt{\mathbb{E}\left(\langle x, x\rangle_H\right)} = \|x\|_{\mathcal{L}^2(\mathcal{F})}.$$

It follows that for each $k, N \in \mathbb{N}$

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x), x\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} &\leq \frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x), x\rangle_H - \langle T^n(x_k), x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&+ \frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x_k), x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \\
&\leq \left(\|x\|_H + 1\right)\|x - x_k\|_{\mathcal{L}^2(\mathcal{F})} + \frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x_k), x_k\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})}.
\end{aligned}$$

Since $\{x_k\}_{k\in\mathbb{N}} \subseteq H_{wm}$ and $x_k \overset{\| \|_{\mathcal{L}^2(\mathcal{F})}}{\to} x \in H$, we can conclude that for each $k \in \mathbb{N}$

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x), x\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} \leq \left(\|x\|_H + 1\right)\|x - x_k\|_{\mathcal{L}^2(\mathcal{F})} \to 0,$$

proving that $\lim_N \frac{1}{N}\sum_{n=0}^{N-1}\left\|\langle T^n(x), x\rangle_H\right\|_{\mathcal{L}^2(\mathcal{F})} = 0$, that is, $x \in H_{wm}$. Since H is self-dual and $\{x_k\}_{k\in\mathbb{N}} \subseteq B_H$, we have that $x_k \overset{\| \|_{\mathcal{L}^2(\mathcal{F})}}{\to} x \in B_H$. By Theorem 1, equation (10) follows. ■

Remark 4 Inter alia, in a mildly different setting, Zhao [31] obtains the same decomposition contained in equation (10). He first shows that $H_{wm}^{\perp}$ coincides with the set of conditionally almost periodic elements. He then proceeds, by direct arguments, to show that $H = H_{wm} \oplus H_{wm}^{\perp}$. Instead here, we obtain the latter as a consequence of an Hilbertian decomposition.

4.3 Stochastic processes and Hilbert modules

Consider a discrete-time filtered space $\left\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, P\right\}$. We denote the conditional expectation $\mathbb{E}(\cdot||\mathcal{F}_t)$ by $\mathbb{E}_t(\cdot)$ for all $t \in \mathbb{N}_0$. We consider three spaces of processes $x = (x_t)_{t\in\mathbb{N}_0}$:

1. $\mathcal{S}_0$ which denotes the space of semimartingales with initial value 0, that is, $x_0 = 0$ and x is adapted, (i.e., $x_t \in \mathcal{L}^0(\mathcal{F}_t)$ for all $t \in \mathbb{N}_0$);

2. M_0^{loc} which denotes the space of local martingales with initial value 0, that is, $x \in M_0^{loc}$ if and only if $x \in \mathcal{S}_0$, $\mathbb{E}_{t-1}(|x_t|) \in \mathcal{L}^0(\mathcal{F}_{t-1})$, and $\mathbb{E}_{t-1}(x_t) = x_{t-1}$ for all $t \in \mathbb{N}$.

3. $M_0^{2,loc}$ which denotes the space of *conditionally* square integrable local martingales with initial value 0, that is, $x \in M_0^{2,loc}$ if and only if $x \in \mathcal{S}_0$, $\mathbb{E}_{t-1}(x_t^2) \in \mathcal{L}^0(\mathcal{F}_{t-1})$, and $\mathbb{E}_{t-1}(x_t) = x_{t-1}$ for all $t \in \mathbb{N}$.

Our terminology and notation is justified by the fact that, in discrete time, being a semimartingale is conceptually equivalent to be an adapted process (see Jacod and Shiryaev [19, p. 62]). The elements in M_0^{loc} are generalized martingales as defined in Shiryaev [28, p. 476] (see also Kabanov and Safarian [20, p. 255]), where the usual integrability condition is weakened to "conditional" integrability. At the same time, the integrability of the initial value x_0 guarantees that x is a generalized martingale if and only if it is a local martingale (see [28, p. 478] and [20, Proposition 5.3.2]). Finally, using the conditional Jensen's inequality, it is immediate to see that $M_0^{2,loc} \subseteq M_0^{loc}$, since $\mathbb{E}_{t-1}\left(x_t^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right)$ implies $\mathbb{E}_{t-1}\left(|x_t|\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right)$.

We say that a process $a = (a_t)_{t\in\mathbb{N}}$ is predictable if and only if $a_t \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right)$ for all $t \in \mathbb{N}$. We denote by A the space of predictable processes (*that start at* $t = 1$). It is not hard to show that A is an f-algebra of $\mathcal{L}^0$ type where the operations of sum, scalar product, and multiplication are the usual pointwise ones and so is the $\geq$ binary relation.[22]

Given $x \in \mathcal{S}_0$, we define $\Delta_t x = x_t - x_{t-1}$ for all $t \in \mathbb{N}$. We focus our attention on the following two spaces:

$$H = \left\{x \in \mathcal{S}_0 : \mathbb{E}_{t-1}\left((\Delta_t x)^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\right\}$$

and

$$H^{mar} = \left\{x \in M_0^{loc} : \mathbb{E}_{t-1}\left((\Delta_t x)^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\right\}.$$

Proposition 5 *$H^{mar} \subseteq H$ and $H^{mar} = H \cap M_0^{loc} = M_0^{2,loc}$. Moreover,*

$$H = \left\{x \in \mathcal{S}_0 : \mathbb{E}_{t-1}\left(x_t^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\right\}. \tag{11}$$

Proof. We only prove (11) and $H^{mar} = M_0^{2,loc}$, being the other inclusions trivial. Observe that, clearly, $\Delta_t x \in \mathcal{L}^0\left(\mathcal{F}_t\right)$ for all $t \in \mathbb{N}$. Note that $x \in H$ if and only if $x \in \mathcal{S}_0$ and $\Delta_t x \in \mathcal{L}^{2,0}\left(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P\right)$ for all $t \in \mathbb{N}$. Define $\tilde{H} = \{x \in \mathcal{S}_0 : \mathbb{E}_{t-1}\left(x_t^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\}$. Assume that $x \in H$. Since $x \in \mathcal{S}_0$ and $x_{t-1} \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \subseteq \mathcal{L}^{2,0}\left(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P\right)$ for all $t \in \mathbb{N}$, it follows that $x_t = \Delta_t x + x_{t-1} \in \mathcal{L}^{2,0}\left(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P\right)$

[22] In particular, we have that A_e is the space of uniformly bounded predictable processes where e is the constant process $e_t = 1_\Omega$ for all $t \in \mathbb{N}$, $\bar{\varphi} : A_e \to \mathbb{R}$ is such that

$$\bar{\varphi}(a) = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \mathbb{E} a_t \qquad \forall a \in A_e,$$

and

$$d(a,b) = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \mathbb{E}\left(|b_t - a_t| \wedge 1_\Omega\right) \quad \forall a, b \in A.$$

for all $t \in \mathbb{N}$, and in particular, $\mathbb{E}_{t-1}(x_t^2) \in \mathcal{L}^0(\mathcal{F}_{t-1})$ for all $t \in \mathbb{N}$, proving that $x \in \tilde{H}$. Viceversa, assume that $x \in \tilde{H}$. This implies that $x \in \mathcal{S}_0$ and $x_t \in \mathcal{L}^{2,0}(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P)$ for all $t \in \mathbb{N}$. Since $x_{t-1} \in \mathcal{L}^0(\mathcal{F}_{t-1}) \subseteq \mathcal{L}^{2,0}(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P)$ for all $t \in \mathbb{N}$, it follows that $\Delta_t x \in \mathcal{L}^{2,0}(\Omega, \mathcal{F}_{t-1}, \mathcal{F}_t, P)$ for all $t \in \mathbb{N}$ and, in particular, $\mathbb{E}_{t-1}\left((\Delta_t x)^2\right) \in \mathcal{L}^0(\mathcal{F}_{t-1})$ for all $t \in \mathbb{N}$, proving that $x \in H$. Since, clearly, $H^{mar} = H \cap M_0^{loc} = \tilde{H} \cap M_0^{loc}$ and $M_0^{2,loc} \subseteq M_0^{loc}$, it follows that $H^{mar} = \tilde{H} \cap M_0^{loc} = M_0^{2,loc}$. ■

In other words, Proposition 5 shows that H is the space of *conditionally* square integrable semimartingales.

Example 2 Let z be such that $z_t = t$ for all $t \in \mathbb{N}_0$. Clearly, $z \in \mathcal{S}_0$ and

$$\Delta_t z = 1 \quad \forall t \in \mathbb{N} \implies \mathbb{E}_{t-1}\left((\Delta_t z)^2\right) \in \mathcal{L}^0(\mathcal{F}_{t-1}) \quad \forall t \in \mathbb{N}.$$

Example 3 Let M_0^2 be the space of square integrable martingales with initial value 0, that is, $x \in M_0^2$ if and only if $x_0 = 0$, x is a martingale, and $\|x_t\|_{\mathcal{L}^2(\mathcal{F})} < \infty$ for all $t \in \mathbb{N}$. It is immediate to see that $M_0^2 \subseteq M_0^{2,loc}$.

We first restrict our attention to H. On H we have two operations: one internal of sum, denoted $+$, and one external of outer product, denoted $\cdot$. We define:

- $+ : H \times H \to H$ to be such that $(x+y)_t = x_t + y_t$ for all $t \in \mathbb{N}_0$.
- $\cdot : A \times H \to H$ to be such that

  $$(a \cdot x)_0 = 0 \text{ and } (a \cdot x)_t = \sum_{s=1}^{t} a_s (x_s - x_{s-1}) = \sum_{s=1}^{t} a_s \Delta_s x \quad \forall t \in \mathbb{N}.$$

  In other words, the outer product is the transform of x by a (see Shiryaev [28, p. 478] and Jacod and Shiryaev [19, p. 62]). Observe that this transform satisfies the following properties: For all $a, b \in A$ and all $x, y \in H$:

  1. $a \cdot (x+y) = a \cdot x + a \cdot y$;
  2. $(a+b) \cdot x = a \cdot x + b \cdot x$;
  3. $a \cdot (b \cdot x) = (ab) \cdot x$;
  4. $e \cdot x = x$.

We can also define a generalized inner product $\langle\ ,\ \rangle_H : H \times H \to A$ by $(x,y) \mapsto \langle x, y \rangle_H$ where the latter is the process

$$(\langle x, y \rangle_H)_t = \mathbb{E}_{t-1}((\Delta_t x)(\Delta_t y)) \qquad \forall t \in \mathbb{N}. \tag{12}$$

By the conditional Cauchy-Schwarz inequality, we have that for each $t \in \mathbb{N}$

$$\left|\left(\langle x, y\rangle_H\right)_t\right| = \left|\mathbb{E}_{t-1}\left(\left(\Delta_t x\right)\left(\Delta_t y\right)\right)\right| \leq \sqrt{\mathbb{E}_{t-1}\left(\left(\Delta_t x\right)^2\right)}\sqrt{\mathbb{E}_{t-1}\left(\left(\Delta_t y\right)^2\right)} \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right). \tag{13}$$

We next show that $+$, $\cdot$, and $\langle\ ,\ \rangle_H$ are well defined given the domains and target spaces we have chosen. Moreover, this will allow us to conclude that H is a pre-Hilbert A-module.[23]

Proposition 6 *$(H, +, \cdot, \langle\ ,\ \rangle_H)$ is a pre-Hilbert A-module.*

Proposition 7 *$\left(M_0^{2,loc}, +, \cdot, \langle\ ,\ \rangle_H\right)$ is a pre-Hilbert A-module. In particular, $M_0^{2,loc}$ is a submodule of H.*

We next show that H and $M_0^{2,loc}$ are self-dual pre-Hilbert A-modules. To show this, we need to consider the metric d_H of equation (3), which, in this case, is equal to

$$d_H(x, y) = \sum_{s=1}^{\infty} \left(\frac{1}{2}\right)^s d_s\left(\Delta_s x, \Delta_s y\right)$$

where $d_s : \mathcal{L}^{2,0}\left(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P\right) \times \mathcal{L}^{2,0}\left(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P\right) \to [0, \infty)$ for all $s \in \mathbb{N}$ is the metric in equation (8), that is,

$$d_s\left(\Delta_s x, \Delta_s y\right) = \mathbb{E}\left(\sqrt{\mathbb{E}_{s-1}\left(\left(\Delta_s x - \Delta_s y\right)^2\right)} \wedge 1_\Omega\right) \qquad \forall s \in \mathbb{N}.$$

Theorem 5 *H is self-dual.*

Theorem 6 *$M_0^{2,loc}$ is self-dual. In particular, $M_0^{2,loc}$ is d_H closed.*

4.3.1 Martingales decompositions

In what follows, we show how our orthogonal decomposition results in pre-Hilbert A-modules, in the setting of semimartingales $\mathcal{S}_0$, yield a pair of famous decomposition results for our class of processes: Doob's and Kunita-Watanabe's. To get an intuition of why this is the case, observe that, if x and y are two square integrable martingales, then they are orthogonal in our sense, that is $\langle x, y\rangle_H = 0$, if and only if they are strongly orthogonal (see Follmer and Schied [12, p. 375]).

We start by obtaining Doob's decomposition result. By Theorems 3 and 6, it follows that $M_0^{2,loc}$ is complemented in H. Define

$$H^{pre} = \left\{x \in \mathcal{S}_0 : x_t \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\right\},$$

[23] The proofs of Propositions 6 and 7 as well as the proofs of Theorems 5 and 6 are in Appendix B.

that is, $x \in H^{pre}$ if and only if x is a predictable process (that starts at $t = 0$) with initial value 0. It is immediate to see that H^{pre} is a submodule of H and that the former set is d_H closed.

Corollary 3 *$H^{pre} = \left(M_0^{2,loc}\right)^{\perp}$ and $H = H^{pre} \oplus M_0^{2,loc}$. In particular, for each $x \in H$ there exists a predictable process $x_{pre} \in H^{pre}$ and a conditionally square integrable martingale $x_{mar} \in M_0^{2,loc}$ such that $x = x_{pre} + x_{mar}$. Moreover, this decomposition is unique.*

Proof. By Theorem 3 and since H^{pre} is a d_H closed submodule, we have that $H^{pre} = (H^{pre})^{\perp\perp}$ and $H = H^{pre} \oplus (H^{pre})^{\perp}$. We next show that $H^{pre} = \left(M_0^{2,loc}\right)^{\perp}$. Consider $y \in H^{pre}$. It follows that for each $x \in M_0^{2,loc}$

$$(\langle x, y\rangle_H)_t = \mathbb{E}_{t-1}\left((\Delta_t x)(\Delta_t y)\right) = (\Delta_t y)\,\mathbb{E}_{t-1}(\Delta_t x) = 0 \qquad \forall t \in \mathbb{N},$$

proving that $H^{pre} \subseteq \left(M_0^{2,loc}\right)^{\perp}$. Viceversa, consider $y \in (H^{pre})^{\perp}$. By Proposition 5, we have that $y \in \mathcal{S}_0$ and $\mathbb{E}_{t-1}(y_t^2) \in \mathcal{L}^0(\mathcal{F}_{t-1})$ for all $t \in \mathbb{N}$. Moreover, we have that for each $x \in H^{pre}$

$$0 = (\langle x, y\rangle_H)_t = \mathbb{E}_{t-1}\left((\Delta_t x)(\Delta_t y)\right) = (\Delta_t x)\,\mathbb{E}_{t-1}(\Delta_t y) \qquad \forall t \in \mathbb{N}.$$

Set $x = z$ as in Example 2, it follows that $x \in H^{pre}$ and

$$\mathbb{E}_{t-1}(\Delta_t y) = 0 \qquad \forall t \in \mathbb{N},$$

proving that $y \in M_0^{2,loc}$. We can conclude that $(H^{pre})^{\perp} \subseteq M_0^{2,loc}$. By point 3 of Lemma 1, $\left(M_0^{2,loc}\right)^{\perp} \subseteq (H^{pre})^{\perp\perp} = H^{pre}$, yielding that $H^{pre} = \left(M_0^{2,loc}\right)^{\perp}$. Finally, by Theorem 3 and since $M_0^{2,loc}$ is d_H closed, it follows that $(H^{pre})^{\perp} = \left(M_0^{2,loc}\right)^{\perp\perp} = M_0^{2,loc}$ which yields the rest of the statement. ■

Remark 5 Note that the above result is a version of Doob's decomposition result (see, e.g., [30, p. 120]). Recall that the classic version of this result requires x to be such that each x_t is integrable. Instead here, in light of Proposition 5, we require x to be such that each x_t is *conditionally* square integrable. Of course, if x is such that each x_t is square integrable, then x satisfies the hypotheses of both versions of the result.

We conclude by proving the Kunita-Watanabe decomposition and by merging the two decompositions together in Corollary 5.

Corollary 4 *Let* $\{x_i\}_{i=1}^n \in M_0^{2,loc}$*. For each* $x \in M_0^{2,loc}$*, there exist* $\{a_i\}_{i=1}^n \subseteq A$ *and* $y \in M_0^{2,loc}$ *such that*

$$x = \sum_{i=1}^{n} a_i \cdot x_i + y \text{ and } \langle x_i, y \rangle_H = 0 \quad \forall i \in \{1, ..., n\}.$$

Moreover, this decomposition is unique, in the sense that y *is uniquely determined.*

Proof. Consider $\operatorname{span}_A \{x_i\}_{i=1}^n$. By Theorem 6 and Corollary 2, it follows that $\operatorname{span}_A \{x_i\}_{i=1}^n$ is d_H closed and complemented, proving the statement. ■

Remark 6 We conjecture that other decomposition results of the stochastic processes' literature, such as the Follmer-Schweizer decomposition, could be obtained as decomposition results in an opportune pre-Hilbert A-module.

Corollary 5 *Let* $\{x_i\}_{i=1}^n \in M_0^{2,loc}$*. For each* $x \in H$*, there exist* $x_{pre} \in H^{pre}$*,* $\{a_i\}_{i=1}^n \subseteq A$*, and* $y \in M_0^{2,loc}$ *such that*

$$x = x_{pre} + \sum_{i=1}^{n} a_i \cdot x_i + y \text{ and } \langle y, x_{pre} \rangle_H = \langle x_i, x_{pre} \rangle_H = \langle x_i, y \rangle_H = 0 \quad \forall i \in \{1, ..., n\}.$$

Moreover, this decomposition is unique, in the sense that x_{pre} *and* y *are uniquely determined.*

A Self-duality

Given a pre-Hilbert A-module H where A is an Arens algebra of $\mathcal{L}^\infty$ type, recall that the dual module is the set

$$H^\sim = \left\{ f \in A^H : f \text{ is } A\text{-linear and bounded} \right\}.$$

We can define a vector norm on $H^\sim$, $N_* : H^\sim \to A_+$, to be such that

$$N_*(f) = \sup_{x \in H} \left(\sup_{n \in \mathbb{N}} \frac{|f(x)|}{N(x) + \frac{1}{n} e} \right) \qquad \forall f \in H^\sim.$$

We also define a metric $d_{H^\sim} : H^\sim \times H^\sim \to [0, \infty)$ by

$$d_{H^\sim}(f, g) = d(0, N_*(f - g)) \qquad \forall f, g \in H^\sim.$$

Lemma 6 *If* A *is an Arens algebra of* $\mathcal{L}^\infty$ *type and* H *a pre-Hilbert* A*-module, then* $d_{H^\sim}$ *is an invariant metric.*

We omit the proof of this fact, since it follows by replicating exactly the same arguments used in proving Lemma 3 in [9].

Theorem 7 *Let A be an Arens algebra of $\mathcal{L}^{\infty}$ type and H a pre-Hilbert A-module. The following statements are equivalent:*

(i) B_H is d_H complete;

(ii) H is self-dual.

Proof. (i) implies (ii). It is [9, Proposition 16].

(ii) implies (i). Consider a d_H Cauchy sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq B_H$. Define $\{f_n\}_{n\in\mathbb{N}}$ as $f_n(x) = \langle x, y_n\rangle_H$ for all $x \in H$ and for all $n \in \mathbb{N}$.

Step 1. There exists $f : H \to A$ such that $f_n(x) \xrightarrow{d} f(x)$ for all $x \in H$.

Proof of the Step. By (2), we have that for each $n, m \in \mathbb{N}$ and for each $x \in H$

$$|f_n(x) - f_m(x)| = |\langle x, y_n\rangle_H - \langle x, y_m\rangle_H| \le N(x)\,N(y_n - y_m). \tag{14}$$

Fix $x \in B_H$. We can conclude that

$$|f_n(x) - f_m(x)| \wedge e \le (N(x)\,N(y_n - y_m)) \wedge e \le N(y_n - y_m) \wedge e,$$

yielding that

$$\begin{aligned} d(f_n(x), f_m(x)) &= \bar{\varphi}(|f_n(x) - f_m(x)| \wedge e) \\ &\le \bar{\varphi}(N(y_n - y_m) \wedge e) = d_H(y_n, y_m) \quad \forall n, m \in \mathbb{N}. \end{aligned}$$

We thus have that $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq A$ is a d Cauchy sequence. Moreover, by (2) and since $x, y_n \in B_H$ for all $n \in \mathbb{N}$, we have that

$$|f_n(x)| \le N(x)\,N(y_n) \le e \qquad \forall n \in \mathbb{N}.$$

Since $[-e, e]$ is d complete, this yields that $f_n(x) \xrightarrow{d} a_x \in [-e, e]$. Next, note that if $x \in H \backslash B_H$, then $\bar{x} = \frac{x}{\|x\|_H} \in B_H$. We have that $f_n(\bar{x}) \xrightarrow{d} a_{\bar{x}}$. Thus, we can conclude that there exists $a_x \in A$ such that

$$f_n(x) = \|x\|_H\, f_n(\bar{x}) \xrightarrow{d} \|x\|_H\, a_{\bar{x}} = a_x.$$

Since the limit is unique and x was arbitrarily chosen, we can define a map $f : H \to A$ such that $f(x) = a_x$ for all $x \in H$. □

Step 2. The map f is A-linear.

Proof of the Step. Consider $a, b \in A$ and $x, y \in H$. By Step 1 and since each f_n is A-linear, this implies that

$$af_n(x) + bf_n(y) = f_n(a \cdot x + b \cdot y) \xrightarrow{d} f(a \cdot x + b \cdot y).$$

At the same time, since $f_n(x) \xrightarrow{d} f(x)$ and $f_n(y) \xrightarrow{d} f(y)$, we have that

$$af_n(x) + bf_n(y) \xrightarrow{d} af(x) + bf(y).$$

Since the limit is unique, we can conclude that $f(a \cdot x + b \cdot y) = af(x)+bf(y)$, proving the statement. $\square$

Step 3. The map f is bounded. In particular, $f \in H^{\sim}$ and there exists $y \in B_H$ such that $f(x) = \langle x, y\rangle_H$ for all $x \in H$.

Proof of the Step. Let $x \in H$. Since $\{y_n\}_{n\in\mathbb{N}} \subseteq B_H$, it follows that

$$|f_n(x)| \leq N(x) N(y_n) \leq N(x) \qquad \forall n \in \mathbb{N}.$$

Since $f_n(x) \xrightarrow{d} f(x)$ and the topology induced by d is locally solid, we can conclude that

$$|f(x)| \xleftarrow{d} |f_n(x)| \leq N(x).$$

Since x was arbitrarily chosen, we have that

$$|f(x)| \leq N(x) \qquad \forall x \in H, \tag{15}$$

proving boundedness. Since H is self-dual, it follows that there exists $y \in H$ such that $f(x) = \langle x, y\rangle_H$ for all $x \in H$. By (15), we have that

$$\|y\|_H^2 = \|\langle y, y\rangle_H\|_A = \|f(y)\|_A \leq \|N(y)\|_A \leq \|y\|_H,$$

proving that $\|y\|_H \leq 1$. $\square$

Step 4. $f_n \xrightarrow{d_{H^{\sim}}} f$.

Proof of the Step. By (14), we have that

$$\frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e} \leq N(y_n - y_m) \qquad \forall k, m, n \in \mathbb{N}, \forall x \in H.$$

This yields that

$$d\left(0, \frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq d_H(y_n, y_m) \qquad \forall k, m, n \in \mathbb{N}, \forall x \in H.$$

Consider $\varepsilon > 0$. Since $\{y_n\}_{n\in\mathbb{N}}$ is a d_H Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d\left(0, \frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq \varepsilon \qquad \forall k \in \mathbb{N}, \forall m, n \geq n_\varepsilon, \forall x \in H.$$

By taking the limit in n, we have that

$$d\left(0, \frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq \varepsilon \qquad \forall k \in \mathbb{N}, \forall m \geq n_\varepsilon, \forall x \in H.$$

Next, consider the sequence $\left\{\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\wedge e\right\}_{k\in\mathbb{N}}$. This sequence is bounded by e and increasing. Thus, $\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\wedge e\uparrow\sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\wedge e\right)=\sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\right)\wedge e$. Since $\bar{\varphi}$ is order continuous, we can conclude that

$$\begin{aligned}d\left(0,\sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\right)\right)&=\bar{\varphi}\left(\sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\right)\wedge e\right)\\&=\bar{\varphi}\left(\sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\wedge e\right)\right)\\&=\sup_{k\in\mathbb{N}}\bar{\varphi}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\wedge e\right)\leq\varepsilon\qquad\forall m\geq n_\varepsilon,\forall x\in H.\end{aligned}$$

We also have that for each $m\in\mathbb{N}$

$$N_*(f-f_m)=\sup_{k\in\mathbb{N}}\frac{|\langle y-y_m,y-y_m\rangle_H|}{N(y-y_m)+\frac{1}{k}e}=N(y-y_m).^{24}\tag{16}$$

We can conclude that

$$\begin{aligned}d_{H^\sim}(f,f_m)&=d(0,N_*(f-f_m))\\&=d\left(0,\sup_{k\in\mathbb{N}}\left(\frac{|f(y-y_m)-f_m(y-y_m)|}{N(y-y_m)+\frac{1}{k}e}\right)\right)\leq\varepsilon\quad\forall m\geq n_\varepsilon,\end{aligned}$$

proving the statement. □

By (16), we have that $d_H(y,y_m)=d_{H^\sim}(f,f_m)$ for all $m\in\mathbb{N}$. By Step 4 and since $d_H(y,y_m)=d_{H^\sim}(f,f_m)$ for all $m\in\mathbb{N}$, we can conclude that $y_m\overset{d_H}{\to}y$, proving that B_H is d_H complete. ■

[24] Note that if there exists $z\in H$ such that $g:H\to A$ is defined by $g(x)=\langle x,z\rangle_H$ for all $x\in H$, then for all $k\in\mathbb{N}$ and for all $x\in H$

$$|g(x)|=|\langle x,z\rangle_H|\leq N(x)N(z)\leq\left(N(x)+\frac{1}{k}e\right)N(z).$$

This implies that

$$\frac{|g(x)|}{N(x)+\frac{1}{k}e}\leq N(z)\qquad\forall k\in\mathbb{N},\forall x\in H,$$

yielding that

$$N_*(g)=\sup_{x\in H}\left(\sup_{k\in\mathbb{N}}\frac{|g(x)|}{N(x)+\frac{1}{k}e}\right)\leq N(z).$$

On the other hand, we have that

$$N_*(g)\geq\sup_{k\in\mathbb{N}}\frac{|g(z)|}{N(z)+\frac{1}{k}e}=\sup_{k\in\mathbb{N}}\frac{\langle z,z\rangle_H}{N(z)+\frac{1}{k}e}=\sup_{k\in\mathbb{N}}\frac{N(z)^2}{N(z)+\frac{1}{k}e}.$$

By the same arguments contained in [9, Footnote 13], we have that

$$\sup_{k\in\mathbb{N}}\frac{N(z)^2}{N(z)+\frac{1}{k}e}=N(z),$$

yielding that $N_*(g)=N(z)$. If we set $g=f-f_m$, then $z=y-y_m$ and (16) follows.

Remark 7 This self-duality result should clarify the connection between pre-Hilbert $\mathcal{L}^\infty$-modules and pre-Hilbert $\mathcal{L}^0$-modules, where self-duality, for the latter ones, amounts to d_H completeness (see also [9, Theorem 5]). Moreover, for the former ones, it confirms the intuition provided by Frank [13, Remark 3.9] for (complex) W^*-algebras. In a nutshell, Frank conjectures that the "combination" of a topology over A that makes the unit ball of A complete with either the vector-valued norm N or the functionals in $H^\sim$ might provide a topology on H with respect to which B_H is complete, allowing for the possibility of characterizing self-duality.

B Stochastic processes

Proof of Proposition 6. We already argued that H is nonempty. Let $x, y \in H$. It is obvious that the sum of two adapted processes is an adapted process and clearly $(x+y)_0 = 0$, proving that $x + y \in \mathcal{S}_0$. Call $z = x + y \in \mathcal{S}_0$. Clearly, $\Delta_t z = \Delta_t x + \Delta_t y$ for all $t \in \mathbb{N}$. By (13), we have that for each $t \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}_{t-1}\left((\Delta_t z)^2\right) &= \mathbb{E}_{t-1}\left((\Delta_t x)^2 + (\Delta_t y)^2 + 2(\Delta_t x)(\Delta_t y)\right) \\ &= \mathbb{E}_{t-1}\left((\Delta_t x)^2\right) + \mathbb{E}_{t-1}\left((\Delta_t y)^2\right) + 2\mathbb{E}_{t-1}\left((\Delta_t x)(\Delta_t y)\right),\end{aligned}$$

where the latter belongs to $\mathcal{L}^0(\mathcal{F}_{t-1})$, proving that $x + y = z \in H$. At the same time, it is immediate to see that for each $\alpha \in \mathbb{R}$, $\alpha x \in H$. It follows that H is a vector subspace of the space of adapted processes. In particular, this yields that $(H, +)$ is an abelian group. Next, let $a \in A$ and $x \in H$. It is immediate to verify that $a \cdot x \in \mathcal{S}_0$. Finally, since a is predictable, we have that

$$\mathbb{E}_{t-1}\left((\Delta_t(a \cdot x))^2\right) = \mathbb{E}_{t-1}\left(a_t^2(\Delta_t x)^2\right) = a_t^2 \mathbb{E}_{t-1}\left((\Delta_t x)^2\right) \in \mathcal{L}^0(\mathcal{F}_{t-1}),$$

proving that $a \cdot x \in H$. Properties 1–4 of the sequence transform yield that $(H, +, \cdot)$ is an A-module. Next, let $x, y, z \in H$ and $a \in A$. First, by (13), recall that $(\langle x, y\rangle_H)_t \in \mathcal{L}^0(\mathcal{F}_{t-1})$ for all $t \in \mathbb{N}$. It follows that the process $\left\{(\langle x, y\rangle_H)_t\right\}_{t\in\mathbb{N}}$ is predictable, that is, $\langle x, y\rangle_H \in A$.

1. Clearly, $\langle x, x\rangle_H \geq 0$. At the same time, we have that

$$\begin{aligned}\langle x, x\rangle_H = 0 \iff (\langle x, x\rangle_H)_t = 0 \quad \forall t \in \mathbb{N} \iff \mathbb{E}_{t-1}\left((\Delta_t x)^2\right) = 0 \quad \forall t \in \mathbb{N} \\ \iff \Delta_t x = 0 \quad \forall t \in \mathbb{N} \iff x_t = x_{t-1} \quad \forall t \in \mathbb{N}.\end{aligned}$$

Since $x_0 = 0$, we have that $\langle x, x\rangle_H = 0 \iff x = 0$.

2. Consider $x, y \in H$. We have that for each $t \in \mathbb{N}$

$$(\langle x, y\rangle_H)_t = \mathbb{E}_{t-1}\left((\Delta_t x)(\Delta_t y)\right) = \mathbb{E}_{t-1}\left((\Delta_t y)(\Delta_t x)\right) = (\langle y, x\rangle_H)_t,$$

that is, $\langle x,y\rangle_H=\langle y,x\rangle_H$.

3. Consider $x,y,z\in H$. We have that for each $t\in\mathbb{N}$

$$\begin{aligned}\left(\langle x+y,z\rangle_H\right)_t&=\mathbb{E}_{t-1}\left(\left(\Delta_t\left(x+y\right)\right)\left(\Delta_t z\right)\right)=\mathbb{E}_{t-1}\left(\left(\Delta_t x+\Delta_t y\right)\left(\Delta_t z\right)\right)\\&=\mathbb{E}_{t-1}\left(\left(\Delta_t x\right)\left(\Delta_t z\right)+\left(\Delta_t y\right)\left(\Delta_t z\right)\right)\\&=\mathbb{E}_{t-1}\left(\left(\Delta_t x\right)\left(\Delta_t z\right)\right)+\mathbb{E}_{t-1}\left(\left(\Delta_t y\right)\left(\Delta_t z\right)\right)\\&=\left(\langle x,z\rangle_H\right)_t+\left(\langle y,z\rangle_H\right)_t,\end{aligned}$$

proving that $\langle x+y,z\rangle_H=\langle x,z\rangle_H+\langle y,z\rangle_H$.

4. Let $a\in A$ and $x\in H$. We have that for each $t\in\mathbb{N}$

$$\begin{aligned}\left(\langle a\cdot x,y\rangle_H\right)_t&=\mathbb{E}_{t-1}\left(\left(\Delta_t\left(a\cdot x\right)\right)\left(\Delta_t y\right)\right)=\mathbb{E}_{t-1}\left(\left(a_t\Delta_t x\right)\left(\Delta_t y\right)\right)\\&=a_t\mathbb{E}_{t-1}\left(\left(\Delta_t x\right)\left(\Delta_t y\right)\right)=a_t\left(\langle x,y\rangle_H\right)_t,\end{aligned}$$

proving that $\langle a\cdot x,y\rangle_H=a\langle x,y\rangle_H$. ■

Proof of Proposition 7. By Proposition 5 and since $M_0^{2,loc}=H^{mar}\subseteq H$, it is enough to show that $M_0^{2,loc}$ is closed under the sum and the outer product. Since $M_0^{2,loc}=H\cap M_0^{loc}$, this amounts to show that $x+y$ and $a\cdot x$ are local martingales whenever $x,y\in M_0^{2,loc}$ and $a\in A$. Clearly, the sum of two local martingales is a local martingale. At the same time, since a is predictable, we have that $a_s\Delta_s x\in\mathcal{L}^0\left(\mathcal{F}_s\right)$ and $a_s x_{s-1}\in\mathcal{L}^0\left(\mathcal{F}_{s-1}\right)$ for all $s\in\mathbb{N}$. It follows that $(a\cdot x)_0=0$ and $(a\cdot x)_t=\sum_{s=1}^t a_s\left(x_s-x_{s-1}\right)\in\mathcal{L}^0\left(\mathcal{F}_t\right)$ for all $t\in\mathbb{N}$. This implies that

$$\begin{aligned}\mathbb{E}_{t-1}\left(\left|(a\cdot x)_t\right|\right)&=\mathbb{E}_{t-1}\left(\left|\sum_{s=1}^t a_s\Delta_s x\right|\right)\leq\mathbb{E}_{t-1}\left(\left|\sum_{s=1}^{t-1} a_s\Delta_s x\right|\right)+\mathbb{E}_{t-1}\left(\left|a_t\Delta_t x\right|\right)\\&=\left|\sum_{s=1}^{t-1} a_s\Delta_s x\right|+\left|a_t\right|\mathbb{E}_{t-1}\left(\left|\Delta_t x\right|\right)\\&\leq\left|\sum_{s=1}^{t-1} a_s\Delta_s x\right|+\left|a_t\right|\mathbb{E}_{t-1}\left(\left|x_t\right|+\left|x_{t-1}\right|\right)\\&=\left|\sum_{s=1}^{t-1} a_s\Delta_s x\right|+\left|a_t\right|\mathbb{E}_{t-1}\left(\left|x_t\right|\right)+\left|a_t\right|\left|x_{t-1}\right|\in\mathcal{L}^0\left(\mathcal{F}_{t-1}\right)\quad\forall t\in\mathbb{N},\end{aligned}$$

proving that $a\cdot x$ is conditionally integrable. Finally, observe that for each $t\in\mathbb{N}$

$$\mathbb{E}_{t-1}\left((a\cdot x)_t-(a\cdot x)_{t-1}\right)=\mathbb{E}_{t-1}\left(a_t\Delta_t x\right)=a_t\mathbb{E}_{t-1}\left(\Delta_t x\right)=0,$$

proving that $a\cdot x\in M_0^{loc}$. ■

Proof of Theorem 5. By [9, Theorem 5], it is enough to show that H is d_H complete. Consider the product metric space $\times_{s=1}^{\infty}\left(\mathcal{L}^{2,0}\left(\Omega,\mathcal{F}_{s-1},\mathcal{F}_s,P\right),d_s\right)$. Endow this space with the metric $d_\infty = \sum_{s=1}^{\infty} \left(\frac{1}{2}\right)^s d_s$. By [25, p. 196] and since $\mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P)$ is d_s complete for all $s \in \mathbb{N}$, $\times_{s=1}^{\infty} \left(\mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P), d_s\right)$ is d_∞ complete. Given $x \in H$, define Δx to be the sequence $\{\Delta_s x\}_{s\in\mathbb{N}}$. By definition of H, it follows that $\{\Delta_s x\}_{s\in\mathbb{N}} \in \times_{s=1}^{\infty} \left(\mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P), d_s\right)$.

Consider a d_H Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H$. Since $d_H(x_n, x_m) = d_\infty(\Delta x_n, \Delta x_m)$ for all $n, m \in \mathbb{N}$. This implies that $\{\Delta x_n\}_{n\in\mathbb{N}} \subseteq \times_{s=1}^{\infty} \left(\mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P), d_s\right)$ is a d_∞ Cauchy sequence, yielding that $\Delta x_n \overset{d_\infty}{\to} w$, that is, $\Delta_s x_n \overset{d_s}{\to} w_s \in \mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P)$ for all $s \in \mathbb{N}$. Define x to be such that $x_0 = 0$ and $x_t = \sum_{s=1}^{t} w_s$. It is immediate to see that $x \in \mathcal{S}_0$. Moreover, $\Delta_s x = w_s \in \mathcal{L}^{2,0}(\Omega, \mathcal{F}_{s-1}, \mathcal{F}_s, P)$ for all $s \in \mathbb{N}$, proving that $x \in H$. Finally, we have that

$$d_H(x_n, x) = d_\infty(\Delta x_n, \Delta x) = d_\infty(\Delta x_n, w) \to 0,$$

proving completeness. ∎

Proof of Theorem 6. By [9, Theorem 5], it is enough to show that $H^{mar} = M_0^{2,loc}$ is d_H complete. By Proposition 7 and Theorem 5 and since $M_0^{2,loc} \subseteq H$, it is enough to show that $M_0^{2,loc}$ is d_H closed. Thus, consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M_0^{2,loc}$ such that $x_n \overset{d_H}{\to} x \in H$. We only need to show that $x \in M_0^{2,loc}$, in particular, that x is conditionally square integrable and satisfies the martingale property. By Proposition 5 and since $x \in H$, we have that $\mathbb{E}_{t-1}(x_t^2) \in \mathcal{L}^0(\mathcal{F}_{t-1})$ for all $t \in \mathbb{N}$. By definition of d_H, we have that $x_n \overset{d_H}{\to} x$ implies $\Delta_s x_n \overset{d_s}{\to} \Delta_s x$ for all $s \in \mathbb{N}$. Since $\mathbb{E}_{s-1}(\Delta_s x_n) = 0$ for all $s \in \mathbb{N}$ and for all $n \in \mathbb{N}$, this yields that $0 = \mathbb{E}_{s-1}(\Delta_s x_n)$ converges in probability to $\mathbb{E}_{s-1}(\Delta_s x)$, proving that $\mathbb{E}_{s-1}(\Delta_s x) = 0$ for all $s \in \mathbb{N}$, that is, $\mathbb{E}_{s-1}(x_s) = x_{s-1}$ for all $s \in \mathbb{N}$, which yields $x \in M_0^{2,loc}$. ∎

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