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Weak time-derivatives and no arbitrage pricing*

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Abstract

The risk-neutral pricing formula provides the valuation of European derivatives in absence of arbitrages. Despite the variety of option payoffs, price dynamics are driven by the same (possibly stochastic) interest rate process. We formalize this intuition by showing that no arbitrage prices constitute the weak solution of a differential equation, where interest rates have prominent role. To achieve this goal, we introduce the notion of weak time-derivative, which permits to differentiate functions of Markov processes. Importantly, martingales have null weak time-derivative. Finally, we reformulate the eigenvalue problem of Hansen and Scheinkman (2009) by using weak time-derivatives.

Keywords: no arbitrage pricing; weak time-derivative; martingale component; stochastic interest rates.

Mathematics Subject Classification (2010): 60G07, 91G80, 49J40.

JEL Classification: C02.

1 Introduction

The no arbitrage pricing formula for the valuation of European derivatives is a milestone of asset pricing theory. Roughly speaking, it states that the *proper* price of a European option at time t is given by the conditional expectation of the discounted future payoff under a

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risk-neutral probability Q. If the derivative has payoff h_T at maturity T, the risk-neutral price $\pi(t)$ is

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} h_T \right],$$

where r is a (possibly stochastic) interest rate. From this formulation it is clear that the short-term rate r plays a fundamental role in the derivative pricing. Indeed, price dynamics are determined by the same interest rate process, whatever the terminal payoff of the option. This intuition goes back to Cox and Ross (1976), who derived Black and Scholes (1973) result by exploiting the accounting relations between bonds, stocks and options. This line of reasoning actually stems from the original approach of Modigliani and Miller (1958).

In this paper we formalize the intuition that risk-neutral valuation is driven by the process of interest rates by proving that the no arbitrage pricing formula is the *weak solution* of the differential equation

$$\frac{d\pi}{dt}(t) = r(t)\pi(t) \tag{1}$$

with the terminal condition $\pi(T) = h_T$. In case we consider, instead of an option, a bond with deterministic payoff, eq. (1) is an ordinary differential equation with exponential solution. However, since in general $\pi(t)$ and $r(t)$ are random processes, we develop the mathematical tools needed to give a precise meaning to eq. (1). In particular, we introduce the notion of *weak time-derivative* for functions of Markov processes.

The definition of weak time-derivative requires a suitable set of test functions (see Definitions 1 and 14) and involves the conditional expectation of functions of stochastic processes. This instrument provides a handy characterization of martingales. Indeed, up to technical conditions, the weak time-derivative of a function is null if and only if the function is a martingale process. Hence, the weak time-derivative allows us to qualify martingales as the stochastic counterpart of constants in deterministic settings. In terms of interpretation, the weak time-derivative provides an indication of the growth rate of the conditional expectation of the process. Similar results hold for submartingality and supermartingality, which are related to positive or negative signs of weak time-derivatives.

The notion of weak time-derivative generalizes that of *infinitesimal generator* for Feller processes: under suitable assumptions, the weak time-derivative specializes to the infinitesimal generator. More in general, the weak time-derivative allows us to deal with generalized formulations of problems that usually employ the infinitesimal generator in their stronger form.

The main results of the paper are summarized by Theorems 12 and 15, which show existence and uniqueness of the weak solution of eq. (1) where $d\pi/dt$ is the weak time-derivative. Specifically, Theorem 12 deals with the case of deterministic short-term rates, while Theorem 15 involves stochastic interest rates. In addition, Proposition 13 provides an equation for no arbitrage pricing of cashflows.

When the risk-free rate is constant, by rewriting eq. (1) in an operator form, we obtain a reformulation of the eigenvalue-eigenvector problem analyzed by Hansen and Scheinkman (2009), that employs the weak time-derivative in place of the *extended generator* of the underlying Markov process. Following the same reasoning of Hansen and Scheinkman, we also obtain a decomposition of the stochastic discount factor into a martingale and a transient component.

Our work combines different areas of mathematical analysis and stochastic calculus. The overall approach comes from variational calculus, it exploits the theory of Sobolev spaces and weak formulations of differential equations. See, for example, Brezis (2010), Adams and Fournier (2003) and Lions (1971) for a comprehensive introduction about variational calculus, and Revuz and Yor (1999) for stochastic calculus.

From a financial point of view, our work builds directly on the foundations of no arbitrage asset pricing theory illustrated, for instance, in Björk (2004), Hansen and Richard (1987) and Föllmer and Schied (2011). In addition, our eigenvalue formulation refers to the long-term risk literature, in particular to Hansen and Scheinkman (2009) and related works, like Alvarez and Jermann (2005).

The paper is organized as follows. Subsection 1.1 introduces the general semigroup framework. Section 2 develops the mathematical formalism of the weak time-derivative, proves its main properties (Subsection 2.1) and relates it to the infinitesimal generator (Subsection 2.2). In Sections 3 and 5 we solve the no arbitrage pricing equation with deterministic and stochastic interest rates, respectively. A brief discussion of the special case of Black-Scholes model is presented in Subsection 3.1, while Subsection 3.2 discusses the risk-neutral pricing of cashflows. Section 4 deals with the eigenvalue-eigenvector problem and the decomposition of the stochastic discount factor. In particular, Subsection 4.1 compares the roles of weak time-derivative and infinitesimal generator in the eigenvalue-eigenvector formulation. Finally, additional results and proofs are included in Appendix.

1.1 The semigroup framework

Given a probability space $(\Omega, \mathcal{F}, P)$, fix $T > 0$ and consider a Markov process $\mathbf{X} = \{X_t\}_{t\in[0,T]}$ such that $X_t : \Omega \longrightarrow \mathbb{R}$ for all $t \in [0, T]$. In our application X_t is a stock price at time t. Let $\mathbb{R}$ be endowed with the Borel σ-algebra and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by $\mathbf{X}$.

Following Hansen and Scheinkman (2009), we consider the valuation operator

$$\mathbb{M}_T h_T = \mathbb{E}\left[M_T h_T\right],$$

where h_T is a $\mathcal{F}_T$-measurable payoff, $M_T = S_T/S_0$ and $\mathbf{S} = \{S_t\}_{t\in[0,T]}$ is a strictly positive *stochastic discount factor* with $S_0 = 1$ a.s. By convenience we will equivalently use both the notations S_t and $S(t)$. Here $\mathbb{M}_T h_T$ represents the price at time 0 associated with the payoff h_T. For instance, h_T may be the payoff at maturity T of a European option. In addition, we require that the stochastic discount factor $\mathbf{S}$ satisfies the intertemporal consistency given by the multiplicative relation

$$S_{t+u} = S_u(\theta_t)\, S_t \qquad \forall u > 0,$$

where θ_t is a shift operator. As a consequence, the process $\mathbf{M} = \{M_t\}_{t\in[0,T]}$ satisfies

$$M_0 = 1 \quad a.s., \qquad\qquad M_{t+u} = M_u(\theta_t)\, M_t$$

and this relation guarantees the *semigroup property* of the valuation operator $\mathbb{M}_T$, i.e.

$$\mathbb{M}_0 = \mathbb{I}, \qquad\qquad \mathbb{M}_{T+U} = \mathbb{M}_T \mathbb{M}_U.$$

The introduction of semigroups in asset pricing is due to Garman (1985).

In the following, we will deal with the valuation operator where the conditional expectation is taken at time $t \leqslant T$:

$$\mathbb{M}_{t,T} h_T = \mathbb{E}_t\left[M_{t,T} h_T\right]$$

with $M_{t,T} = S_T/S_t$. We call $M_{t,T}$ *pricing kernel* in the interval $[t,T]$. This terminology can be retrieved, for instance, in Hansen and Renault (2009). Moreover,

$$M_{T,T} = 1 \quad a.s., \qquad\qquad M_{s,T} = M_{t,T} M_{s,t} \quad \text{for} \quad t \leqslant s \leqslant T.$$

We assume that the physical measure P is strictly positive and there exists a risk-neutral probability Q equivalent to P. We consider Q as given and we do not discuss market completeness. Accordingly, stochastic discount factor and pricing kernel are uniquely determined by the given Q. We denote by L_T the Radon-Nikodym derivative of Q with respect to P and $L_t = \mathbb{E}_t\left[L_T\right]$ for all $t \in [0,T]$. We observe that $L_0 = 1$ and, moreover, we define $L_{t,T} = L_T/L_t$. If a riskless bond with interest rate r is traded, with continuous compounding the stochastic discount factor and the pricing kernel in $[t,T]$ satisfy

$$S_t = e^{-rt} L_t, \qquad\qquad M_{t,T} = e^{-r(T-t)} L_{t,T}.$$

See Hansen and Richard (1987) and Björk (2004) as references on risk-neutral pricing.

2 The weak time-derivative

Given a separable Banach space B, we consider functions $f : [0,T] \longrightarrow B$. We denote as $C([0,T],B)$ the space of continuous functions from $[0,T]$ to B while $C_c^n([0,T],B)$ contains n-times continuously differentiable functions from $[0,T]$ to B with compact support. All the derivatives are defined from $[0,T]$ to B, too. We denote by B' the topological dual of B and we exploit the notion of *weak Lebesgue measurability*, that can be retrieved in Diestel and Uhl (1977). A function $f : [0,T] \longrightarrow B$ is said to be *weakly Lebesgue measurable* (w.L.m. for brevity) when the function

$$\tau \in [0,T] \longmapsto \ell[f](\tau)$$

is Lebesgue measurable for any $\ell \in B'$. For example, a function $u : [0,T] \longrightarrow L^2(\mathcal{F}_T)$ is weakly Lebesgue measurable when the function

$$\tau \in [0,T] \longrightarrow \mathbb{E}\left[z_T u(\tau)\right]$$

is Lebesgue measurable for any $z_T \in L^2(\mathcal{F}_T)$. Hence, we can discuss the integrability of this function over the interval $[0,T]$.

For our purposes we introduce the notion of *conditional weak Lebesgue measurability.* We say that a function $u : [0,T] \longrightarrow \mathbb{R}^\Omega$ is *conditionally weakly Lebesgue measurable* when, for every $t \in [0,T]$, the function

$$\tau \in [t,T] \longmapsto \mathbb{E}\left[L_\tau u(\tau)\, \mathbf{1}_{A_t}\right]$$

is Lebesgue measurable for any $\mathcal{F}_t$-measurable set A_t. The presence of indicator functions reveals that conditional weak Lebesgue measurability is actually a property of the conditional expectation of $L_\tau u(\tau)$ at any given time t. The term L_τ entails the change of measure from the probability P to Q. Depending on the context, in case a change of measure is not needed, it is enough to set L_τ constantly equal to 1 (for brevity we will write $L_\tau \equiv 1$).

We define the space $\mathcal{V}$ as

$$\begin{aligned}\mathcal{V} = \Big\{ u : [0,T] \longrightarrow \mathbb{R}^\Omega : \quad & L_\tau u(\tau) \in L^2(\mathcal{F}_\tau) \quad \forall \tau, \\ u \quad \text{conditionally} \quad \text{w.L.m.}, \quad & \int_0^T \mathbb{E}\left[L_\tau^2 u^2(\tau)\right] d\tau < +\infty \Big\}.\end{aligned}$$

In addition, if $u \in \mathcal{V}$ is such that $L(\cdot)u(\cdot)$ is Lebesgue measurable, then $L(\cdot)u(\cdot)$ is also Bochner integrable. In particular, the Bochner integral of $L(\cdot)u(\cdot)$ is the element of $L^2(\mathcal{F}_T)$ denoted by

$$\int_0^T L_\tau u(\tau)\, d\tau.$$

To be precise, we should require that $L(\cdot)u(\cdot)$ is *strongly measurable.* However, this is equivalent to the fact that $L(\cdot)u(\cdot)$ is Lebesgue measurable and its values $L(\tau)u(\tau)$ belong, for a.e. $\tau \in [0,T]$, to a separable closed subspace of $\mathbb{R}^{\Omega}$, a property that we always assume. Moreover, the fact that

$$\int_0^T \mathbb{E}\left[L_\tau^2 u^2(\tau)\right] d\tau < +\infty$$

ensures that

$$\int_0^T \left(\mathbb{E}\left[L_\tau^2 u^2(\tau)\right]\right)^{\frac{1}{2}} d\tau < +\infty.$$

The last condition is necessary and sufficient for a strongly measurable function $L(\cdot)u(\cdot)$ to be Bochner integrable. See Diestel and Uhl (1977) and Aliprantis and Border (2006) as references.

We have now all the instruments to introduce the concept of *weak time-differentiability* for functions in $\mathcal{V}$.

Definition 1 *Given $u \in \mathcal{V}$, we say that u is* weakly time-differentiable *when there exists a function $w \in \mathcal{V}$ such that for every $t \in [0,T]$*

$$\int_t^T \mathbb{E}\left[L_\tau w(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = -\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1([t,T],\mathbb{R}).$$

In this case we call w a weak time-derivative *of u.*

Similarly, $u \in \mathcal{V}$ is said to be *twice weakly time-differentiable* when there exists $z \in \mathcal{V}$ such that for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = \int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right]\frac{d^2\varphi}{dt^2}(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^2([t,T],\mathbb{R}).$$

Observe that, if $u \in \mathcal{V}$, the integrals

$$\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau, \qquad \int_t^T \mathbb{E}\left[L_\tau w(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau$$

are finite for any choice of $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$. As a result, Definition 1 is well-posed.

We identify two functions $u, \hat{u}$ in $\mathcal{V}$ when $u(\tau) = \hat{u}(\tau)$ almost surely for a.e. $\tau \in [0,T]$. After this identification the weak time-derivative turns out to be unique.

Proposition 2 *Let $u \in \mathcal{V}$ be weakly time-differentiable. Then, the weak time-derivative of u is unique.*

With a little abuse of notation we denote du/dt one of the versions of the weak time-derivative of u. Moreover, we introduce the space

$$\mathcal{W} = \{\text{weakly} \quad \text{time-differentiable} \quad u \in \mathcal{V}\}$$

with the norm

$$\|u\|_{\mathcal{W}} = \left(\int_0^T \mathbb{E}\left[L_\tau^2 u^2(\tau)\right] d\tau + \int_0^T \mathbb{E}\left[L_\tau^2 \left(\frac{du}{dt}(\tau)\right)^2\right] d\tau \right)^{\frac{1}{2}}.$$

2.1 Calculus of the weak time-derivative

We start by proving that a weakly time-differentiable function u has null weak time-derivative when, for a.e. $t \in [0,T]$ and $\tau \in [t,T]$, $\mathbb{E}_t[u(\tau)]$ depends only on t.

Proposition 3 *A function $u \in \mathcal{V}$ is weakly time-differentiable with $du/dt = 0$ if and only if, for a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$*

$$\mathbb{E}_t[L_\tau u(\tau)] = f_t \qquad a.s.$$

with $f_t \in L^2(\mathcal{F}_t)$.

For example, suppose that $u \in \mathcal{V}$ and $L_t u(t)$ has zero mean and independent increments with respect to the given filtration. Then, for every $t \in [0,T]$,

$$\mathbb{E}_t[L_\tau u(\tau)] = \mathbb{E}_t[L_\tau u(\tau) - L_t u(t)] + \mathbb{E}_t[L_t u(t)] = L_t u(t) \qquad \forall \tau \in [t,T].$$

Therefore, u is weakly time-differentiable with $du/dt = 0$.

More precisely, in case $u \in \mathcal{V}$ and the process $\mathbf{Lu} = \{L_t u(t)\}_{t \in [0,T]}$ is a martingale, for every $t \in [0,T]$

$$\mathbb{E}_t[L_\tau u(\tau)] = L_t u(t) \qquad \forall \tau \in [t,T].$$

Hence, u is weakly time-differentiable with $du/dt = 0$. For the converse implication, however, we need an additional assumption.

Corollary 4 *Let $u \in \mathcal{V}$.*

i) If the process $\mathbf{Lu}$ is a martingale, then u is weakly time-differentiable with $du/dt = 0$.

ii) If u is weakly time-differentiable with $du/dt = 0$ and, for a.e. $t \in [0, T]$,

$$L_\tau u(\tau) \xrightarrow{L^1} L_t u(t) \qquad \tau \longrightarrow t^+,$$

then, for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t [L_\tau u(\tau)] = L_t u(t) \qquad a.s.$$

Under the assumptions of *ii*), it is easy to characterize the case of *correct expectations*, in which for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t [L_\tau u(\tau)] = L_\tau u(\tau) \qquad a.s.$$

Indeed, a function $u \in \mathcal{V}$ satisfying *ii*) has correct expectations if and only if $L_t u(t)$ is a.s. constant for a.e. $t \in [0, T]$. In particular, by setting $u(t) = X_t$ we see that forecasts about future prices X_τ are accurate if and only if prices are a.s. constant over time. See the discussion in Samuelson (1965).

Another simple corollary of Proposition 3 shows that, given a weak time-derivative w, all the functions $u \in \mathcal{W}$ such that $du/dt = w$ satisfy a precise property. Indeed, their conditional expectations at any instant t differ by a function $f_t \in L^2(\mathcal{F}_t)$.

Corollary 5 *Let $w \in \mathcal{V}$ be the weak time-derivative of $u_1 \in \mathcal{W}$. If w is also the weak time-derivative of $u_2 \in \mathcal{W}$, then, for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$*

$$\mathbb{E}_t [L_\tau u_2(\tau)] = \mathbb{E}_t [L_\tau u_1(\tau)] + f_t \qquad a.s.$$

with $f_t \in L^2(\mathcal{F}_t)$.

It is also interesting to study the case in which $L_\tau \equiv 1$ and the weak time-derivative is deterministic.

Proposition 6 *Let $u \in \mathcal{V}$, $L_\tau \equiv 1$, $g : [0, T] \longrightarrow \mathbb{R}$ square-integrable and, for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$*

$$\mathbb{E}_t [u(\tau)] = \int_t^\tau g(s) ds + f_t \qquad a.s.$$

with $f_t \in L^2(\mathcal{F}_t)$. Then, $du/dt = g$.

For example, in case $L_\tau \equiv 1$ and for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t [u(\tau)] = \alpha\tau - \alpha t + f_t \qquad a.s. \tag{2}$$

with $\alpha \in \mathbb{R}$ and $f_t \in L^2(\mathcal{F}_t)$, it is easy to see that $du/dt = \alpha$. In addition, by Corollary 5 all the functions $u \in \mathcal{W}$ such that $du/dt = \alpha$ may be written as in eq. (2). For instance, if $\mathbf{m}$ is a martingale in $\mathcal{W}$, the function

$$u(t) = \alpha t + m_t$$

has $du/dt = \alpha$. Indeed, a process in $\mathcal{W}$ which is the sum of a deterministic trend and a martingale component has constant weak time-derivative and the value of du/dt identifies the drift.

We now discuss the connection between monotonicity and sign of the weak time-derivative. Any notion of monotonicity in this framework must account of conditional expectation.

Proposition 7 *Let $u \in \mathcal{W}$. For a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$*

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \geqslant 0 \qquad a.s.$$

if and only if, for a.e. $t \in [0,T]$, for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 \geqslant \tau_1$

$$\mathbb{E}_t\left[L_{\tau_2} u(\tau_2)\right] \geqslant \mathbb{E}_t\left[L_{\tau_1} u(\tau_1)\right] \qquad a.s.$$

The analogous result holds with $\leqslant$ instead of $\geqslant$.

Hence, in conditional terms, we can associate the positivity of $L_\tau du/dt(\tau)$ with the increasing monotonicity of u. Similarly, a negative weak time-derivative reveals the decreasing monotonicity of u, after taking the conditional expectation. Interestingly, these features can be related to *submartingality* and *supermartingality* respectively.

Corollary 8 *Let $u \in \mathcal{W}$ be such that, for a.e. $t \in [0,T]$*

$$\mathbb{E}_t\left[L_\tau u(\tau)\right] \xrightarrow{a.s.} L_t u(t) \qquad \tau \longrightarrow t^+.$$

i) If, for a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \geqslant 0 \qquad a.s.,$$

then, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau u(\tau)\right] \geqslant L_t u(t).$$

ii) If, for a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \leqslant 0 \qquad a.s.,$$

then, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau u(\tau)\right] \leqslant L_t u(t).$$

Point $i)$ of Corollary 8 reminds of the submartingale property of the process $\mathbf{Lu}$, while point $ii)$ is reminiscent of supermartingality.

To conclude, we focus on the increments of the weak time-derivative, i.e. we deal with the convexity (or concavity) of u. The function u satisfies a convexity property when $L_\tau du/dt\,(\tau)$ is increasing, after taking the conditional expectation.

Proposition 9 *Let $u \in \mathcal{W}$. For a.e. $t \in [0,T]$, for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$*

$$\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right] \leqslant \mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right] \qquad a.s.$$

if and only if, for a.e. $t \in [0,T]$, for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$

$$\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right] \leqslant \frac{\mathbb{E}_t\left[L_{\tau_2}u(\tau_2)\right] - \mathbb{E}_t\left[L_{\tau_1}u(\tau_1)\right]}{\tau_2 - \tau_1} \leqslant \mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right] \qquad a.s.$$

The analogous result holds with $\geqslant$ instead of $\leqslant$.

Additional results about weak time-differentiability are presented in Appendix A. In these further results, we focus on Bochner integrable functions because they can be integrated directly, bypassing the expectation.

2.2 Comparison with the infinitesimal generator

We relate the notion of weak time-derivative with the one of *infinitesimal generator*, widely known in stochastic calculus. A further comparison of the applications of both instruments in option pricing is discussed in Subsection 4.1.

We begin with considering the infinitesimal incremental ratios of conditional expectations. We provide conditions that guarantee the convergence of these quantities to the weak time-derivative.

Proposition 10 *Let $u \in \mathcal{W}$ and $t \in [0,T]$. If for a.e. $\tau \in [t,T]$*

$$\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h}$$

is convergent in L^1 when $h \longrightarrow 0^+$, then for a.e. $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[L_\tau\frac{du}{dt}(\tau)\right] \qquad h \longrightarrow 0^+.$$

If, in addition, when $\tau \longrightarrow t^+$

$$L_\tau u(\tau) \xrightarrow{L^1} L_t u(t), \qquad L_\tau\frac{du}{dt}(\tau) \xrightarrow{L^1} L_t\frac{du}{dt}(t),$$

then

$$\frac{\mathbb{E}_t\left[L_{t+h}u(t+h)\right] - L_t u(t)}{h} \xrightarrow{L^1} L_t\frac{du}{dt}(t) \qquad h \longrightarrow 0^+.$$

Fixing $t \in [0,T]$, the outcome of Proposition 10 can be restated as

$$\mathbb{E}_t \left[L_{t+h} u\left(t+h\right) \right] - L_t u\left(t\right) - L_t \frac{du}{dt}\left(t\right) h \xrightarrow{L^1} 0 \qquad h \longrightarrow 0^+$$

or, equivalently,

$$\mathbb{E}_t \left[L_\tau u\left(\tau\right) \right] - L_t u\left(t\right) - L_t \frac{du}{dt}\left(t\right) \left(\tau - t\right) \xrightarrow{L^1} 0 \qquad \tau \longrightarrow t^+,$$

which is a first-order expansion of $\mathbb{E}_t \left[L_\tau u\left(\tau\right) \right]$ in a right neighbourhood of t, with the limit taken in L^1. This is the natural way to use weak time-derivatives for first-order approximations.

As described in Revuz and Yor (1999), the infinitesimal generator of a Feller process $\mathbf{X} = \{X_t\}_{t \in [0,T]}$ is the operator $\mathcal{A}$ that maps any continuous bounded function f belonging to the domain of $\mathcal{A}$ into the function $\mathcal{A}f$ such that, for any $t \in [0,T]$,

$$\mathcal{A}f\left(X_t\right) = \lim_{h \to 0^+} \frac{\mathbb{E}_t \left[f\left(X_{t+h}\right) \right] - f\left(X_t\right)}{h}.$$

The limit, here, is in the uniform topology and so $\mathcal{A}f$ is continuous and bounded.

If L_τ is constantly equal to 1, Proposition 10 shows that the weak time-derivative generally works as the infinitesimal generator with the limit $h \longrightarrow 0^+$ taken in the L^1-norm. However, the weak time-derivative is a more general notion than the infinitesimal generator. Indeed, under suitable assumptions the weak time-derivative turns out to be equal to the infinitesimal generator.

Corollary 11 *Let $L_\tau \equiv 1$ and $u \in \mathcal{W}$ be such that, for every $t \in [0,T]$*

$$u\left(\tau\right) \xrightarrow{L^1} u\left(t\right), \qquad \frac{du}{dt}\left(\tau\right) \xrightarrow{L^1} \frac{du}{dt}\left(t\right) \qquad \tau \longrightarrow t^+.$$

Assume that $u\left(t\right)$ is not explicitly dependent on t and it defines a continuous bounded function of X_t, i.e. $u\left(t\right) = f\left(X_t\right)$ for any $t \in [0,T]$, with f in the domain of $\mathcal{A}$. Then, for every $t \in [0,T]$,

$$\frac{du}{dt}\left(t\right) = \mathcal{A}f\left(X_t\right) \qquad a.s.$$

As we will see in Subsection 4.1, the weak time-derivative provides a generalized formulation of operator equations that involve the infinitesimal generator, such as the eigenvalue-eigenvector problem $\mathcal{A}f = rf$, with $r > 0$. This generalization is made possible by the fact that both instruments provide a similar characterization of martingales. Indeed, the process $\{f\left(X_t\right)\}_{t \in [0,T]}$ is a martingale when the infinitesimal generator of f is null, as ensured by Proposition 1.6 in Chapter VII of Revuz and Yor (1999). This result actually shares the same insight of Proposition 3 and Corollary 4 that relate the martingale property to the nullity of weak time-derivatives.

However, the last remark holds when $L_\tau \equiv 1$. If L_τ is arbitrary, the weak time-derivative parallels the *extended generator* used by Hansen and Scheinkman (2009).

3 No arbitrage pricing

We consider a continuous-time market with two assets: a bond with constant interest rate r and price process $\mathbf{B} = \{B_t\}_{t\in[0,T]}$ and a risky asset with price process $\mathbf{X} = \{X_t\}_{t\in[0,T]}$. We take into consideration the filtered probability space $(\Omega, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by $\mathbf{X}$ and P is a physical probability.

We call *arbitrage* possibility any self-financing portfolio whose value V satisfies:

$$V_0 = 0, \qquad P\left(V_T \geqslant 0\right) = 1, \qquad P\left(V_T > 0\right) > 0.$$

As in Subsection 1.1, we assume that there exists a risk-neutral probability Q. In this case, the no arbitrage price process of a European derivative with $\mathcal{F}_T$-measurable payoff function h_T is $\boldsymbol{\pi} = \{\pi(t)\}_{t\in[0,T]}$ such that

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t^Q\left[h_T\right].$$

The Radon-Nikodym derivative of Q with respect to P, denoted by L_T, allows us to compute $\mathbb{E}^Q\left[h_T\right] = \mathbb{E}\left[L_T h_T\right]$. By the Bayes' rule of conditional expectation, since $L_t = \mathbb{E}_t\left[L_T\right]$, we obtain

$$\mathbb{E}_t^Q\left[h_T\right] = \mathbb{E}_t\left[\frac{L_T}{L_t}h_T\right] = \mathbb{E}_t\left[L_{t,T}h_T\right],$$

where $L_{t,T} = L_T/L_t$. As a result,

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t\left[L_{t,T}h_T\right]. \tag{3}$$

We refer to $\pi(t)$ as the *no arbitrage pricing function* (or *risk-neutral pricing function*) of the derivative with terminal payoff h_T. Note also that the discounted price process $\left\{e^{-rt}L_t\pi(t)\right\}_{t\in[0,T]}$ satisfies the *martingale property*

$$\frac{L_t\pi(t)}{e^{rt}} = \mathbb{E}_t\left[\frac{L_\tau\pi(\tau)}{e^{r\tau}}\right] \qquad \forall\tau\in[t,T].$$

The use of Radon-Nikodym derivative shows that martingality holds under the risk-neutral probability measure.

The price of the riskless bond can be obtained by taking a constant terminal payoff in eq. (3). In particular, the bond price satisfies the boundary problem

$$\begin{cases} \frac{dB(t)}{dt} = rB(t) & t \in [0,T) \\ B(T) = e^{rT} \end{cases}$$

In words, with continuous compounding, the rate of change of $B(t)$ is proportional to $B(t)$ and the coefficient of proportionality coincides with r.

In the following we show that the no arbitrage pricing function $\pi(t)$ of a European derivative satisfies the analogous boundary problem

$$\begin{cases} \frac{d\pi}{dt}(t) = r\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases}$$

where $L_T h_T \in L^2(\mathcal{F}_T)$ and du/dt is the weak time-derivative defined in Section 2. In particular, the equality $\frac{d\pi}{dt}(t) = r\pi(t)$ means that for a.e. $t \in [0,T]$ the weak time-derivative of π equals $r\pi$ almost surely.

The financial interpretation of the problem is straightforward once we recall that $d\pi/dt\,(t)$ is $\mathcal{F}_t$-measurable for every $t \in [0,T]$. Then, the infinitesimal variation $d\pi/dt\,(t)$ is known at time t and the no arbitrage condition imposes that $d\pi/dt$ behaves as the deterministic bond. In other words, the rate of change of $\pi(t)$ must be proportional to $\pi(t)$ as it is for the riskless asset price.

In our setting the boundary condition alone is not enough to guarantee uniqueness of the solution. Therefore, we require that $L_t\pi(t)$ converges to $L_T h_T$ in L^1 as t approaches T. Specifically, we solve

$$\begin{cases} \frac{d\pi}{dt}(t) = r\pi(t) & t \in [0,T) \\ \pi(T) = h_T & \\ L_t\pi(t) \xrightarrow{L^1} L_T h_T & t \longrightarrow T \end{cases} \tag{4}$$

with $L_T h_T \in L^2(\mathcal{F}_T)$.

Theorem 12 $\pi(t) = e^{-r(T-t)}\mathbb{E}_t[L_{t,T}h_T]$ *is a solution of Problem* (4) *in* $\mathcal{W}$. *Moreover, it is the unique solution in* $\mathcal{W}$.

The proof of uniqueness exploits the fact that the weak time-derivative of $e^{-rt}\pi(t)$ is null a.s., which is the counterpart of the martingale property of the discounted price process in our framework.

Suppose now that the interest rate is time dependent and that $r(t)$ is Lebesgue measurable and bounded. Then, Theorem 12 applies accordingly and the no arbitrage pricing function

$$\pi(t) = e^{-\int_t^T r(s)ds}\mathbb{E}_t[L_{t,T}h_T]$$

is the unique solution of problem

$$\begin{cases} \frac{d\pi}{dt}(t) = r(t)\pi(t) & t \in [0,T) \\ \pi(T) = h_T & \\ L_t \pi(t) \xrightarrow{L^1} L_T h_T & t \longrightarrow T. \end{cases}$$

A more general treatment of the same problem with stochastic interest rates is provided in Section 5.

3.1 Example: Black-Scholes model

Black and Scholes (1973) model involves a continuous-time market with a riskless bond and a risky asset, as the one we have described. In the filtered probability space $(\Omega, \mathbb{F}, P)$, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by a P-Wiener process $\overline{\mathbf{W}} = \{\overline{W}_t\}_{t\in[0,T]}$. The bond and stock prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t d\overline{W}_t,$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and $r \in \mathbb{R}$ is the risk-free rate. Girsanov Theorem ensures that there exists a probability measure Q equivalent to P under which the discounted stock price process is a martingale. According to the First Fundamental Theorem of Asset Pricing the market is, then, arbitrage-free. In particular, the dynamics of the stock price under Q are

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

where $\mathbf{W} = \{W_t\}_{t\in[0,T]}$ is a Q-Wiener process. Hence, the risky asset and the bond must share the same drift given by the interest rate r in order to exclude any arbitrage possibility. See Björk (2004) as a reference.

As before, the no arbitrage price process of a European derivative with $\mathcal{F}_T$-measurable payoff h_T is $\boldsymbol{\pi} = \{\pi(t)\}_{t\in[0,T]}$ such that

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t^Q[h_T],$$

where the conditional expectation is computed according to Q. Specifically, in Black-Scholes model $\pi(t)$ is a deterministic function of t and X_t. Since the discounted price process $\{e^{-rt}\pi(t)\}_{t\in[0,T]}$ is also a Q-martingale, the drift of $\boldsymbol{\pi}$ will be equal to r, too. This is the crucial property which is captured, more in general, by Problem (4), where no special price dynamics are assumed. This is also the intuition that drives Cox and Ross (1976) derivation of Black-Scholes equation, that is based on a hedging argument.

3.2 Valuation of cashflows

The payoff of a European derivative with maturity T can be seen as a special cashflow in which there is a unique random payment at time T. Indeed, the no arbitrage theory described so far generalizes to the pricing of payoff streams.

In particular, we consider an adapted cashflow $\mathbf{h} = \{h_t\}_{t\in[0,T]}$ which defines a function $h : [0,T] \longrightarrow \mathbb{R}^{\Omega}$. We suppose that $L^2(\cdot)h^2(\cdot)$ is Bochner integrable with respect to a finite measure μ on $[0,T]$ that weighs cashflows over time.

The no arbitrage price process $\boldsymbol{\pi} = \{\pi(t)\}_{t\in[0,T]}$ of $\mathbf{h}$ is the expected discounted value of future cashflows under the risk-neutral probability, i.e.

$$\pi(t) = \mathbb{E}_t\left[\int_t^T e^{-r(m-t)} L_{t,m} h_m \mu(dm)\right]. \tag{5}$$

For example, if μ is a counting measure, the previous formula evaluates a finite number or a sequence of future payments. In case μ is absolutely continuous, we are pricing instead a continuous stream of payoffs. In general, it is convenient to write $\mu(dm) = p_m dm$. In fact, if μ is absolutely continuous, p_m denotes the Radon-Nikodym derivative of μ with respect to Lebesgue measure, otherwise p_m has to be intended in the sense of distributions.

We also assume that the function $h(\cdot)p(\cdot)$ belongs to $\mathcal{V}$. Hence, we are able to show that the risk-neutral pricing formula for cashflows satisfies the differential equation

$$\frac{d\pi}{dt}(t) = r\pi(t) - h_t p_t \qquad\qquad t \in [0,T],$$

where $d\pi/dt$ is the weak time-derivative of π. If h_t is null except for the time T and μ has mass concentrated in T, we retrieve as special case the differential equation of Problem (4) about European options with maturity T.

Proposition 13 $\pi(t) = \mathbb{E}_t\left[\int_t^T e^{-r(m-t)} L_{t,m} h_m \mu(dm)\right]$ *belongs to* $\mathcal{W}$ *and it is a solution of the equation*

$$\frac{d\pi}{dt}(t) = r\pi(t) - h_t p_t \qquad\qquad t \in [0,T].$$

Observe that a term analogous to $-h_t p_t$ is added in Feynman-Kac equation when a stream of dividends is present. See Duffie (2010) as a reference.

4 An operator approach

In this section we define the spaces and operators that allow us to formalize Problem (4) as an eigenvalue-eigenvector problem.

We start with observing that the no arbitrage pricing function π is weakly time-differentiable infinitely many times. Indeed, $d\pi/dt$ belongs to $\mathcal{V}$ and it equals the original π except for the multiplicative constant r. Hence, $d\pi/dt$ is weakly time-differentiable, too. By defining the subspace of $\mathcal{V}$

$$\mathcal{Z} = \{\text{infinitely} \quad \text{weakly} \quad \text{time} - \text{differentiable} \quad u \in \mathcal{V}\},$$

we have that $\pi \in \mathcal{Z}$. Moreover, the weak time-derivative defines a linear operator $\mathbb{A} : \mathcal{Z} \longrightarrow \mathcal{Z}$ by

$$\mathbb{A} : \quad u \longmapsto \frac{du}{dt}.$$

$\mathbb{A}$ allows us to rewrite the differential equation of Problem (4) in the operator form

$$\mathbb{A}\pi = r\pi, \qquad \pi \in \mathcal{Z}. \tag{6}$$

Hence, we obtain an eigenvalue-eigenvector problem, which is the same problem faced by Hansen and Scheinkman (2009) where, instead of $\mathbb{A}$, the *extended generator* of the underlying Markov process is involved. In our setting the no arbitrage pricing function π is an eigenfunction of the operator $\mathbb{A}$, associated with the eigenvalue r. Moreover, the process $\left\{e^{-rt} L_t \pi\left(t\right)\right\}_t$ is a martingale.

Following Hansen and Scheinkman (2009), we choose a positive payoff h_T. The positivity of h_T is related to the requirement of π to be an eigenfunction related to the *principal eigenvalue* in Hansen and Scheinkman (2009). Indeed, Hansen and Scheinkman generalize the Perron-Frobenius theory (see Meyer (2000)) from the finite-state Markov chain setting to more abstract frameworks.

Then, we define

$$\hat{L}_t = e^{-rt} L_t \frac{\pi\left(t\right)}{\pi\left(0\right)},$$

which still satisfies the martingale property and the multiplicative property

$$\hat{L}_0 = 1, \qquad \hat{L}_{t+u} = \hat{L}_u\left(\theta_t\right)\hat{L}_t.$$

In addition, the stochastic discount factor S_t decomposes as

$$S_t = \hat{L}_t \frac{\pi\left(0\right)}{\pi\left(t\right)} = e^{-rt}\hat{L}_t \frac{\tilde{\pi}\left(0\right)}{\tilde{\pi}\left(t\right)},$$

where we define $\tilde{\pi}\left(t\right) = \mathbb{E}_t\left[L_{t,T} h_T\right]$. In the last decomposition $-r$ is referred to as the *growth rate* of S_t, $\hat{L}_t$ is the *martingale component* and $\tilde{\pi}\left(0\right)/\tilde{\pi}\left(t\right)$ is the *transient component*. However, the decomposition is not unique.

This kind of results has proved to be fruitful in the macro-financial literature. For instance, Alvarez and Jermann (2005) employ the last decomposition to quantify the dynamics of stochastic discount factors. Moreover, an application to the study of long-term risk-return trade-off for the valuation of cash flows is described in Hansen, Heaton, and Li (2008).

4.1 Comparison with the infinitesimal generator

As we saw in Proposition 10, in case $L_t \equiv 1$ the weak time-derivative provides a way to differentiate random processes which generalizes the traditional infinitesimal generator for a Feller process $\mathbf{X} = \{X_t\}_{t\in[0,T]}$. Moreover, if the infinitesimal generator of f is null, then the process $\{f(X_t)\}_{t\in[0,T]}$ is a martingale, a fact that parallels Proposition 3 and Corollary 4. In particular, simple computations show that the no arbitrage pricing function

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t[h_T]$$

satisfies the eigenvalue-eigenvector problem $\mathcal{A}\pi = r\pi$. We refer to $\mathcal{A}\pi = r\pi$ as a *strong form* eigenvalue-eigenvector problem, while Problem (4), rewritten as (6), defines its *generalized form*.

In addition, it holds that $\mathcal{A}\left(e^{-rt}\pi(t)\right) = 0$, hence the discounted price process $\left\{e^{-rt}\pi(t)\right\}_{t\in[0,T]}$ is a martingale. By exploiting the terminal condition $\pi(T) = h_T$, this fact ensures that π is the unique solution of the problem in strong form.

Finally, we observe that the convergence requirement of Problem (4), namely $\pi(t) \xrightarrow{L^1} h_T$ as t approaches T, is replaced here by the more general Feller property.

5 No arbitrage pricing with stochastic interest rates

We provide a refinement of the theory described so far in order to solve the no arbitrage pricing equation when interest rates are stochastic. In this case we have two sources of randomness. Indeed, we consider two Markov processes $\mathbf{X} = \{X_t\}_{t\in[0,T]}$ and $\mathbf{Y} = \{Y_t\}_{t\in[0,T]}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$. As for the interpretation, $\mathbf{X}$ is associated with the underlying stock, while $\mathbf{Y}$ affects the interest rates. For any instant t, $X_t, Y_t : \Omega \longrightarrow \mathbb{R}$ and $\mathbb{R}$ is endowed with the Borel σ-algebra. We consider the filtration generated by the pair $(\mathbf{X}, \mathbf{Y})$ and denote it by $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. As before, we assume that there exists a risk-neutral probability Q equivalent to P and we consider the process $\mathbf{L} = \{L_t\}_{t\in[0,T]}$ such that $L_t = \mathbb{E}_t[L_T]$, where L_T is the Radon-Nikodym derivative of Q with respect to P.

In this section we assign a stronger meaning to the notions of conditional weak Lebesgue measurability and weak time-differentiability. To distinguish the new definitions from the analogous ones of Section 2 we will write r-, that stays for *robust*.

Specifically, we say that a function $u : [0,T] \longrightarrow \mathbb{R}^\Omega$ is *conditionally r-weakly Lebesgue measurable* when, for every $t \in [0,T]$, the function

$$\tau \in [t,T] \longmapsto \mathbb{E}\left[L_\tau u(\tau)\, \mathbf{1}_{A_t} \psi(\tau)\right]$$

is Lebesgue measurable for any $\mathcal{F}_t$-measurable set A_t and any adapted function $\psi \in C_c\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right)$. By saying that ψ is adapted, we mean that $\psi(\tau)$ is $\mathcal{F}_\tau$-measurable for all τ in $[t,T]$. If ψ is constantly equal to 1, we retrieve the requirement of conditional weak Lebesgue measurability discussed in Section 2. Accordingly, we define the space

$$\begin{aligned}\mathcal{V}_r = \Bigg\{ u : [0,T] \longrightarrow \mathbb{R}^\Omega : \quad & L_\tau u(\tau) \in L^2\left(\mathcal{F}_\tau\right) \quad \forall \tau, \\ u \quad \text{conditionally} \quad \text{r-w.L.m.}, \quad & \int_0^T \mathbb{E}\left[L_\tau^2 u^2(\tau)\right] d\tau < +\infty \Bigg\},\end{aligned}$$

which is a subspace of the space $\mathcal{V}$.

In this context *r-weak time-differentiability* involves a larger set of test functions than weak time-differentiability of Definition 1. Hence, r-weak time-differentiability turns out to be a stronger requirement.

Definition 14 *Given $u \in \mathcal{V}_r$, we say that u is* r-weakly time-differentiable *when there exists a function $w \in \mathcal{V}_r$ such that for every $t \in [0,T]$*

$$\int_t^T \mathbb{E}\left[L_\tau w(\tau)\, \mathbf{1}_{A_t} \varphi(\tau)\right] d\tau = -\int_t^T \mathbb{E}\left[L_\tau u(\tau)\, \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right) \quad \textit{adapted.}$$

In this case we call w a r-weak time-derivative *of u.*

Definition 14 is well-posed because the integrals

$$\int_t^T \mathbb{E}\left[L_\tau u(\tau)\, \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau, \qquad \int_t^T \mathbb{E}\left[L_\tau \frac{du}{dt}(\tau)\, \mathbf{1}_{A_t} \varphi(\tau)\right] d\tau$$

are finite for any choice of A_t and φ as required. Indeed, φ and $d\varphi/dt$ are continuous functions that take values in $L^\infty\left(\mathcal{F}_T\right)$, hence their image is bounded.

We finally define the space

$$\mathcal{W}_r = \left\{\text{r-weakly} \quad \text{time-differentiable} \quad u \in \mathcal{V}_r\right\}.$$

If u is r-weakly time-differentiable, it is also weakly time-differentiable because the test functions φ, that are random variables, may specialize to deterministic functions. This simple observation allows us to inherit some of the results of Section 2. For instance, the r-weak time-derivative is still unique. Moreover, if a function $u \in \mathcal{V}_r$ is r-weakly time-differentiable with $du/dt = 0$, then for a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t \left[L_\tau u (\tau) \right] = f_t \qquad a.s.$$

with $f_t \in L^2 (\mathcal{F}_t)$.

We assume that the interest rate is a Lebesgue measurable function $r : [0,T] \longrightarrow L^\infty (\mathcal{F}_T)$ which is adapted. In addition, we impose that interest rates are uniformly bounded over time, i.e. there is a positive R such that

$$|r(t)| \leqslant R \qquad a.s. \qquad \forall t \in [0,T].$$

Lebesgue measurability and boundedness ensure the Bochner integrability of r. As a result, the Bochner integral $\int_0^T r(\tau) d\tau$ is a well-defined object in $L^\infty (\mathcal{F}_T)$.

Furthermore, given any state $\omega \in \Omega$, consider the restriction $r_\omega : [0,T] \to \mathbb{R}$ of r on the path induced by ω. We assume that the map from $[0,T] \times \Omega$ to $\mathbb{R}$ such that $(t, \omega) \longmapsto r_\omega(t)$ is measurable. This assumption allows us to compute the Bochner integral of r as the pathwise Lebesgue integral of r_ω. Indeed, since r is Bochner integrable, Bochner integral and pathwise Lebesgue integral coincide almost surely. See Lemma 3 in Appendix B.

Since now interest rates are stochastic, the no arbitrage pricing differential equation is

$$\begin{cases} \frac{d\pi}{dt} (t) = r(t) \pi (t) & t \in [0,T) \\ \pi (T) = h_T & \\ L_t \pi (t) \xrightarrow{L^1} L_T h_T & t \longrightarrow T \end{cases} \tag{7}$$

with $L_T h_T \in L^2 (\mathcal{F}_T)$. Differently from Problem (4), now each $r(t) \in L^\infty (\mathcal{F}_t)$ and $d\pi/dt$ represents the r-weak time-derivative. The unique solution of this problem in $\mathcal{W}_r$ is

$$\pi (t) = \mathbb{E}_t \left[L_{t,T} e^{-\int_t^T r(s) ds} h_T \right].$$

Theorem 15 $\pi (t) = \mathbb{E}_t \left[L_{t,T} e^{-\int_t^T r(s) ds} h_T \right]$ *is a solution of Problem* (7) *in* $\mathcal{W}_r$. *Moreover, it is the unique solution in* $\mathcal{W}_r$.

The proof is more involved than that of Theorem 12 and it exploits the relation between Bochner and pathwise Lebesgue integrability.

6 Conclusion

We introduced the weak time-derivative, a novel mathematical tool that allows us to differentiate stochastic processes in a more general way than the infinitesimal generator. It provides easy characterizations of martingales and permits to formulate differential equations for random processes in weak form. Therefore, we expect this instrument to be suitable for different kinds of differential problems, beyond the ones discussed in this work.

As we described in the body of the paper, a fruitful application of the weak time-derivative involves the solution of the no arbitrage pricing equation for European options. In particular, the generalized form that we solve clarifies the central role of interest rates in driving the asset prices, with both deterministic and stochastic short-term rates. In addition, constant interest rates deliver an eigenvalue-eigenvector formulation of the risk-neutral pricing equation in full agreement with the long-term risk literature. Nevertheless, how to set up the analogous eigenvalue-eigenvector problem when interest rates are time-varying or stochastic still remains an open problem. Indeed, the candidate eigenvalue would be a function or a random process. Moreover, such a formulation should be able to generate a term structure of interest rates. We leave this and other interesting questions for future research.

A Additional results about weak time-differentiability

The following results assume Lebesgue measurability and involve Bochner integrability. In this case we can integrate a function $u \in \mathcal{V}$ directly because we are not forced to employ the expectation to ensure measurability.

Lemma 1 *Let $u \in \mathcal{V}$ such that for a.e. $t \in [0,T]$*

$$L_t u(t) = \int_0^t L_s g(s)\, ds,$$

where $g \in \mathcal{V}$ and $L(\cdot)g(\cdot)$ is Lebesgue measurable. Then, u is weakly time-differentiable with $du/dt = g$.

Proof. First, observe that the function $L(\cdot)g(\cdot)$ is Bochner integrable, as discussed in Section 2. We want to show that $g \in \mathcal{V}$ satisfies the definition of weak time-derivative of u. Given $t \in [0,T]$, consider any $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$. Then

$$\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau = \int_t^T \mathbb{E}\left[\int_0^\tau L_s g(s)\, ds \mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau.$$

As the expectation defines a bounded operator, according to Lemma 11.45 in Aliprantis and Border (2006) we can exchange the order of expectation and Bochner integral. After that, we apply Fubini's Theorem and exploit the compact support of φ:

$$\begin{aligned}
\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau &= \int_t^T \left(\int_0^\tau \mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\frac{d\varphi}{dt}(\tau)\right]ds\right)d\tau \\
&= \int_0^t \left(\int_t^T \mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau\right)ds \\
&\quad + \int_t^T \left(\int_s^T \mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau\right)ds \\
&= \int_0^t \left(\mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\right]\int_t^T \frac{d\varphi}{dt}(\tau)d\tau\right)ds \\
&\quad + \int_t^T \left(\mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\right]\int_s^T \frac{d\varphi}{dt}(\tau)d\tau\right)ds \\
&= -\int_t^T \mathbb{E}\left[L_s g(s)\mathbf{1}_{A_t}\right]\varphi(s)ds.
\end{aligned}$$

In consequence, g is the weak time-derivative of u. ■

The next result is reminiscent of the Fundamental Theorem of Integral Calculus.

Proposition 16 *Let $u \in \mathcal{W}$ such that $L(\cdot)du/dt(\cdot)$ is Lebesgue measurable. Then, for a.e. $t \in [0,T]$ there exists $\bar{u}_t \in \mathcal{W}$ such that $L(\cdot)\bar{u}_t(\cdot) \in C([t,T], L^\infty(\mathcal{F}_T))$, for a.e. $\tau \in [t,T]$*

$$\mathbb{E}_t\left[L_\tau u(\tau)\right] = \mathbb{E}_t\left[L_\tau \bar{u}_t(\tau)\right]$$

and, for any $\tau_1, \tau_2 \in [t, T]$,

$$L_{\tau_2}\bar{u}_t(\tau_2) - L_{\tau_1}\bar{u}_t(\tau_1) = \int_{\tau_1}^{\tau_2} L_s \frac{du}{dt}(s)\, ds.$$

Proof. du/dt belongs to $\mathcal{V}$ and $L(\cdot)du/dt(\cdot)$ is Lebesgue measurable, therefore $L(\cdot)du/dt(\cdot)$ is Bochner integrable and we can consider the function $\hat{u}(\cdot) : [0, T] \longrightarrow L^\infty(\mathcal{F}_T)$ defined by

$$\hat{u}(\tau) = \frac{1}{L_\tau} \int_0^\tau L_s \frac{du}{dt}(s)\, ds.$$

As $L(\cdot)\hat{u}(\cdot)$ is a primitive of a Bochner integrable function, $L(\cdot)\hat{u}(\cdot)$ belongs to $C([0, T], L^\infty(\mathcal{F}_T))$. See Hille and Phillips (1996) as a reference.

By Lemma 1, du/dt is the weak time-derivative of $\hat{u}$. As du/dt is also the weak time-derivative of u, the weak time-derivative of $u - \hat{u}$ is null. By Proposition 3, for a.e. $t \in [0, T]$, for a.e. $\tau, s \in [t, T]$

$$\mathbb{E}_t[L_\tau u(\tau)] - \mathbb{E}_t[L_\tau \hat{u}(\tau)] = \mathbb{E}_t[L_s u(s)] - \mathbb{E}_t[L_s \hat{u}(s)].$$

Choose $s = s_t$ in a way that the above equality is satisfied and consider the function $\bar{u}_t(\cdot) : [t, T] \longrightarrow L^\infty(\mathcal{F}_T)$ defined by

$$\bar{u}_t(\tau) = \hat{u}(\tau) + \frac{L_{s_t}}{L_\tau} u(s_t) - \frac{L_{s_t}}{L_\tau} \hat{u}(s_t).$$

$L(\cdot)\bar{u}_t(\cdot) \in C([t, T], L^\infty(\mathcal{F}_T))$ because $L(\cdot)\hat{u}(\cdot) \in C([0, T], L^\infty(\mathcal{F}_T))$. Moreover, for a.e. $\tau \in [t, T]$

$$\begin{aligned}
\mathbb{E}_t[L_\tau u(\tau)] &= \mathbb{E}_t[L_\tau \hat{u}(\tau)] + \mathbb{E}_t[L_{s_t} u(s_t)] - \mathbb{E}_t[L_{s_t} \hat{u}(s_t)] \\
&= \mathbb{E}_t[L_\tau \hat{u}(\tau)] + \mathbb{E}_t\left[L_\tau \frac{L_{s_t}}{L_\tau} u(s_t)\right] - \mathbb{E}_t\left[L_\tau \frac{L_{s_t}}{L_\tau} \hat{u}(s_t)\right] \\
&= \mathbb{E}_t\left[L_\tau \left\{\hat{u}(\tau) + \frac{L_{s_t}}{L_\tau} u(s_t) - \frac{L_{s_t}}{L_\tau} \hat{u}(s_t)\right\}\right] \\
&= \mathbb{E}_t[L_\tau \bar{u}(\tau)].
\end{aligned}$$

Since

$$L_\tau \bar{u}_t(\tau) = L_\tau \hat{u}(\tau) + L_{s_t} u(s_t) - L_{s_t} \hat{u}(s_t),$$

we have that, for any $\tau_1, \tau_2 \in [t, T]$,

$$\begin{aligned}
L_{\tau_2}\bar{u}_t(\tau_2) - L_{\tau_1}\bar{u}_t(\tau_1) &= L_{\tau_2}\hat{u}_t(\tau_2) - L_{\tau_1}\hat{u}_t(\tau_1) \\
&= \int_0^{\tau_2} L_s \frac{du}{dt}(s)\, ds - \int_0^{\tau_1} L_s \frac{du}{dt}(s)\, ds \\
&= \int_{\tau_1}^{\tau_2} L_s \frac{du}{dt}(s)\, ds.
\end{aligned}$$

■

B Proofs

Lemma 2 *Let* $f : [t,T] \longrightarrow \mathbb{R}$.

i) If f *is bounded, nonnegative, with compact support and* $\int_t^T f(\tau)g(\tau)d\tau = 0$ *for any* $g \in C_c([t,T],\mathbb{R})$*, then* $f = 0$ *a.e.*

ii) If f *is measurable and* $\int_t^T f(\tau)g(\tau)d\tau = 0$ *for any* $g \in C_c([t,T],\mathbb{R})$*, then* $f = 0$ *a.e.*

Proof.

$i)$ If f is strictly positive on a set A with positive measure, consider the indicator function $\mathbf{1}_A$ and a sequence $\{U_n\}_n$ of continuous positive approximations of $\mathbf{1}_A$, obtained by convolution with a smooth positive kernel. As U_n converges to $\mathbf{1}_A$ in L^2,

$$0 \leqslant \int_t^T f(\tau)\mathbf{1}_A(\tau)d\tau = \lim_n \int_t^T f(\tau)U_n(\tau)d\tau = 0.$$

In consequence, f is null a.e.

$ii)$ Suppose that f is positive with compact support. For any $N > 0$ consider $f_N(s) = \min\{f(\tau), N\}$. Then

$$0 \leqslant \int_t^T f_N(\tau)g(\tau)d\tau \leqslant \int_t^T f(\tau)g(\tau)d\tau = 0.$$

Therefore, each f_N is null a.e. by $i)$ and so f is.

■

Proof of Proposition 2

Let w and $\hat{w}$ be two weak time-derivatives of u. Then, for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[L_\tau \left\{w(\tau) - \hat{w}(\tau)\right\} \mathbf{1}_{A_t}\right] \varphi(\tau)d\tau = 0$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1([t,T],\mathbb{R}).$$

By Lemma 2, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}\left[L_\tau \left\{w(\tau) - \hat{w}(\tau)\right\} \mathbf{1}_{A_t}\right] = \mathbb{E}\left[0\mathbf{1}_{A_t}\right] = 0 \qquad \forall A_t \in \mathcal{F}_t.$$

Hence, the a.s. null function fits the definition of conditional expectation of $L_\tau \left\{w(\tau) - \hat{w}(\tau)\right\}$ with respect to $\mathcal{F}_t$. Therefore, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}_t\left[L_\tau w(\tau)\right] = \mathbb{E}_t\left[L_\tau \hat{w}(\tau)\right] \qquad a.s.$$

Now fix $\tau \in [0,T]$. Except for a set with measure zero of values of τ, we can find an increasing sequence $\{t_i\}_{i\in\mathbb{N}} \subset [t,\tau]$ such that $t_i \to \tau$ and

$$\mathbb{E}_{t_i}\left[L_\tau w\left(\tau\right)\right] = \mathbb{E}_{t_i}\left[L_\tau \hat{w}\left(\tau\right)\right] \qquad a.s.$$

By Lévy's Upward Theorem, i.e. Theorem 14.2 in Williams (1991), when t_i approaches τ,

$$\mathbb{E}_{t_i}\left[L_\tau w\left(\tau\right)\right] \longrightarrow \mathbb{E}_{\tau}\left[L_\tau w\left(\tau\right)\right] = L_\tau w\left(\tau\right) \qquad a.s.$$

Since the last relation holds also for $\hat{w}$ and L_τ is strictly positive, we deduce that $w\left(\tau\right) = \hat{w}\left(\tau\right)$ almost surely for a.e. $\tau \in [0,T]$.

Proof of Proposition 3

Let $t \in [0,T]$ be such that for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right] = f_t \qquad a.s.$$

Then, for all $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$,

$$\begin{aligned}
\int_t^T \mathbb{E}\left[L_\tau u\left(\tau\right)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau &= \int_t^T \mathbb{E}\left[\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right]\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau \\
&= \int_t^T \mathbb{E}\left[f_t\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau \\
&= \mathbb{E}\left[f_t\mathbf{1}_{A_t}\right]\int_t^T \frac{d\varphi}{dt}(\tau)d\tau \\
&= 0
\end{aligned}$$

because φ is a function in $C_c^1\left([t,T],\mathbb{R}\right)$. As a result, $w = 0$ a.s. for a.e. $t \in [0,T]$ satisfies the definition of weak time-derivative of u. By uniqueness of the weak time-derivative, we claim that $du/dt = 0$.

Conversely, suppose that $u \in \mathcal{V}$ is weakly time-differentiable with $du/dt = 0$. We show that, given $t \in [0,T]$, $\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take a continuous function $\eta : [t,T] \longrightarrow \mathbb{R}$ with compact support such that

$$\int_t^T \eta(\tau)d\tau = 1.$$

Given a continuous function $\xi : [t,T] \longrightarrow \mathbb{R}$ with compact support, we define the function $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(s) = \xi(s) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(s).$$

k_ξ is continuous with compact support and

$$\int_t^T k_\xi(\tau)d\tau = 0.$$

Hence, k_ξ has a primitive K_ξ which is continuous with compact support. As $K_\xi \in C_c^1([t,T],\mathbb{R})$, we employ it as a test function in the definition of weak time-derivative of u. Since $du/dt = 0$, for all $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}
0 &= \int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \left\{\xi(s) - \left(\int_t^T \xi(\tau) d\tau\right) \eta(s)\right\} ds \\
&= \int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \xi(s) ds - \int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \left(\int_t^T \xi(\tau) d\tau\right) \eta(s) ds \\
&= \int_t^T \mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right] \xi(\tau) d\tau - \int_t^T \left\{\int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \eta(s) ds\right\} \xi(\tau) d\tau \\
&= \int_t^T \left\{\mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \eta(s) ds\right\} \xi(\tau) d\tau.
\end{aligned}$$

By the density of continuous functions ξ, Lemma 2 ensures that for a.e. $\tau \in [t,T]$

$$\mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] \eta(s) ds.$$

Since $\int_t^T \eta(s) ds = 1$, we can rewrite the left-hand side as $\int_t^T \mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right] \eta(s) ds$ so that

$$\int_t^T \left\{\mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right] - \mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right]\right\} \eta(s) ds = 0.$$

As the last equality holds for any continuous function η with compact support in $[t,T]$, we have that, for a.e. $s \in [t,T]$,

$$\mathbb{E}\left[L_s u\left(s\right) \mathbf{1}_{A_t}\right] = \mathbb{E}\left[L_\tau u\left(\tau\right) \mathbf{1}_{A_t}\right].$$

Since A_t is any $\mathcal{F}_t$-measurable set, we deduce that, for a.e. $s \in [t,T]$,

$$\mathbb{E}_t\left[L_s u\left(s\right)\right] = \mathbb{E}_t\left[L_\tau u\left(\tau\right)\right] \qquad a.s.$$

As a result, $\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and so we can state that

$$\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right] = f_t \qquad a.s.$$

for some $\mathcal{F}_t$-measurable function f_t. By Jensen's inequality, $\mathbb{E}_t\left[L_\tau u\left(\tau\right)\right] \in L^2\left(\mathcal{F}_t\right)$ since

$$\mathbb{E}\left[\left(\mathbb{E}_t\left[L_\tau u(\tau)\right]\right)^2\right] \leqslant \mathbb{E}\left[\mathbb{E}_t\left[L_\tau^2 u^2(\tau)\right]\right] = \mathbb{E}\left[L_\tau^2 u^2(\tau)\right],$$

which is finite because $L_\tau u(\tau) \in L^2\left(\mathcal{F}_\tau\right)$. Hence, $f_t \in L^2\left(\mathcal{F}_t\right)$, too.

Proof of Corollary 4

i) Since for every $t \in [0, T]$

$$\mathbb{E}_t \left[L_\tau u(\tau) \right] = L_t u(t) \qquad \forall \tau \in [t, T],$$

Proposition 3 ensures that u is weakly time-differentiable with $du/dt = 0$.

ii) By Proposition 3 we know that for a.e. $t \in [0, T]$ there exists $f_t \in L^2(\mathcal{F}_t)$ such that, for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t \left[L_\tau u(\tau) \right] = f_t \qquad a.s.$$

Since $L_\tau u(\tau)$ converges in L^1 to $L_t u(t)$ as τ goes to t from the right, we have that

$$\begin{aligned} \mathbb{E}\left[\left| \mathbb{E}_t \left[L_\tau u(\tau) - L_t u(t) \right] \right| \right] &\leqslant \mathbb{E}\left[\mathbb{E}_t \left[\left| L_\tau u(\tau) - L_t u(t) \right| \right] \right] \\ &= \mathbb{E}\left[\left| L_\tau u(\tau) - L_t u(t) \right| \right] \longrightarrow 0, \end{aligned}$$

i.e.

$$\mathbb{E}_t \left[L_\tau u(\tau) \right] \xrightarrow{L^1} \mathbb{E}_t \left[L_t u(t) \right] = L_t u(t) \qquad\qquad \tau \longrightarrow t^+.$$

Since for a.e. $\tau \in [t, T]$, $\mathbb{E}_t \left[L_\tau u(\tau) \right]$ coincides a.s. with f_t, which is not dependent on τ, the uniqueness of the L^1-limit implies that $f_t = L_t u(t)$ a.s. Therefore, for a.e. $t \in [0, T]$, for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t \left[L_\tau u(\tau) \right] = L_t u(t) \qquad a.s.$$

Proof of Corollary 5

Consider the function $u_2 - u_1 \in \mathcal{W}$. The weak time-derivative of $u_2 - u_1$ is null, hence, by Proposition 3, for a.e. $t \in [0, T]$ we can find $f_t \in L^2(\mathcal{F}_t)$ such that for a.e. $\tau \in [t, T]$

$$\mathbb{E}_t \left[L_\tau u_2(\tau) \right] = \mathbb{E}_t \left[L_\tau u_1(\tau) \right] + f_t \qquad a.s.$$

Proof of Proposition 6

As g is deterministic and square-integrable, $g \in \mathcal{V}$. Moreover, denote for a.e. $\tau \in [t, T]$,

$$G(\tau) = \int_t^\tau g(s) ds = \mathbb{E}_t \left[u(\tau) \right] - f_t.$$

Taken any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1([t,T],\mathbb{R})$, for every $t \in [0,T]$ we have

$$
\begin{aligned}
\int_t^T \mathbb{E}\Big[g(\tau)\,\mathbf{1}_{A_t}\Big]\varphi(\tau)d\tau &= \int_t^T g(\tau)\,\mathbb{E}\left[\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\
&= P(A_t)\int_t^T g(\tau)\,\varphi(\tau)d\tau \\
&= P(A_t)\left\{[G(\tau)\varphi(\tau)]_t^T - \int_t^T G(\tau)\frac{d\varphi}{dt}(\tau)d\tau\right\} \\
&= -P(A_t)\int_t^T G(\tau)\frac{d\varphi}{dt}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[\mathbf{1}_{A_t}\right]G(\tau)\frac{d\varphi}{dt}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[G(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[\mathbb{E}_t\left[u(\tau)\right]\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau + \int_t^T \mathbb{E}\left[f_t\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau + \mathbb{E}\left[f_t\mathbf{1}_{A_t}\right]\int_t^T \frac{d\varphi}{dt}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau.
\end{aligned}
$$

In consequence, g is the weak time-derivative of u.

Proof of Proposition 7

As in the proof of Proposition 3, we consider the continuous with compact support functions $\eta, \xi : [t,T] \longrightarrow \mathbb{R}$, with $\int_t^T \eta(s)ds = 1$ and we define $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(\tau) = \xi(\tau) - \left(\int_t^T \xi(s)ds\right)\eta(\tau).$$

The primitive

$$K_\xi(\tau) = \int_t^\tau \xi(s)ds - \left(\int_t^T \xi(s)d\tau\right)\int_t^\tau \eta(s)ds$$

belongs to $C_c^1([t,T],\mathbb{R})$ and we employ it as a test function in the definition of weak time derivative of u. In addition, we require that $\int_t^T \xi(s)ds = 1$, so that we consider

$$K_\xi(\tau) = \int_t^\tau (\xi(s) - \eta(s))\,ds, \qquad\qquad k_\xi(\tau) = \xi(\tau) - \eta(\tau).$$

As $u \in \mathcal{W}$, for all $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$, we have

$$\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right](\xi(\tau)-\eta(\tau))\,d\tau = -\int_t^T \mathbb{E}\left[L_\tau \frac{du}{dt}(\tau)\mathbf{1}_{A_t}\right]\left(\int_t^\tau (\xi(s)-\eta(s))\,ds\right)d\tau.$$

By Fubini's Theorem,

$$\int_t^T \mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right](\xi(\tau)-\eta(\tau))\,d\tau = -\int_t^T \left(\int_s^T \mathbb{E}\left[L_\tau \frac{du}{dt}(\tau)\mathbf{1}_{A_t}\right]d\tau\right)(\xi(s)-\eta(s))\,ds$$

so that

$$\int_t^T \left(\mathbb{E}\big[L_\tau u(\tau)\mathbf{1}_{A_t}\big] + \int_\tau^T \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds\right)\xi(\tau)d\tau = \int_t^T \left(\mathbb{E}\left[L_\tau u(\tau)\mathbf{1}_{A_t}\right] + \int_\tau^T \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds\right)\eta(\tau)d\tau.$$

By the density of continuous functions ξ, η, Lemma 2 implies that for a.e. $\tau_1, \tau_2 \in [t,T]$

$$\mathbb{E}\left[L_{\tau_1} u(\tau_1)\mathbf{1}_{A_t}\right] + \int_{\tau_1}^T \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds = \mathbb{E}\left[L_{\tau_2} u(\tau_2)\mathbf{1}_{A_t}\right] + \int_{\tau_2}^T \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds,$$

namely

$$\mathbb{E}\left[L_{\tau_2}\left(u(\tau_2) - L_{\tau_1}u(\tau_1)\right)\mathbf{1}_{A_t}\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds.$$

Note that we can rewrite the equality as

$$\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2}u(\tau_2) - L_{\tau_1}u(\tau_1)\right]\mathbf{1}_{A_t}\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathbb{E}_t\left[L_s \frac{du}{dt}(s)\right]\mathbf{1}_{A_t}\right]ds.$$

If for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \geqslant 0 \qquad a.s.,$$

then, for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 \geqslant \tau_1$,

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[L_s \frac{du}{dt}(s)\mathbf{1}_{A_t}\right]ds \geqslant 0$$

for any $A_t \in \mathcal{F}_t$ and so

$$\mathbb{E}\left[\left(L_{\tau_2}u(\tau_2) - L_{\tau_1}u(\tau_1)\right)\mathbf{1}_{A_t}\right] \geqslant 0.$$

Since this holds for any $\mathcal{F}_t$-measurable set A_t, we infer that

$$\mathbb{E}_t\left[L_{\tau_2} u\left(\tau_2\right)-L_{\tau_1} u\left(\tau_1\right)\right] \geqslant 0 \qquad a.s.,$$

as we wanted to prove.

Conversely, if

$$\mathbb{E}_t\left[L_{\tau_2} u\left(\tau_2\right)-L_{\tau_1} u\left(\tau_1\right)\right] \geqslant 0 \qquad a.s.$$

for a.e. $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 \geqslant \tau_1$, then

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[L_s \frac{du}{dt}(s) \mathbf{1}_{A_t}\right] ds \geqslant 0.$$

Since this holds for a.e. $\tau_1, \tau_2 \in [t, T]$, it follows that, for a.e. $\tau \in [t, T]$

$$\mathbb{E}\left[L_\tau \frac{du}{dt}(\tau) \mathbf{1}_{A_t}\right] \geqslant 0.$$

As A_t is any $\mathcal{F}_t$-measurable set, we have

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \geqslant 0 \qquad a.s.$$

Proof of Corollary 8

We prove only $i)$ since $ii)$ is analogous.

By Proposition 7, for a.e. $t \in [0, T]$, for a.e. $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 \geqslant \tau_1$

$$\mathbb{E}_t\left[L_{\tau_2} u\left(\tau_2\right)\right] \geqslant \mathbb{E}_t\left[L_{\tau_1} u\left(\tau_1\right)\right] \qquad a.s.$$

When τ_1 tends to t from the right, $\mathbb{E}_t\left[L_{\tau_1} u\left(\tau_1\right)\right] \xrightarrow{a.s.} L_t u(t)$ and so

$$\mathbb{E}_t\left[L_{\tau_2} u\left(\tau_2\right)\right] \geqslant L_t u(t),$$

as we wanted to show.

Proof of Proposition 9

Following the proof of Proposition 7, given $t \in [0, T]$, for a.e. $\tau_1, \tau_2 \in [t, T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2} u\left(\tau_2\right)-L_{\tau_1} u\left(\tau_1\right)\right] \mathbf{1}_{A_t}\right]=\int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathbb{E}_t\left[L_s \frac{du}{dt}(s)\right] \mathbf{1}_{A_t}\right] ds.$$

Let $\tau_1 \leqslant \tau_2$. If

$$\mathbb{E}_t\left[L_{\tau_1} \frac{du}{dt}\left(\tau_1\right)\right] \leqslant \mathbb{E}_t\left[L_{\tau_2} \frac{du}{dt}\left(\tau_2\right)\right] \qquad a.s.$$

we have

$$\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right]\mathbf{1}_{A_t}\right]\leqslant\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right]\mathbf{1}_{A_t}\right]$$

and this monotonicity ensures that

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right]\mathbf{1}_{A_t}\right]&\leqslant\frac{\int_{\tau_1}^{\tau_2}\mathbb{E}\left[\mathbb{E}_t\left[L_s\frac{du}{dt}(s)\right]\mathbf{1}_{A_t}\right]ds}{\tau_2-\tau_1}\\&\leqslant\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

By the initial equality,

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right]\mathbf{1}_{A_t}\right]&\leqslant\frac{\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2}u(\tau_2)-L_{\tau_1}u(\tau_1)\right]\mathbf{1}_{A_t}\right]}{\tau_2-\tau_1}\\&\leqslant\mathbb{E}\left[\mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

As this holds for any $A_t\in\mathcal{F}_t$, we deduce that

$$\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right]\leqslant\frac{\mathbb{E}_t\left[L_{\tau_2}u(\tau_2)\right]-\mathbb{E}_t\left[L_{\tau_1}u(\tau_1)\right]}{\tau_2-\tau_1}\leqslant\mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right]\qquad a.s.$$

Conversely, if the last inequality holds, it is clear that

$$\mathbb{E}_t\left[L_{\tau_1}\frac{du}{dt}(\tau_1)\right]\leqslant\mathbb{E}_t\left[L_{\tau_2}\frac{du}{dt}(\tau_2)\right]\qquad a.s.$$

Proof of Proposition 10

By following the same steps of the proof of Proposition 7 we find that, for a.e. $\tau,\hat{\tau}\in[t,T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[L_{\hat{\tau}}u(\hat{\tau})-L_\tau u(\tau)\right]\mathbf{1}_{A_t}\right]=\int_\tau^{\hat{\tau}}\mathbb{E}\left[\mathbb{E}_t\left[L_s\frac{du}{dt}(s)\right]\mathbf{1}_{A_t}\right]ds.$$

By setting $\hat{\tau}=\tau+h$ for some $h>0$, we have

$$\mathbb{E}\left[\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h)-L_\tau u(\tau)\right]}{h}\mathbf{1}_{A_t}\right]=\frac{1}{h}\int_\tau^{\tau+h}\mathbb{E}\left[L_s\frac{du}{dt}(s)\,\mathbf{1}_{A_t}\right]ds.$$

Now we take the limit as $h\longrightarrow 0^+$. By Lebesgue Differentiation Theorem, the right-hand side converges to $\mathbb{E}\left[L_\tau du/dt(\tau)\,\mathbf{1}_{A_t}\right]$. Moreover, if $w(\tau)$ denotes the $\mathcal{F}_t$-measurable L^1-limit of $\frac{\mathbb{E}_t[L_{\tau+h}u(\tau+h)-L_\tau u(\tau)]}{h}$, the left-hand side converges to $\mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\right]$. Consequently,

$$\mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\right]=\mathbb{E}\left[L_\tau\frac{du}{dt}(\tau)\,\mathbf{1}_{A_t}\right]$$

for every $\mathcal{F}_t$-measurable set A_t. Hence, by definition of conditional expectation,

$$w(\tau) = \mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right].$$

As a result, by uniqueness of the L^1-limit, we conclude that, for a.e. $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \qquad h \longrightarrow 0^+$$

and this proves the first part of the statement.

Now suppose that, as $\tau \longrightarrow t^+$,

$$L_\tau u(\tau) \xrightarrow{L^1} L_t u(t), \qquad L_\tau \frac{du}{dt}(\tau) \xrightarrow{L^1} L_t \frac{du}{dt}(t).$$

This implies that

$$\mathbb{E}_t\left[L_\tau u(\tau)\right] \xrightarrow{L^1} \mathbb{E}_t\left[L_t u(t)\right] = L_t u(t) \qquad \tau \longrightarrow t^+$$

and

$$\mathbb{E}_t\left[L_\tau \frac{du}{dt}(\tau)\right] \xrightarrow{L^1} \mathbb{E}_t\left[L_t \frac{du}{dt}(t)\right] = L_t \frac{du}{dt}(t) \qquad \tau \longrightarrow t^+.$$

Also, the fact that, for a.e. $\tau \in [t,T]$,

$$\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h} \xrightarrow{L^1} w(\tau) \qquad h \longrightarrow 0^+$$

ensures that, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}_t\left[L_{\tau+h}u(\tau+h)\right] \xrightarrow{L^1} \mathbb{E}_t\left[L_\tau u(\tau)\right] \qquad h \longrightarrow 0^+.$$

Indeed, for every $h > 0$,

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}_t[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)]\right|\right] &= h\mathbb{E}\left[\frac{\left|\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]\right|}{h}\right] \\ &\leqslant h\left\{\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h} - w(\tau)\right|\right] + \mathbb{E}\left[\left|w(\tau)\right|\right]\right\}\end{aligned}$$

and this quantity converges to 0 as $h \longrightarrow 0^+$ since w is integrable. In particular, for any fixed $h > 0$, we have

$$\mathbb{E}_t\left[L_{\tau+h}u(\tau+h)\right] \xrightarrow{L^1} \mathbb{E}_t\left[L_{t+h}u(t+h)\right] \qquad \tau \longrightarrow t^+$$

and so

$$\frac{\mathbb{E}_t\left[L_{\tau+h}u(\tau+h) - L_\tau u(\tau)\right]}{h} \xrightarrow{L^1} \frac{\mathbb{E}_t\left[L_{t+h}u(t+h)\right] - L_t u(t)}{h} \qquad \tau \longrightarrow t^+.$$

Putting things together, for a.e. $\tau \in [t,T], h>0$ we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[L_{t+h}u\left(t+h\right)\right]-L_tu\left(t\right)}{h}-L_t\frac{du}{dt}\left(t\right)\right|\right]\\
&\quad\leqslant \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[L_{t+h}u\left(t+h\right)\right]-L_tu\left(t\right)}{h}-\frac{\mathbb{E}_t\left[L_{\tau+h}u\left(\tau+h\right)-L_\tau u\left(\tau\right)\right]}{h}\right|\right]\\
&\quad+\mathbb{E}\left[\left|-L_t\frac{du}{dt}\left(t\right)+\mathbb{E}_t\left[L_\tau\frac{du}{dt}\left(\tau\right)\right]\right|\right]\\
&\quad+\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[L_{\tau+h}u\left(\tau+h\right)-L_\tau u\left(\tau\right)\right]}{h}-\mathbb{E}_t\left[L_\tau\frac{du}{dt}\left(\tau\right)\right]\right|\right].
\end{aligned}$$

The previous convergences allow us to choose $\tau \in [t,T]$ in a way that the first two terms in the right-hand side are arbitrarily small and the first part of the Proposition allows us to choose h so that the last term is arbitrarily little. Hence, when $h \longrightarrow 0^+$,

$$\frac{\mathbb{E}_t\left[L_{t+h}u\left(t+h\right)\right]-L_tu\left(t\right)}{h}\xrightarrow{L^1}L_t\frac{du}{dt}\left(t\right).$$

Proof of Corollary 11

The function $f\left(X_t\right)$ is continuous and bounded and $\mathcal{A}f\left(X_t\right)$ is continuous, bounded and integrable. Since, for every $\tau \in [t,T]$,

$$\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}$$

converges to $\mathcal{A}f\left(X_\tau\right)$ as $h \longrightarrow 0^+$ in the uniform topology,

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\xrightarrow{L^1}\mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right]\qquad h\longrightarrow 0^+.$$

Indeed, since f is in the domain of the infinitesimal generator $\mathcal{A}$, we can find an arbitrary small $\varepsilon>0$ such that

$$\begin{aligned}
\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right| &\leqslant \left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}-\mathcal{A}f\left(X_\tau\right)\right|+\left|\mathcal{A}f\left(X_\tau\right)\right|\\
&\leqslant \sup_{X_\tau}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}-\mathcal{A}f\left(X_\tau\right)\right|+\left|\mathcal{A}f\left(X_\tau\right)\right|\\
&\leqslant \varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|.
\end{aligned}$$

By the Conditional Dominated Convergence Theorem, when $h \longrightarrow 0^+$

$$\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right]\longrightarrow\mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right]\qquad a.s.$$

that is

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\longrightarrow\mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right]\qquad a.s.$$

Moreover,

$$\begin{aligned}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\right| &= \left|\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right]\right| \\ &\leqslant \mathbb{E}_t\left[\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right|\right] \\ &\leqslant \mathbb{E}_t\left[\varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|\right] \\ &= \varepsilon+\mathbb{E}_t\left[\left|\mathcal{A}f\left(X_\tau\right)\right|\right].\end{aligned}$$

Therefore, by the Dominated Convergence Theorem, for every $t \in [0,T]$, for every $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right] \qquad h \longrightarrow 0^+.$$

In particular,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \mathcal{A}f\left(X_t\right) \qquad h \longrightarrow 0^+.$$

Since $\frac{\mathbb{E}_t[f(X_{\tau+h})]-f(X_\tau)}{h}$ is convergent in L^1 as $h \longrightarrow 0^+$ for every $t \in [0,T]$ and every $\tau \in [t,T]$, Proposition 10 applies. In consequence,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \frac{du}{dt}(t) \qquad h \longrightarrow 0^+.$$

By uniqueness of the L^1-limit, we infer that

$$\frac{du}{dt}(t)=\mathcal{A}f\left(X_t\right) \qquad a.s.$$

Proof of Theorem 12

• EXISTENCE

In order to show that $\pi \in \mathcal{W}$, we prove that π belongs to $\mathcal{V}$ and that it is weakly time-differentiable.

First, for all $\tau \in [0,T]$, $L_\tau\pi(\tau) \in L^2\left(\mathcal{F}_\tau\right)$. Indeed, by Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\left[L_\tau^2\pi^2\left(\tau\right)\right] &= e^{-2r(T-\tau)}\mathbb{E}\left[L_\tau^2\left(\mathbb{E}_\tau\left[L_{\tau,T}h_T\right]\right)^2\right] \\ &= e^{-2r(T-\tau)}\mathbb{E}\left[\left(\mathbb{E}_\tau\left[L_T h_T\right]\right)^2\right] \\ &\leqslant e^{-2r(T-\tau)}\mathbb{E}\left[L_T^2 h_T^2\right] < +\infty\end{aligned}$$

because $L_T h_T \in L^2\left(\mathcal{F}_T\right)$.

As for the conditional weak Lebesgue measurability of π, fix $t \in [0,T]$ and consider for any $A_t \in \mathcal{F}_t$ the function

$$\begin{aligned}\tau \in [t,T] \longmapsto \mathbb{E}\left[L_\tau\pi\left(\tau\right)\mathbf{1}_{A_t}\right] &= \mathbb{E}\left[L_\tau\mathbb{E}_\tau\left[L_{\tau,T}h_T\right]\mathbf{1}_{A_t}\right]e^{-r(T-\tau)} \\ &= \mathbb{E}\left[L_T h_T\mathbf{1}_{A_t}\right]e^{-r(T-\tau)},\end{aligned}$$

where we exploited the fact that $L_\tau \mathbf{1}_{A_t}$ is $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since $\mathbb{E}\left[L_T h_T \mathbf{1}_{A_t}\right]$ is not dependent on τ, we easily conclude that $\mathbb{E}\left[L_\tau \pi\left(\tau\right) \mathbf{1}_{A_t}\right]$ is Lebesgue measurable and so π is conditionally weakly Lebesgue measurable.

In addition,

$$\begin{aligned}
\int_0^T \mathbb{E}\left[L_\tau^2 \pi^2\left(\tau\right)\right] d\tau &\leqslant \int_0^T e^{-2r(T-\tau)} \mathbb{E}\left[L_T^2 h_T^2\right] d\tau \\
&= \left(\int_0^T e^{-2r(T-\tau)} ds\right) \mathbb{E}\left[L_T^2 h_T^2\right],
\end{aligned}$$

which is finite because $L_T h_T \in L^2\left(\mathcal{F}_T\right)$. Therefore π belongs to $\mathcal{V}$.

Now we look for the weak time-derivative of π. We consider any set $A_t \in \mathcal{F}_t$ and any test function $\varphi \in C_c^1\left([t,T], \mathbb{R}\right)$. Recall that, for any set A_t, the functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$, so that

$$\begin{aligned}
-\int_t^T \mathbb{E}\left[L_\tau \pi\left(\tau\right) \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau) d\tau &= -\int_t^T \mathbb{E}\left[L_\tau e^{-r(T-\tau)} \mathbb{E}_\tau\left[L_{\tau,T} h_T\right] \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau) d\tau \\
&= -\int_t^T \mathbb{E}\left[e^{-r(T-\tau)} \mathbb{E}_\tau\left[L_T h_T\right] \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau) d\tau \\
&= -\int_t^T \mathbb{E}\left[e^{-r(T-\tau)} L_T h_T \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau) d\tau \\
&= -\mathbb{E}\left[L_T h_T \mathbf{1}_{A_t}\right] \int_t^T e^{-r(T-\tau)} \frac{d\varphi}{dt}(\tau) d\tau \\
&= \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t}\right] \int_t^T r e^{-r(T-\tau)} \varphi(\tau) d\tau \\
&= \int_t^T r \mathbb{E}\left[e^{-r(T-\tau)} L_T h_T \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau \\
&= \int_t^T r \mathbb{E}\left[L_\tau e^{-r(T-\tau)} L_{\tau,T} h_T \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau \\
&= \int_t^T \mathbb{E}\left[L_\tau r \pi\left(\tau\right) \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau.
\end{aligned}$$

Therefore, the candidate weak time-derivative of π is $r\pi$ and $L_t r\pi(t)$ belongs to $L^2\left(\mathcal{F}_t\right)$ for all t. Clearly, $r\pi$ is also conditionally weakly Lebesgue measurable and it belongs to $\mathcal{V}$. Hence, $r\pi$ is the weak time-derivative of π:

$$\frac{d\pi}{dt}\left(t\right) = r\pi\left(t\right).$$

As for the L^1-convergence to the boundary, Lévy's Upward Theorem, that is Theorem 14.2 in Williams (1991), guarantees that

$$e^{-r(T-t)} \mathbb{E}_t\left[L_T h_T\right] \xrightarrow{L^1} \mathbb{E}_T\left[L_T h_T\right] = L_T h_T \qquad t \longrightarrow T$$

and so $L_t\pi(t)$ converges in L^1 to $L_T h_T$.

Summing up, we showed that $\pi \in \mathcal{W}$ and it solves Problem (4).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}$ be two solutions of Problem (4), that is for a.e. $t \in [0,T]$

$$\frac{d\pi_i}{dt}(t) = r\pi_i(t) \qquad i = 1, 2,$$

$\pi_i(T) = h_T$ and $L_t\pi_i(t) \xrightarrow{L^1} L_T h_T$ as t goes to T. By defining $z = \pi_1 - \pi_2 \in \mathcal{W}$, we have that, for a.e. $t \in [0,T]$,

$$\frac{dz}{dt}(t) = rz(t),$$

$z(T) = 0$ and $L_t z(t) \xrightarrow{L^1} 0$ as t goes to T.

Now we show that the weak time-derivative of the function $e^{-rt}z(t)$ is

$$e^{-rt}\left(\frac{dz}{dt}(t) - rz(t)\right).$$

Indeed, for every $t \in [0,T]$, we have that, for any $\varphi \in C_c^1([t,T],\mathbb{R})$ and $A_t \in \mathcal{F}_t$,

$$\begin{aligned}\int_t^T \mathbb{E}\left[L_\tau \frac{dz}{dt}(\tau)\mathbf{1}_{A_t}\right] e^{-r\tau}\varphi(\tau)d\tau &= -\int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}\right]\frac{d}{dt}\left(e^{-rt}\varphi(t)\right)(\tau)d\tau \\ &= -\int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}\right] e^{-r\tau}\frac{d\varphi}{dt}(\tau)d\tau \\ &\quad + \int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}\right] e^{-r\tau} r\varphi(\tau)d\tau\end{aligned}$$

that is

$$\begin{aligned}&\int_t^T \mathbb{E}\left[L_\tau e^{-r\tau}\left(\frac{dz}{dt}(\tau) - rz(\tau)\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\ &\qquad\qquad = -\int_t^T \mathbb{E}\left[L_\tau e^{-r\tau} z(\tau)\mathbf{1}_{A_t}\right]\frac{d\varphi}{dt}(\tau)d\tau.\end{aligned}$$

This means that the weak time-derivative of $e^{-rt}z(t)$ is

$$e^{-rt}\left(\frac{dz}{dt}(t) - rz(t)\right).$$

However this function is null a.s. Therefore, $e^{-rt}z(t)$ has null weak time-derivative. Consequently, by Proposition 3, for a.e. $t \in [0,T]$ there exists a function $f_t \in L^2(\mathcal{F}_t)$ such that, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau e^{-r\tau} z(\tau)\right] = f_t$$

or, equivalently,

$$\mathbb{E}_t\left[L_\tau z(\tau)\right] = e^{r\tau} f_t.$$

Letting τ go to T, we have that

$$\mathbb{E}_t\left[L_\tau z\left(\tau\right)\right] \longrightarrow e^{rT} f_t \qquad \textit{pointwise.}$$

In addition, the fact that $L_\tau z\left(\tau\right)$ converges to zero in L^1 as τ approaches T ensures that

$$\mathbb{E}\left[\left|\mathbb{E}_t\left[L_\tau z\left(\tau\right) - 0\right]\right|\right] \leqslant \mathbb{E}\left[\mathbb{E}_t\left[\left|L_\tau z\left(\tau\right)\right|\right]\right] = \mathbb{E}\left[\left|L_\tau z\left(\tau\right)\right|\right] \longrightarrow 0,$$

i.e. $\mathbb{E}_t\left[L_\tau z\left(\tau\right)\right]$ tends to zero in L^1. By uniqueness of the L^1-limit, we infer that $f_t = 0$ a.s. As a result, for a.e. $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[L_\tau z\left(\tau\right)\right] = 0 \qquad a.s.$$

An application of Lévy's Upward Theorem as in Proposition 2 ensures that, for a.e. $\tau \in [0,T]$, $z\left(\tau\right) = 0$ a.s. This proves uniqueness of the solution of Problem (4).

Proof of Proposition 13

We show that $\pi(t) = \mathbb{E}_t\left[\int_t^T e^{-r(m-t)} L_{t,m} h_m \mu(dm)\right]$ belongs to $\mathcal{V}$ and it is weakly time-differentiable.

First, for all $\tau \in [0,T]$, $L_\tau \pi(\tau) \in L^2\left(\mathcal{F}_\tau\right)$. By exploiting Jensen's inequality twice, indeed, we find

$$\begin{aligned}
\mathbb{E}\left[L_\tau^2 \pi^2\left(\tau\right)\right] &= \mathbb{E}\left[L_\tau^2\left(\mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_{\tau,m} h_m \mu(dm)\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_m h_m \mu(dm)\right]\right)^2\right] \\
&\leqslant \mathbb{E}\left[\left(\int_\tau^T e^{-r(m-\tau)} L_m h_m \mu(dm)\right)^2\right] \\
&\leqslant \mathbb{E}\left[\int_\tau^T e^{-2r(m-\tau)} L_m^2 h_m^2 \mu(dm)\right] \mu([\tau,T]) \\
&\leqslant \mathbb{E}\left[\int_\tau^T L_m^2 h_m^2 \mu(dm)\right] \mu([\tau,T]) < +\infty
\end{aligned}$$

because μ is a finite measure and $L^2(\cdot)h^2(\cdot)$ is Bochner integrable with respect to μ.

As for the conditional weak Lebesgue measurability of π, fix $t \in [0,T]$ and take into consideration for any $A_t \in \mathcal{F}_t$ the function

$$\begin{aligned}
\tau \in [t,T] \longmapsto \mathbb{E}\left[L_\tau \pi\left(\tau\right) \mathbf{1}_{A_t}\right] &= \mathbb{E}\left[L_\tau \mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_{\tau,m} h_m \mu(dm)\right] \mathbf{1}_{A_t}\right] \\
&= \mathbb{E}\left[\mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_m h_m \mu(dm)\right] \mathbf{1}_{A_t}\right] \\
&= e^{r\tau} \mathbb{E}\left[\int_\tau^T e^{-rm} L_m h_m \mu(dm) \mathbf{1}_{A_t}\right].
\end{aligned}$$

The last integral is a well-defined Bochner integral and it defines a continuous (and so Lebesgue measurable) function of τ. As the expectation is a continuous operator, it preserves measurability and so the whole function is Lebesgue measurable. As a result, π is conditionally weakly Lebesgue measurable.

Moreover,

$$\begin{aligned}\int_0^T \mathbb{E}\left[L_\tau^2 \pi^2(\tau)\right] d\tau &\leqslant \int_0^T \mathbb{E}\left[\int_\tau^T L_m^2 h_m^2 \mu(dm)\right] \mu([\tau,T]) d\tau \\ &\leqslant \mu([0,T]) \int_0^T \mathbb{E}\left[\int_0^T L_m^2 h_m^2 \mu(dm)\right] d\tau \\ &= \mu^2([0,T]) \mathbb{E}\left[\int_0^T L_m^2 h_m^2 \mu(dm)\right],\end{aligned}$$

which is finite as argued before. In consequence, π belongs to $\mathcal{V}$.

Now we compute the weak time-derivative of π. We consider any set $A_t \in \mathcal{F}_t$ and any test function $\varphi \in C_c^1([t,T],\mathbb{R})$. For any set A_t, the functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$ and so

$$\begin{aligned}&-\int_t^T \mathbb{E}\Big[L_\tau \pi(\tau) \mathbf{1}_{A_t}\Big] \frac{d\varphi}{dt}(\tau) d\tau \\ &\qquad = -\int_t^T \mathbb{E}\left[L_\tau \mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_{\tau,m} h_m \mu(dm)\right] \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau) d\tau \\ &\qquad = -\int_t^T \mathbb{E}\left[\int_\tau^T e^{-r(m-\tau)} L_m h_m \mathbf{1}_{A_t} p_m dm\right] \frac{d\varphi}{dt}(\tau) d\tau.\end{aligned}$$

because $\mu(dm) = p_m dm$. Since the expectation is a bounded operator, by Lemma 11.45 in Aliprantis and Border (2006) we can exchange it with the integral. Later we apply integration by parts:

$$\begin{aligned}&-\int_t^T \mathbb{E}[L_\tau \pi(\tau) \mathbf{1}_{A_t}] \frac{d\varphi}{dt}(\tau) d\tau \\ &= -\int_t^T \left(\int_\tau^T e^{-r(m-\tau)} \mathbb{E}\left[L_m h_m p_m \mathbf{1}_{A_t}\right] dm\right) \frac{d\varphi}{dt}(\tau) d\tau \\ &= 0 + \int_t^T \frac{d}{d\tau}\left(\int_\tau^T e^{-r(m-\tau)} \mathbb{E}\left[L_m h_m p_m \mathbf{1}_{A_t}\right] dm\right) \varphi(\tau) d\tau \\ &= \int_t^T \left(0 - e^{-r(\tau-\tau)} \mathbb{E}\left[L_\tau h_\tau p_\tau \mathbf{1}_{A_t}\right] + r \int_\tau^T e^{-r(m-\tau)} \mathbb{E}\left[L_m h_m p_m \mathbf{1}_{A_t}\right] dm\right) \varphi(\tau) d\tau \\ &= \int_t^T \left(\mathbb{E}\left[L_\tau(-h_\tau p_\tau) \mathbf{1}_{A_t}\right] + \mathbb{E}\left[r \int_\tau^T e^{-r(m-\tau)} L_m h_m p_m \mathbf{1}_{A_t} dm\right]\right) \varphi(\tau) d\tau \\ &= \int_t^T \mathbb{E}\left[L_\tau(-h_\tau p_\tau) \mathbf{1}_{A_t} + L_\tau r \mathbb{E}_\tau\left[\int_\tau^T e^{-r(m-\tau)} L_{\tau,m} h_m \mu(dm)\right] \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau \\ &= \int_t^T \mathbb{E}\left[L_\tau(r\pi(\tau) - h_\tau p_\tau) \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau.\end{aligned}$$

Since both $\pi(\cdot)$ and $h(\cdot)p(\cdot)$ belong to $\mathcal{V}$, it follows that $\pi(\cdot) - h(\cdot)p(\cdot)$ is included in $\mathcal{V}$. Therefore, the latter is the weak time-derivative of π:

$$\frac{d\pi}{dt}(t) = r\pi(t) - h_t p_t.$$

Lemma 3 *Let $r : [0,T] \longrightarrow L^\infty(\mathcal{F}_T)$ be a Bochner integrable function and let $r_\omega : [0,T] \longrightarrow \mathbb{R}$ denote its restriction on the path induced by any $\omega \in \Omega$. If the map $[0,T] \times \Omega \longrightarrow \mathbb{R}$ such that $(t,\omega) \longmapsto r_\omega(t)$ is measurable, the Bochner integral of r coincides a.s. with the pathwise Lebesgue integral obtained by integrating each restriction r_ω.*

Proof. The Bochner integral construction involves simple functions $s : [0,T] \longrightarrow L^\infty(\mathcal{F}_T)$ that are finite linear combinations of terms as $\mathbf{1}_E w$, where E is a measurable subset of $[0,T]$ and $w \in L^\infty(\mathcal{F}_T)$. Since r is Bochner integrable, there exists a sequence of simple functions s_n such that

$$\int_0^T \mathbb{E}\left[(r(\tau) - s_n(\tau))^2\right]^{\frac{1}{2}} d\tau \leqslant \frac{1}{2^n}$$

and the Bochner integral of r is the L^2-limit of the integral of simple functions:

$$\int_0^T s_n(\tau)d\tau \xrightarrow{L^2} \int_0^T r(\tau)d\tau.$$

Hence the convergence is also in probability.

On the other hand,

$$\sum_{n=1}^{+\infty} \int_0^T \mathbb{E}\left[|r(\tau) - s_n(\tau)|\right] \leqslant \sum_{n=1}^{+\infty} \int_0^T \mathbb{E}\left[(r(\tau) - s_n(\tau))^2\right]^{\frac{1}{2}} d\tau \leqslant \sum_{n=1}^{+\infty} \frac{1}{2^n},$$

which is finite. Therefore, by Fubini's Theorem,

$$\mathbb{E}\left[\int_0^T \sum_{n=1}^{+\infty} |r(\tau) - s_n(\tau)|\, d\tau\right] = \sum_{n=1}^{+\infty} \int_0^T \mathbb{E}\left[|r(\tau) - s_n(\tau)|\right] d\tau$$

is finite, too. As a result, the random variable

$$\int_0^T \sum_{n=1}^{+\infty} |r(\tau) - s_n(\tau)|\, d\tau$$

is finite a.s. Consequently, for a.e. $\omega \in \Omega$ the pathwise restriction satisfies

$$|r_\omega(\tau) - s_{n,\omega}(\tau)| \longrightarrow 0$$

for a.e. $\tau \in [0,T]$. Moreover, since $|r_\omega(\tau) - s_{n,\omega}(\tau)|$ is dominated by its sum over n, the Dominated Convergence Theorem ensures that, for a.e. $\omega \in \Omega$

$$\int_0^T s_{n,\omega}(\tau)d\tau \longrightarrow \int_0^T r_\omega(\tau)d\tau,$$

which is the Lebesgue integral along the path induced by ω. By collecting all the trajectories we find the limit a.s. of the random variable $\int_0^T s_n(\tau)d\tau$. This convergence holds in probability, too. But before we showed the convergence in probability of $\int_0^T s_n(\tau)d\tau$ to the Bochner integral $\int_0^T r(\tau)d\tau$. Therefore, by uniqueness of the limit, the Bochner integral coincides a.s. with the pathwise Lebesgue integral. ■

Proof of Theorem 15

• EXISTENCE

In order to show that $\pi \in \mathcal{W}_r$, we prove that π belongs to $\mathcal{V}_r$ and that it is r-weakly time-differentiable.

First, for all $\tau \in [0,T]$, $L_\tau \pi(\tau) \in L^2(\mathcal{F}_\tau)$. Indeed, $\int_t^T r(s)ds$ is a continuous, and then bounded, function of $t \in [0,T]$. Hence, there exists $K > 0$ such that $e^{-\int_\tau^T r(s)ds} \leqslant K$ a.s. for all t. In addition, Jensen's inequality ensures that

$$\begin{aligned}
\mathbb{E}\left[L_\tau^2 \pi^2(\tau)\right] &= \mathbb{E}\left[L_\tau^2 \left(\mathbb{E}_\tau\left[L_{\tau,T} e^{-\int_\tau^T r(s)ds} h_T\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}_\tau\left[L_T e^{-\int_\tau^T r(s)ds} h_T\right]\right)^2\right] \\
&\leqslant \mathbb{E}\left[L_T^2 e^{-2\int_\tau^T r(s)ds} h_T^2\right] \\
&\leqslant K^2 \mathbb{E}\left[L_T^2 h_T^2\right] < +\infty
\end{aligned}$$

because $L_T h_T \in L^2(\mathcal{F}_T)$.

As for the conditional r-weak Lebesgue measurability of π, fix $t \in [0,T]$, take any $A_t \in \mathcal{F}_t$ and any adapted function $\psi \in C_c([t,T], L^\infty(\mathcal{F}_T))$. Then, consider the function

$$\begin{aligned}
\tau \in [t,T] \longmapsto \mathbb{E}\left[L_\tau \pi(\tau) \mathbf{1}_{A_t} \psi(\tau)\right] &= \mathbb{E}\left[L_\tau \mathbb{E}_\tau\left[L_{\tau,T} e^{-\int_\tau^T r(s)ds} h_T\right] \mathbf{1}_{A_t} \psi(\tau)\right] \\
&= \mathbb{E}\left[L_T e^{-\int_\tau^T r(s)ds} h_T \mathbf{1}_{A_t} \psi(\tau)\right],
\end{aligned}$$

where we exploited the fact that $L_\tau \mathbf{1}_{A_t} \psi(\tau)$ is $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since $\int_\tau^T r(s)ds$ is a well-defined Bochner integral, $e^{-\int_\tau^T r(s)ds}$ is a continuous function of τ, as well as $\psi(\tau)$. Therefore, the quantity

$$L_T e^{-\int_\tau^T r(s)ds} h_T \mathbf{1}_{A_t} \psi(\tau)$$

is Lebesgue measurable. The expectation is a continuous operator and so it preserves Lebesgue measurability. Consequently, π is conditionally r-weakly Lebesgue measurable.

In addition,

$$\int_0^T \mathbb{E}\left[L_\tau^2 \pi^2(\tau)\right] d\tau \leqslant \int_0^T K^2 \mathbb{E}\left[L_T^2 h_T^2\right] d\tau = K^2 T \mathbb{E}\left[L_T^2 h_T^2\right],$$

which is finite because $L_T h_T \in L^2(\mathcal{F}_T)$. Therefore π belongs to $\mathcal{V}_r$.

Now we look for the r-weak time-derivative of π. We consider any set $A_t \in \mathcal{F}_t$ and any adapted function $\varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T))$. Recall that, for any $\mathcal{F}_t$-measurable set A_t, the functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since $d\varphi/dt$ is adapted too, we deduce that

$$
\begin{aligned}
-\int_t^T \mathbb{E}\left[L_\tau \pi(\tau) \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau \\
&= -\int_t^T \mathbb{E}\left[L_\tau \mathbb{E}_\tau\left[L_{\tau,T} e^{-\int_\tau^T r(s)ds} h_T\right] \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau \\
&= -\int_t^T \mathbb{E}\left[\mathbb{E}_\tau\left[L_T e^{-\int_\tau^T r(s)ds} h_T \mathbf{1}_{A_t}\right] \frac{d\varphi}{dt}(\tau)\right] d\tau \\
&= -\int_t^T \mathbb{E}\left[L_T e^{-\int_\tau^T r(s)ds} h_T \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau.
\end{aligned}
$$

$e^{-\int_\tau^T r(s)ds} d\varphi/dt(\tau)$ is a continuous function of $\tau \in [t,T]$, hence it is Bochner integrable. The expectation is a bounded operator, so Lemma 11.45 in Aliprantis and Border (2006) allows us to exchange expectation and integral. Therefore,

$$
\begin{aligned}
-\int_t^T \mathbb{E}\left[L_\tau \pi(\tau) \mathbf{1}_{A_t} \frac{d\varphi}{dt}(\tau)\right] d\tau &= -\mathbb{E}\left[L_T h_T \mathbf{1}_{A_t} \int_t^T e^{-\int_\tau^T r(s)ds} \frac{d\varphi}{dt}(\tau) d\tau\right] \\
&= \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t} \int_t^T \left(1 - e^{-\int_\tau^T r(s)ds}\right) \frac{d\varphi}{dt}(\tau) d\tau\right] \\
&\quad - \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t} \int_t^T \frac{d\varphi}{dt}(\tau) d\tau\right] \\
&= \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t} \int_t^T \left(1 - e^{-\int_\tau^T r(s)ds}\right) \frac{d\varphi}{dt}(\tau) d\tau\right]
\end{aligned}
$$

because φ has compact support. Now consider the function $u \longmapsto r(u) e^{-\int_u^T r(s)ds}$. This function is Bochner integrable (because r is bounded) and its Bochner integral coincides almost surely with the pathwise Lebesgue integral. For any state $\omega \in \Omega$ the restriction r_ω of r satisfies:

$$
\int_\tau^T r_\omega(u) e^{-\int_u^T r_\omega(s)ds} du = \left[e^{-\int_u^T r_\omega(s)ds}\right]_\tau^T = 1 - e^{-\int_\tau^T r_\omega(s)ds}.
$$

In consequence, the Bochner integral is

$$
\int_\tau^T r(u) e^{-\int_u^T r(s)ds} du = 1 - e^{-\int_\tau^T r(s)ds} \qquad a.s.
$$

By exploiting integration by parts (see Craven (1970)), we obtain

$$
\begin{aligned}
-\int_t^T & \mathbb{E}\left[L_\tau \pi\left(\tau\right)\mathbf{1}_{A_t}\frac{d\varphi}{dt}(\tau)\right]d\tau \\
&= \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t}\int_t^T\left(\int_\tau^T r(u)e^{-\int_u^T r(s)ds}du\right)\frac{d\varphi}{dt}(\tau)d\tau\right] \\
&= \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t}\int_t^T r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)d\tau\right] \\
&= \int_t^T \mathbb{E}\left[L_T h_T \mathbf{1}_{A_t} r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)\right]d\tau \\
&= \int_t^T \mathbb{E}\left[r(\tau)\mathbb{E}_\tau\left[L_T e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&= \int_t^T \mathbb{E}\left[L_\tau r(\tau)\mathbb{E}_\tau\left[L_{\tau,T} e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&= \int_t^T \mathbb{E}\left[L_\tau r(\tau)\pi(\tau)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau.
\end{aligned}
$$

Therefore, the candidate r-weak time-derivative of π is $r\pi$ and $L_t r(t)\pi(t)$ belongs to $L^2\left(\mathcal{F}_t\right)$ for all t because r is bounded. Clearly, $r\pi$ is also conditionally r-weakly Lebesgue measurable and it belongs to $\mathcal{V}_r$. Hence, $r\pi$ is the r-weak time-derivative of π:

$$\frac{d\pi}{dt}\left(t\right) = r(t)\pi\left(t\right).$$

As for the L^1-convergence to the boundary of $L_t\pi(t)$, observe that

$$
\begin{aligned}
\mathbb{E}&\left[\left|\mathbb{E}_t\left[L_T e^{-\int_t^T r(s)ds}h_T\right] - L_T h_T\right|\right] \\
&\leqslant \mathbb{E}\left[\left|\mathbb{E}_t\left[L_T e^{-\int_t^T r(s)ds}h_T - L_T h_T\right]\right|\right] + \mathbb{E}\left[\left|\mathbb{E}_t\left[L_T h_T\right] - L_T h_T\right|\right] \\
&= \mathbb{E}\left[\left|L_T h_T\right|\left|e^{-\int_t^T r(s)ds} - 1\right|\right] + \mathbb{E}\left[\left|\mathbb{E}_t\left[L_T h_T\right] - L_T h_T\right|\right] \\
&\leqslant \left(\mathbb{E}\left[L_T^2 h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(e^{-\int_t^T r(s)ds} - 1\right)^2\right]\right)^{\frac{1}{2}} + \mathbb{E}\left[\left|\mathbb{E}_t\left[L_T h_T\right] - L_T h_T\right|\right].
\end{aligned}
$$

$L_T h_T \in L^2\left(\mathcal{F}_t\right)$ and $\mathbb{E}\left[\left(e^{-\int_t^T r(s)ds} - 1\right)^2\right]$ converges to zero as t approaches T because r is bounded. Moreover, $\mathbb{E}_t\left[L_T h_T\right]$ tends to $L_T h_T$ in L^1 by Lévy's Upward Theorem, i.e. Theorem 14.2 in Williams (1991). As a result, the right-hand side of the previous inequality tends to zero as t goes to T and this ensures the L^1-convergence of $L_t\pi(t)$ to $L_T h_T$.

Summing up, we showed that $\pi \in \mathcal{W}_r$ and it solves Problem (7).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}_r$ be two solutions of Problem (7), that is for a.e. $t \in [0, T]$

$$\frac{d\pi_i}{dt}\left(t\right) = r(t)\pi_i\left(t\right) \qquad i = 1, 2,$$

$\pi_i(T) = h_T$ and $L_t\pi_i(t) \xrightarrow{L^1} L_T h_T$ as t goes to T. By defining $z = \pi_1 - \pi_2 \in \mathcal{W}_r$, we have that, for a.e. $t \in [0,T]$,

$$\frac{dz}{dt}(t) = r(t)z(t),$$

$z(T) = 0$ and $L_t z(t) \xrightarrow{L^1} 0$ as t goes to T.

The function $r(t)$ is Bochner integrable. Reasoning state by state, we have

$$\int_t^T r_\omega(s)ds = R_\omega(T) - R_\omega(t),$$

where R_ω is a primitive of r_ω. By denoting with R the random variable that collects all R_ω, it follows that the Bochner integral of r is

$$\int_t^T r(s)ds = R(T) - R(t) \qquad a.s.$$

Now we show that the r-weak time-derivative of the function $e^{-R(t)}z(t)$ is

$$e^{-R(t)}\left(\frac{dz}{dt}(t) - rz(t)\right).$$

For any adapted $\varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T))$, consider the function

$$u \longmapsto e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\frac{d\varphi}{dt}(u).$$

Since r is bounded, this function is Bochner integrable. By reasoning pathwise, it follows that

$$\int_\tau^T \left(e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\frac{d\varphi}{dt}(u)\right)du = e^{-R(\tau)}\varphi(\tau) \qquad a.s.$$

Hence, $e^{-R}\varphi$ is adapted, it belongs to $C_c^1([t,T], L^\infty(\mathcal{F}_T))$ and so we can use it as test function in the definition of r-weak time-derivative of z:

$$\begin{aligned}
&\int_t^T \mathbb{E}\left[L_\tau \frac{dz}{dt}(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}\varphi(\tau)\right]d\tau \\
&\qquad = -\int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}\left(e^{-R(\tau)}\frac{d\varphi}{dt}(\tau) - e^{-R(\tau)}r(\tau)\varphi(\tau)\right)\right]d\tau \\
&\qquad = -\int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}\frac{d\varphi}{dt}(\tau)\right]d\tau \\
&\qquad\quad + \int_t^T \mathbb{E}\left[L_\tau z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}r(\tau)\varphi(\tau)\right]d\tau
\end{aligned}$$

that is

$$\begin{aligned}
&\int_t^T \mathbb{E}\left[L_\tau e^{-R(\tau)}\left(\frac{dz}{dt}(\tau) - r(\tau)z(\tau)\right)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&\qquad\qquad = -\int_t^T \mathbb{E}\left[L_\tau e^{-R(\tau)}z(\tau)\mathbf{1}_{A_t}\frac{d\varphi}{dt}(\tau)\right]d\tau.
\end{aligned}$$

This means that the r-weak time-derivative of $e^{-R(t)}z\left(t\right)$ is

$$e^{-R(t)}\left(\frac{dz}{dt}\left(t\right)-r(t)z\left(t\right)\right).$$

However this function is null a.s. Therefore, $e^{-R(t)}z\left(t\right)$ has null r-weak time-derivative. Consequently, by following the proof of Proposition 3 for test functions in $C_c^1\left([t,T],L^\infty\left(\mathcal{F}_T\right)\right)$, for a.e. $t\in[0,T]$ there exists a function $f_t\in L^2\left(\mathcal{F}_t\right)$ such that, for a.e. $\tau\in[t,T]$

$$\mathbb{E}_t\left[L_\tau e^{-R(\tau)}z\left(\tau\right)\right]=f_t.$$

As τ approaches T, $\mathbb{E}_t\left[L_\tau e^{-R(\tau)}z\left(\tau\right)\right]$ goes to zero in L^1. Indeed, since $e^{-R(\tau)}$ is bounded,

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}_t\left[L_\tau e^{-R(\tau)}z(\tau)\right]-0\right|\right]&\leqslant\mathbb{E}\left[\mathbb{E}_t\left[\left|L_\tau z(\tau)\right|e^{-R(\tau)}\right]\right]\\&\leqslant C\mathbb{E}\left[\left|L_\tau z(\tau)\right|\right]\end{aligned}$$

for some $C>0$. However, the last term converges to zero because $L_\tau z(\tau)$ tends to zero in L^1 as τ approaches T.

By uniqueness of the L^1-limit, we infer that $f_t=0$ a.s. As a result, for a.e. $t\in[0,T]$, for a.e. $\tau\in[t,T]$

$$\mathbb{E}_t\left[L_\tau e^{-R(\tau)}z\left(\tau\right)\right]=0\qquad a.s.$$

An application of Lévy's Upward Theorem as in Proposition 2 ensures that, for a.e. $\tau\in[0,T]$, $L_\tau e^{-R(\tau)}z\left(\tau\right)=0$ a.s. and so $z\left(\tau\right)=0$ a.s. This proves uniqueness of the solution of Problem (7).

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