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# Unique Tarski Fixed Points*

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**Abstract**

We establish sufficient conditions that ensure the uniqueness of Tarski-type fixed points of monotone operators. Several applications are presented.

## 1 Introduction

In this paper we establish sufficient conditions that ensure in ordered vector spaces the uniqueness of fixed points a la Tarski [36], often a highly desirable property in the many applications in economics and operations research in which such fixed points appear (cf. Topkis [38]).

More specifically, our results establish the existence and uniqueness of fixed points of monotone operators that are either order concave or subhomogenous. Their common feature is to require that no fixed points belong to the lower perimeter of the domain. This novel notion, which we introduce in Section 3, is thus a keystone of our analysis.

We establish our main results in Sections 4 and 5. The results of the latter section rely on a close relation between the subhomogeneous case and the contractive property according to a metric introduced by Thompson [37]. This novel connection, elaborated in the Appendix, permits to prove the uniqueness and global attractiveness of fixed points of subhomogeneous operators. Besides the role of lower perimeters, this connection is the other main contribution of this paper.

We illustrate our uniqueness results with some applications on recursive utilities, integral equations, complementary problems, variational inequalities, and operator equations in Section 6. We conclude by discussing the related literature in Section 7.

## 2 Preliminaries

In this introductory section we briefly present a few basic notions that we use in the paper (we refer to [11], [24] and [31] for comprehensive studies).

**Posets** A *poset* $(A, \geq)$ is *chain complete* (resp., *$\sigma$-complete*) if it has a minimum element and if every (resp., countable) chain has a supremum.[1] A lattice is *complete* when every nonempty subset has an infimum and supremum element. A lattice is complete if and only if is chain complete.

If $a \leq b$ are two elements of a poset $A$, then $[a, b] = \{x \in A : a \leq x \leq b\}$ is an *order interval.* A poset is *Dedekind* (*$\sigma$-complete*) *complete* if every order interval is (countably) chain complete.

An element $a \in A$ is: (i) *dominated* if there is $b \in B$ such that $a < b$, (ii) *minimal* if there is no $b \in A$ such that $b < a$, (iii) a *minimum* if $a \leq b$ for all $b \in A$.

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[1]The minimum can be actually regarded as the supremum of the empty chain.

**Spaces** Throughout the paper $V$ is a (partially) ordered vector space with order relation $\geq$ and $K$ will always denote its positive cone. If $V$ is Dedekind $\sigma$-complete, then it is Archimedean.[2] When $V$ is a lattice, it is called *Riesz space*. In this case, to be Dedekind complete amounts to say that the order intervals $[a, b] \subseteq V$ are complete lattices.

**Fixed points** A self-map $T : A \to A$ is *monotone* (or *order preserving*) if $a \leq b$ implies $T(a) \leq T(b)$ for all $a, b \in A$.

A fixed point theorem due to Tarski [36] p. 286 says that the set of fixed points of a monotone self-map defined on a complete lattice is a nonempty complete lattice. A generalized version of this result says that set of fixed points of a monotone self-map defined on a chain complete poset is a nonempty chain complete poset.[3]

A self-map $T : A \to A$ is *order continuous* if, given any countable chain $\{a_n\} \subseteq A$ for which $\sup a_n$ exists, we have $T(\sup a_n) = \sup T(a_n)$. Clearly, order continuous self-maps are monotone. A fixed point theorem, essentially due to Kantorovich [17] p. 68, says that the set of fixed points of a order continuous self-map defined on a chain $\sigma$-complete poset has a least fixed point.

**Concavity** A subset $A$ of an $V$ is *order convex* if $a \leq c \leq b$ and $a, b \in A$ imply $c \in A$. This amounts to say that $A$ contains all order intervals (and so all segments) determined by its elements.

A self-map $T : A \to A$ defined on an order convex subset is *order concave* if

$$T(ta + (1-t)b) \geq tT(a) + (1-t)T(b)$$

for all $t \in [0, 1]$ and all $a, b \in A$ with $a \leq b$. Order concave and order convex operators are studied in Amann [3, Chapter V], along with their differential characterizations.

**Subhomogeneity** The study of subhomogeneity for operators was pioneered by Krasnoselskii [19]. An operator $T : K \to K$ is called:[4]

(i) *subhomogeneous* if $T(\alpha x) \geq \alpha T(x)$ for all $x \in K$ and all $\alpha \in [0, 1]$;[5]

(ii) *strictly subhomogeneous* if the inequality is strict when $\alpha \in (0, 1)$ and $x \neq 0$;

(iii) *strongly subhomogeneous* if

$$T(\alpha x) \geq \varphi(x, \alpha) T(x) \qquad \forall 0 \neq x \in K, \forall \alpha \in (0, 1) \tag{1}$$

with $\alpha < \varphi(x, \alpha) < 1$;[6]

(iv) *subhomogeneous of order* $p \in (0, 1)$ if $T(\alpha x) \geq \alpha^p T(x)$ for all $x \in K$ and all $\alpha \in [0, 1]$.

Note that for a subhomogeneous operator we have $T(0) \geq 0$. Subhomogeneous operators of order $p$ are strongly subhomogeneous with $\varphi(x, \alpha) = \alpha^p$ (they are, actually, the most convenient class of such operators). For brevity, throughout the paper operators in (iv) will be called $p$-subhomogeneous.[7]

[2] That is, if $x \geq 0$ and $nx \leq y$ for every $n \in \mathbb{N}$, then $x = 0$. In fact, the countable chain $\cdots \geq nx \geq \cdots \geq x$ has a supremum $\sup nx$ in the order interval $[0, y]$. But $2^{-1} \sup(2n)x = \sup nx = \sup(2n)x$, so $0 \leq x \leq \sup nx = 0$.

[3] This result, as stated, can be found in Markowsky [27] p. 65. The existence of a least fixed point is due to Abian and Brown [1] p. 80.

[4] Though for simplicity we consider a cone domain, we can actually consider any star-shaped subset $A$ of $K$, i.e., $\alpha x \in A$ if $x \in A$ and $\alpha \in [0, 1]$. Throughout the paper, we consider these more general domains whenever needed.

[5] Different authors use different terminologies. Subhomogeneous operators are called concave by [19] and sublinear by [3]. For a class of subhomogeneous operators, [9] coined the term "pseudoconcave operators".

[6] Under mild assumptions, $\varphi(x, \alpha)$ can be chosen to be monotone and continuous in $\alpha$. See the proof of Proposition 36.

[7] They are also called $p$-concave operators by some authors. Strongly subhomogeneous operators are also called $\varphi$-concave (see [22]).

**Norms and units** A positive element $e \in V$ is an *order unit* for $V$ if the interval $[-e, e]$ is absorbing, that is, $V = \bigcup_{\lambda > 0} \lambda [-e, e]$. More in general, for every nonzero element $u \in K$ the set $V_u = \bigcup_{\lambda > 0} \lambda [-u, u]$ is a nontrivial vector subspace of $V$ that has $u$ as an order unit. If $V$ is Archimedean, then $V_u$ can be equipped with an *order unit norm* (see [19] and [3]):

$$\|x\|_u = \inf \{\lambda > 0 : -\lambda u \leq x \leq \lambda u\}.$$

This norm is simply called the *$u$-norm*. The cone $K_u = K \cap V_u$ is a closed cone in $(V_u, \|\cdot\|_u)$ with nonempty interior consisting of the order units of $V_u$.

**Links and the Thompson metric** Two elements $x, y \in K$ are *linked* (see [37]), written $x \sim y$, if there exist scalars $\alpha, \beta > 0$ such that

$$\alpha y \leq x \leq \beta y.$$

The binary relation $\sim$ is an equivalence relation that partitions the positive cone $K$ in disjoint components, which form the quotient set $K/\sim$. We denote by $Q(x)$ the equivalence class with representative $x \in K$, i.e., $Q(x) = \{y \in K : x \sim y\}$.

If $x \sim y$ define

$$d(x, y) = \inf \left\{\lambda \geq 0 : e^{-\lambda} x \leq y \leq e^{\lambda} x\right\}. \tag{2}$$

The binary relation $d$ defines a distance, the so-called *Thompson metric*, on each component $Q$ of $K$ provided $V$ is Archimedean (see [37] and [30]). Note that if $x$ and $y$ are comparable, say $x \leq y$, then (2) reduces to $d(x, y) = \inf \left\{\lambda \geq 0 : y \leq e^{\lambda} x\right\}$.

The positive cone $K$ in a normed ordered space is called *normal* if $0 \leq x \leq y$ implies $\|x\| \leq \gamma \|y\|$ for some $\gamma \geq 1$.

**Theorem 1 (Thompson)** *Let $V$ be a normed ordered space. If $K$ is normal, then convergence in the Thompson metric implies convergence in norm. If, in addition, $V$ is Banach, then each metric space $(Q, d)$ is complete.*

# 3 Lower perimeter

## 3.1 Definition and characterizations

Let $A$ be a set in an ordered vector space $V$. The *lower perimeter* $\partial_\diamond A$ of $A$ is defined by

$$\partial_\diamond A = \{x \in A : \exists y \in A,\, y > x \text{ and } tx + (1-t)\, y \notin A \text{ for all } t > 1\}.$$

In words, $\partial_\diamond A$ consists of the dominated elements $a$ of $A$ such that the segments that join them with a dominant element $b$ of $A$ cannot be prolonged beyond $a$ without exiting $A$. In contrast, an element $a \in A$ does not belong to $\partial_\diamond A$ if it is either undominated (i.e., it is maximal) or $ta + (1-t)\, b \in A$ holds for some $t > 1$ whenever $a < b \in A$.

**Proposition 2** *A dominated and minimal element of a convex set $A$ belongs to $\partial_\diamond A$.*

**Proof** Let $x \in A$ be dominated, with $x < z \in A$, and minimal. Suppose by contradiction that $x \notin \partial_\diamond A$. Then, there exists $t > 1$ such that $tx + (1-t)\, z \in A$, which contradicts minimality because $tx + (1-t)\, z < x$. ∎

Of course, $\partial_\diamond A$ may contain non-minimal elements, as the characterizations of lower perimeters that we are about to establish will show. We first characterize lower perimeters of intervals via the link equivalence relation $\sim$.

**Proposition 3** *Let $I = [a, b] \subseteq V$, with $a < b$. An element $x \in I$ does not belong to $\partial_\diamond I$ if and only if $x - a \sim b - a$.*

**Proof** Let $x \in I \backslash \partial_\diamond I$. If $x = b$, the result is obvious. Thus, suppose $a \leq x < b$. By definition, $(1-t)\, b + tx \geq a$ for some $t > 1$. Setting $t = 1 + \delta$, this is equivalent to $(1+\delta)\, x - \delta b \geq a$ for some $\delta > 0$. Namely, $(1+\delta)\, x \geq a + \delta b$. By subtracting $(1+\delta)\, a$ from both sides, we get

$$(1+\delta)(x-a) \geq \delta\,(b-a)\,.$$

Hence,

$$b - a \geq x - a \geq \frac{\delta}{1+\delta}\,(b-a)$$

So, $x - a$ and $b - a$ are linked. Conversely, suppose $a \leq x < b$ and $x - a \sim b - a$. Given that $x - a < b - a$, this means that $\lambda\,(x-a) \geq b - a$ and that $\lambda > 1$. Otherwise, $x - a \geq b - a$ which implies $x = b$. As $\lambda > 1$, we can set $\delta = (\lambda - 1)^{-1} > 0$, that is, $1/\lambda = \delta/\,(1+\delta)$. Consequently,

$$x - a \geq \frac{1}{\lambda}\,(b-a) = \frac{\delta}{1+\delta}\,(b-a) \geq \frac{\delta}{1+\delta}\,(b'-a)$$

for every $b' \leq b$. So, $(1+\delta)\, x - \delta b' \geq a$. By the substitution $t = 1+\delta$, it becomes $tx + (1-t)\, b' \geq a$ for some $t > 1$. This suffices to conclude that $x \in I \backslash \partial_\diamond I$. ■

Next we characterize the lower perimeters of the positive cone.

**Proposition 4** *Let $V$ be Archimedean $V$, with order units. An element $x \in K$ does not belong to $\partial_\diamond K$ if and only if $x \sim e$ for some order unit $e \in K$.*

Namely, $K \backslash \partial_\diamond K$ is the set of the all order units of $V$ which, indeed, is easily seen to be the component $Q\,(e) = \{x \in K : x \sim e\}$.

**Proof** Let $x \in K \backslash \partial_\diamond K$. We have $\lambda e > x$ for some $\lambda$, as $e$ is an order unit. In view of the previous proof $(1+\delta)\, x \geq \delta \lambda e$ for some $\delta > 0$. Hence,

$$\lambda e \geq x \geq \frac{\delta\lambda}{1+\delta} e$$

and so $x \sim e$. Conversely, let $x \sim e$ and $b > x$. Then, $\lambda e \geq b$ for some $\lambda$ and $x \geq \mu e$ because $x \sim e$. Hence,

$$b > x \geq \frac{\mu}{\lambda} b.$$

It follows that $\mu/\lambda < 1$. Therefore, $\mu/\lambda = \delta/\,(1+\delta)$ for some $\delta > 0$. Namely, $(1+\delta)\, x - \delta b \geq 0$ which means $x \in K \backslash \partial_\diamond K$. ■

This proposition establishes a sharp topological characterization of the lower perimeter of $K$. Indeed, the space $V$ can be equipped with the order unit norm $\|\cdot\|_e$, where $e$ is an order unit of $V$. Hence, by Proposition 4 we have

$$K \backslash \partial_\diamond K = \operatorname{int} K$$

according to the topology induced by $\|\cdot\|_e$. So, $\partial_\diamond K$ is the boundary of $K$.

In a similar vein, we have the following geometric version of Proposition 3.

**Proposition 5** *Let $I = [a, b] \subseteq V$, with $a < b$ and $V$ Archimedean. An element $x \in I$ does not belong to $\partial_\diamond I$ if and only if it belongs to $a + \operatorname{int} K_{b-a}$.*

Here, $\text{int}K_{b-a}$ denotes the interior of the cone $K_{b-a} = K \cap V_{b-a}$ in the normed space $\left(V_{b-a}, \|\cdot\|_{b-a}\right)$ generated by the vector $b - a > 0$.

**Proof** By Proposition 3, $I \backslash \partial_\diamond I = a + J \backslash \partial_\diamond J$ where $J = [0, b-a]$. Moreover, $x \in J \backslash \partial_\diamond J$ if and only if $x \sim b - a$ and $x \in [0, b-a]$. On the other hand, $b - a$ is an order unit of $\left(V_{b-a}, \|\cdot\|_{b-a}\right)$. Consequently, the elements of $J \backslash \partial_\diamond J$ are the order unit of $\left(V_{b-a}, \|\cdot\|_{b-a}\right)$ contained in $[0, b-a]$. It is well known that the set of order unit of $\left(V_{b-a}, \|\cdot\|_{b-a}\right)$ agrees with the interior of the cone $K_{b-a}$. Hence, $J \backslash \partial_\diamond J = \text{int}\, K_{b-a} \cap [0, b-a]$, as desired. ■

Finally, a dual notion of upper perimeter $\partial^\diamond A$ can be defined, for which dual results hold. For instance, in the dual version of Proposition 3 we have

$$x \in I \backslash \partial^\diamond I \Longleftrightarrow b - x \sim b - a. \tag{3}$$

In what follows, whenever needed we take for granted such dual results for upper perimeters.

## 3.2 Examples

We now present a few examples of lower perimeters that will be useful in the rest of the paper. We consider the space $\mathbb{R}^X$ of the real-valued functions $f : X \to \mathbb{R}$ defined on a set $X$, endowed with the pointwise order between functions. A piece of notation: if $f, g \in \mathbb{R}^X$, we write $f \ll g$ when $\inf_{x \in X} [g(x) - f(x)] > 0$.[8]

**Proposition 6** *Let $V$ be a vector subspace of $\mathbb{R}^X$. Consider an interval $I = [f, g] \subseteq V$, with $f < g$. Then,*[9]

$$\partial_\diamond I = \left\{ h \in I : \inf_{x \in X} \frac{h(x) - f(x)}{g(x) - f(x)} = 0 \right\}. \tag{4}$$

*In particular, if $f \ll g$ and $\sup_{x \in X} [g(x) - f(x)] < \infty$, then*

$$\partial_\diamond I = \left\{ h \in I : \inf_{x \in X} [h(x) - f(x)] = 0 \right\}. \tag{5}$$

**Proof** By Proposition 3, we have $h \in I \backslash \partial_\diamond I$ if and only if $h - f \geq \varepsilon (g - f)$ for some $\varepsilon > 0$. Note that if $g(x) - f(x) = 0$ then $h(x) - f(x) = 0$. Therefore, we have $h \in I \backslash \partial_\diamond I$ if and only if

$$\inf_{x \in X} \frac{h(x) - f(x)}{g(x) - f(x)} > 0$$

and so (4) holds. The two conditions $f \ll g$ and $\sup_{x \in X} [g(x) - f(x)] < \infty$ mean that $M \geq g - f \geq \varepsilon > 0$. Hence, in this case

$$\frac{1}{\varepsilon}(h - f) \geq \frac{h - f}{g - f} \geq \frac{1}{M}(h - f),$$

which shows the equivalence of (4) and (5). ■

By similar methods one can show that the lower perimeter of the positive cone $B_+(X)$ of the space of bounded functions $B(X)$,

$$\partial_\diamond B_+(X) = \left\{ h \in B_+(X) : \inf_{x \in X} h(x) = 0 \right\}.$$

as well as the following characterization for the space $L^\infty(X, \Sigma, \mu)$ – whose points are, as usual, classes of functions. For simplicity, we consider only the counterpart of (5).

[8] This notation is consistent with the familiar relation $x \ll y$ between vectors $x$ and $y$ of $\mathbb{R}^n$.

[9] In (4) we adopt the convention $0/0 = 0$.

**Proposition 7** *Let $I = [f, g] \subseteq L^{\infty}(X, \Sigma, \mu)$, with* $\operatorname{ess\,inf}_{x \in X}[g(x) - f(x)] > 0$. *Then*

$$\partial_{\diamond} I = \left\{ h \in I : \operatorname*{ess\,inf}_{x \in X} [h(x) - f(x)] = 0 \right\}.$$

We turn now to a real (or complex) Hilbert space $H$ with inner product $(\cdot, \cdot)$. Let $\mathcal{L}_s(H)$ be the real Banach space of all linear self-adjoint operators on $H$, endowed with the usual operator norm $\|\cdot\|$. Endow $\mathcal{L}_s(H)$ with the *Loewner order*, with positive cone

$$\mathcal{L}_s^+(H) = \{A \in \mathcal{L}_s(H) : (Ax, x) \geq 0\}.$$

It is known that $\mathcal{L}_s(H)$ is not a lattice, unless $\dim H = 1$ (see [24, Ch. 8]). However, bounded order intervals of $\mathcal{L}_s(H)$ are chain complete.[10]

The Loewner order is just the pointwise order of quadratic functions on the unit sphere $S$ of $H$, i.e., for all $A, B \in \mathcal{L}_s(H)$,

$$A \geq B \Longleftrightarrow (Ax, x) \geq (Bx, x) \qquad \forall x \in S.$$

So, Proposition 6 applies and leads to the next result, where we set $[A]_{\infty} = \inf_{x \in S}(Ax, x)$ for all $A \in \mathcal{L}_s(H)$.

**Proposition 8** *Let $I = [A, B] \subseteq \mathcal{L}_s(H)$, with $[B - A]_{\infty} \geq \varepsilon > 0$. Then*

$$\partial_{\diamond} I = \{X \in I : [X - A]_{\infty} = 0\}.$$

# 4 Existence and uniqueness: order concavity

Throughout this section, $V$ denotes a Dedekind $\sigma$-complete ordered vector space.

**Lemma 9** *Let $T : A \to A$ be a monotone and order concave self-map defined on an order convex subset $A$ of $V$. Assume that either $A$ is Dedekind complete or $T$ is order continuous. Suppose that:*

*(i) for each $b \in A \backslash \partial_{\diamond} A$, there is $b \geq a \in A$ such that $T(a) > a$;*

*(ii) $T(a) \neq a$ for all $a \in \partial_{\diamond} A$.*

*Then, $T$ has a least fixed point in $A$ if and only if it has a unique fixed point in $A$.*

Observe that $\partial_{\diamond} A$ might be empty. A dual result holds for order convex operators by considering the upper perimeter $\partial^{\diamond} A$. It actually suffices to consider the conjugate map $\tilde{T} : -A \to -A$ defined by $\tilde{T}(x) = -T(-x)$.

**Proof** Let $\xi$ be the least fixed point in $A$. Suppose, per contra, that it is not unique. Let $\zeta$ be another fixed point. Clearly, $\xi < \zeta$ (so $\xi$ is a dominated element of $A$). By (ii), $\xi \notin \partial_{\diamond} A$. By (i), there exists $a \leq \xi$ such that $T(a) > a$. This implies that $a < \xi$. For each $n \in \mathbb{N}$, set

$$x_n = \xi - \frac{1}{n}(\zeta - \xi),$$

i.e.,

$$\xi = \frac{1}{1 + n^{-1}} x_n + \frac{n^{-1}}{1 + n^{-1}} \zeta.$$

[10] An earlier result is due to Vigier (see [24, Th. 53.4]). See also [16].

Clearly, $x_n < \xi$ for each $n \in \mathbb{N}$, and so $x_n \neq T(x_n)$ for all $n \in \mathbb{N}$. Since $\xi \notin \partial_\diamond A$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. Hence, by order concavity of $T$, for all $n \geq n_0$ we have

$$\frac{1}{1+n^{-1}} x_n + \frac{n^{-1}}{1+n^{-1}} \zeta = \xi = T(\xi) \geq \frac{1}{1+n^{-1}} T(x_n) + \frac{n^{-1}}{1+n^{-1}} \zeta.$$

Hence,

$$x_n > T(x_n) \qquad \forall n \geq n_0 \tag{6}$$

As $V$ is Dedekind $\sigma$-complete, the $\sup_n x_n$ exists. Let us show that $\sup_n x_n = \xi$. Suppose not. Then, there exists an element $\eta$ such that

$$\xi - \frac{1}{n}(\zeta - \xi) \leq \eta < \xi.$$

Hence $n(\xi - \eta) \leq \zeta - \xi$ for every $n \in \mathbb{N}$. Since a Dedekind $\sigma$-complete ordered vector space is Archimedean, we have the contradiction $\xi \leq \eta$. We conclude that $\sup_n x_n = \xi$. In turn, this implies $x_{\bar{n}} > a$ for some $\bar{n} \in \mathbb{N}$. Since $A$ is order convex, $[a, x_{\bar{n}}] \subseteq A$. By (6), $T$ maps the interval $[a, x_{\bar{n}}]$ into itself because $T(a) > a$. The set $[a, x_{\bar{n}}]$ is countably chain complete. Consider two cases.

(i) If $A$ is Dedekind complete, then $[a, x_{\bar{n}}]$ is chain complete, so by the generalized Tarski's Theorem there is a fixed point of $T$ that belongs to $[a, x_{\bar{n}}]$;

(ii) if $T$ is order continuous, the same is true by Kantorovich's Theorem. In both cases, since $\xi$ is the least fixed point in $A$, we then have $\xi \in [a, x_{\bar{n}}]$, which contradicts $x_{\bar{n}} < \xi$. We conclude that $\zeta = \xi$. ■

In view of the generalized Tarski's Theorem and of Kantorovich's Theorem, we have the following existence and uniqueness result for fixed points.

**Theorem 10** *Let $T : A \to A$ be a monotone and order concave self-map defined on an order convex and chain $\sigma$-complete subset $A$ of $V$. Assume that either $A$ is chain complete or $T$ is order continuous. If $T(x) \neq x$ for all $x \in \partial_\diamond A$, then $T$ has a unique fixed point.*

**Proof** (i) Assume that $A$ is chain complete. Then, the existence of a least fixed point is guaranteed by the generalized Tarski's Theorem. As $A$ is chain complete, it has a minimum element $a$. In view of Proposition 2 $a \in \partial_\diamond A$ and so $a < T(a)$. The hypotheses of Lemma 9 are then satisfied, so the fixed point is unique.

(ii) Assume that $T$ is order continuous. The existence of a least fixed point is then ensured by Kantorovich's Theorem. Since $A$ is chain $\sigma$-complete, it has a minimum element $a$. Since $a \in \partial_\diamond A$, we have $a < T(a)$. The hypotheses of Lemma 9 are then satisfied, so the fixed point is unique. ■

In Riesz spaces, order convex and chain complete subsets are order intervals. Therefore, the previous theorem is usually applicable to self-maps defined on order intervals, as some examples will show later in the paper. The following result deals, instead, with self-maps defined on positive cones of $\sigma$-chain complete ordered vector spaces, a case not covered by the previous theorem.

**Theorem 11** *Let $T : K \to K$ be a monotone and order concave self-map. Let $e \in K$ be an order unit of $V$. Then, $T$ has a unique fixed point on $K$ provided:*

*(i) $T(x) \neq x$ for all $x \in \partial_\diamond K$;*

*(ii) $T(\lambda e) \leq \lambda e$ for all $\lambda > 0$ sufficiently large;*

*(iii) the intervals $[0, \lambda e]$ are chain complete or $T$ is order continuous.*

**Proof** By (ii), there is $\lambda_0 > 0$ such that $T$ is a monotone self-map on the intervals $[0, \lambda e]$ if $\lambda \geq \lambda_0$. By (iii), there is a least fixed point $\xi \in [0, \lambda_0 e]$. For the same reason, there is a least fixed point $\zeta$ in the interval $[0, \lambda e]$. Hence, $\zeta \leq \xi$ that in turn implies $\xi \geq \zeta$. So, $\xi$ is the least fixed point for every interval $[0, \lambda e]$. If now $\eta$ is any fixed point of $T : K \to K$, we have $\eta \in [0, \lambda e]$ for some $\lambda$ because $e$ is an order unit. Therefore, $\xi \leq \eta$ and so $\xi$ is the least fixed point in $K$. By (i), being $T(0) > 0$, Lemma 9 guarantees the existence of the unique fixed point $\xi$ in $K$. ■

The lower perimeter plays a key role in the previous results. Indeed, the requirement that there are no fixed points on the lower perimeter is needed. For instance, consider the self-map $T(x_1, x_2) = (1/2, \sqrt{x_2})$ defined on the cone $\mathbb{R}^2_+$. It is monotone and concave, with $T(0,0) > (0,0)$, but it has the two fixed points $(1/2, 0)$ and $(1/2, 1)$. Clearly, $(1/2, 0) \in \partial_\diamond \mathbb{R}^2_+$.

**Example 12** Define $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$T(x_1, x_2) = \left(\mu_1 x_1^{\alpha_1} + \lambda_1 x_2^{\beta_1} + \varepsilon_1, \mu_2 x_1^{\alpha_2} + \lambda_2 x_2^{\beta_2} + \varepsilon_2\right)$$

where all parameters are positive and $\varepsilon_1 + \varepsilon_2 > 0$. Clearly, $T$ is monotone. It is also concave when $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$. By Proposition 4,

$$\partial_\diamond \mathbb{R}^2_+ = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 \cdot x_2 = 0\right\}.$$

Observe that

$$T_1(x) \cdot T_2(x) > 0 = x_1 \cdot x_2 \qquad \forall (0,0) \neq x \in \partial_\diamond \mathbb{R}^2_+$$

and that $T(0,0) > (0,0)$ provided $\varepsilon_1 + \varepsilon_2 > 0$. Consequently, $T(x) \neq x$ for every $x \in \partial_\diamond \mathbb{R}^2_+$.

Now set $\alpha = \max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, $\sigma = \max\{\lambda_1 + \mu_1, \lambda_2 + \mu_2\}$ and $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. There exists a scalar $\bar{L}$ such that, for every $t \geq \bar{L} \geq 1$, we have $\sigma t^\alpha + \varepsilon \leq t$. Accordingly, if $x_1, x_2 \leq L$ with $L \geq \bar{L}$, then

$$\lambda_i x_1^{\alpha_i} + \mu_i x_2^{\beta_i} + \varepsilon_i \leq (\lambda_i + \mu_i) L^\alpha + \varepsilon \leq \sigma L^\alpha + \varepsilon \leq L.$$

In turn, this implies $T([0, Le]) \subseteq [0, Le]$ if $L \geq \bar{L}$ and $e = (1, 1)$, which is an order unit of $\mathbb{R}^2$. By Theorem 11, we conclude that $T$ has a unique fixed point $\xi \in \mathbb{R}^2_+$. Since $T$ is order continuous, $\xi$ can be then obtained by iterating the map from $(0,0)$, i.e., $T^n(0,0) \uparrow \xi$. ▲

**Example 13** Define $T : [0,1]^n \to [0,1]^n$ by

$$T(x_1, x_2, .., x_n) = \left(\lambda_i \prod_{k=1}^n x_k^{\alpha_k^i} + \varepsilon_i\right)_{i=1}^n.$$

If $\varepsilon_i \geq 0$, $\lambda_i + \varepsilon_i \leq 1$ and $\alpha_k^i > 0$ for each $i$ and $k$, then $T$ maps monotonically $[0,1]^n$ into itself. Assume now:

(i) $\alpha_k^i > 1$ for every $i, k = 1, ..., n$;

(ii) $\lambda_i + \varepsilon_i < 1$ for some $i = 1, ..., n$.

From (i) it follows that $T$ is order convex, though it is not convex.[11] Clearly, the upper perimeter of $[0,1]^n$ is:

$$\partial^\diamond [0,1]^n = \{x \in [0,1]^n : x_1 \vee x_2 \vee \cdots \vee x_n = 1\}.$$

Therefore, if $x \in \partial^\diamond [0,1]^n$ and $x \neq \mathbf{1}$, then $\lambda_i \prod_{k=1}^n x_k^{\alpha_k^i} + \varepsilon_i < 1$ for all $i$.[12] Hence, $T(x) \neq x$ in these cases. On the other hand, by (ii) we have $T(\mathbf{1}) < \mathbf{1}$. So, Theorem 10 guarantees the existence of a unique fixed point $\xi$ in $[0,1]^n$, which can be computed recursively as $T^n(\mathbf{1}) \downarrow \xi$. ▲

[11] The functions $\varphi(x_1, x_2, .., x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $\alpha_i \geq 1$ are ultramodular (see [25]), so order convex.

[12] Throughout by boldface $\mathbf{k}$ we mean the constant vector or function that assumes value $k \in \mathbb{R}$.

# 5 Existence and uniqueness: subhomogeneity

## 5.1 Subhomogeneity and order concavity

In this section we consider versions of the previous uniqueness results for subhomogeneous operators. The techniques that we use here are altogether different from those of the previous section and rely on a connection with the Thompson metric that will be fully developed in the Appendix. Note that the fixed point problems based on subhomogeneous operators are often defined on cones, while those based on order concave operators are often defined on bounded intervals.

To best appreciate the scope of these results, it is important to understand first the relations between order concavity and subhomogeneity.

**Proposition 14** *An order concave operator $T : K \to K$ is subhomogeneous if and only if $T(0) \geq 0$.*

**Proof** Let $T$ be order concave. If $T(0) \geq 0$, then $T(\alpha x) \geq \alpha T(x) + (1-\alpha) T(0) \geq \alpha T(x)$ for all $x \in K$. So, $T$ is subhomogeneous. The converse is trivially true since for a $T$ subhomogeneous it holds $T(0) \geq 0$. ■

A weak notion of concavity at 0 is relevant for subhomogeneity. Say that an operator $T : K \to K$ is *subconcave at* 0 if, for a fixed element $u \in K$, we have

$$T(\alpha x) \geq \alpha T(x) + (1-\alpha) u \qquad \forall x \in K, \forall \alpha \in [0,1].$$

This definition implies that $T(0) \geq u$. So, order concave operators $T : K \to K$ are subconcave at 0 provided $T(0) \geq 0$. In turn, operators $T : K \to K$ that are subconcave at 0 are easily seen to be subhomogeneous (strictly if $u > 0$).

**Proposition 15** *Let $T : K \to K$ be subconcave at* 0.

*(i) If $T([0,y]) \subseteq [0,y]$ for some $y \in K$ with $u \sim y$, then $T$ is $p$-subhomogeneous on $[0,y]$ for some $p \in (0,1)$.*

*(ii) If $T([0,\lambda_0 e]) \subseteq [0,\lambda_0 e]$ for some $\lambda_0 > 0$ and an order unit $e \sim u$, then $T$ is strongly subhomogeneous on $K$.*

*(iii) If $\operatorname{Im} T \subseteq Q(u)$ for some $0 < u \in K$, then $T$ is strongly subhomogeneous.*

**Proof** (i) By hypothesis, there is a positive scalar $\lambda \leq 1$ for which $u \geq \lambda y$ (note that $u \leq T(0) \leq y$). We can assume $\lambda < 1$ (the case $\lambda = 1$ is trivial). Hence, if $x \in [0,y]$ we have

$$\begin{aligned} T(\alpha x) &\geq \alpha T(x) + (1-\alpha) u \geq \alpha T(x) + (1-\alpha)\lambda y \geq \alpha T(x) + (1-\alpha)\lambda T(x) \\ &= [\alpha + (1-\alpha)\lambda] T(x). \end{aligned} \tag{7}$$

On the other hand, the property of superdifferentiability at $\alpha = 1$ of the concave function $\alpha \to \alpha^{1-\lambda}$ defined on $\mathbb{R}_+$, with $\lambda \in [0,1]$, yields the inequality

$$\alpha^{1-\lambda} \leq 1 + (1-\lambda)(\alpha - 1) = \alpha + (1-\alpha)\lambda. \tag{8}$$

In view of (7), we get $T(\alpha x) \geq \alpha^{1-\lambda} T(x)$ for all $x \in [0,y]$ and $\alpha \in [0,1]$. Hence, $T$ is $(1-\lambda)$-subhomogeneous.

(ii) As $T$ is subconcave, it is subhomogeneous. Hence, if $\mu \geq 1$ then $T(\mu\lambda_0 e) \leq \mu T(\lambda_0 e) \leq \mu\lambda_0 e$. Namely, $T([0,\lambda e]) \subseteq [0,\lambda e]$ holds for every $\lambda \geq \lambda_0$. Now, given a vector $x \in K$, we have $x \in [0,\lambda e]$ and $T([0,\lambda e]) \subseteq [0,\lambda e]$, as long as $\lambda$ is large enough. Point (i) provides the desired result.

(iii) This points follows from point (ii) but here we prove it directly. By hypothesis, $T(\alpha x) = T(\alpha x + (1-\alpha)0) \geq \alpha T(x) + (1-\alpha)T(0)$, and both $T(x)$ and $T(0)$ lie in $Q(u)$. Hence, $T(0) \geq \lambda(x)T(x)$, with $\lambda(x) \in (0,1)$. Therefore,

$$\begin{aligned} T(\alpha x) &\geq \alpha T(x) + (1-\alpha)T(0) \geq (\alpha + (1-\alpha)\lambda(x))T(x) \\ &= \alpha\left[1 + (1-\alpha)\alpha^{-1}\lambda(x)\right]T(x). \end{aligned}$$

as desired. ■

By Proposition 14, order concave operators $T$ cannot be subhomogeneous as soon as $T(0) < 0$. On the other hand, the next example shows that subhomogeneous operators might well be convex, a further stark illustration that order concavity and subhomogeneity are only partially related.

**Example 16** The monotone convex functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\varphi(t) = (1+t^n)^{1/n}$ are strongly subhomogeneous on $\mathbb{R}_+$ and $p$-subhomogeneous on each bounded interval of $\mathbb{R}_+$. Since $\varphi'(t)\,t/\varphi(t) = t^n/(1+t^n) < 1$, this assertion will be a consequence of Corollary 34 (see Appendix). ▲

A more interesting example is the Bellman operator $T : B(X) \to B(X)$ defined by

$$T(v)(x) = \sup_{y \in D(x)} u(x,y) + \beta v(y) \tag{9}$$

where $D : X \rightrightarrows X$ is a nonempty valued correspondence on some set $X$, $u : \mathrm{Gr}\,D \to \mathbb{R}$ is a bounded (short-run) objective function and $\beta \in (0,1)$ is a discount factor.

Though the Bellman operator is convex, it is subhomogeneous when it is positive. Set $\bar{u} = \inf_{x\in X} \inf_{y\in D(x)} u(x,y)$.

**Proposition 17** *If $\bar{u} \geq 0$, the Bellman operator $T : B_+(X) \to B_+(X)$ is subhomogeneous (strongly if $\bar{u} > 0$).*

**Proof** Let $\bar{u} \geq 0$. Then,

$$\begin{aligned} T(\alpha v)(x) &= \sup_{y\in D(x)} u(x,y) + \alpha\beta v(y) = \sup_{y\in D(x)} \alpha u(x,y) + \alpha\beta v(y) + (1-\alpha)u(x,y) \\ &\geq \alpha \sup_{y\in D(x)} [u(x,y) + \beta v(y)] + (1-\alpha)\bar{u} = \alpha T(v)(x) + (1-\alpha)\bar{u}. \end{aligned}$$

Thus, $T$ is subconcave at 0 if $\bar{u} \geq 0$ (though it is not concave at 0). This, in turn, implies that $T$ is subhomogeneous. Now, let $\bar{u} > 0$. The function $\bar{u}1_X$ is trivially linked to $1_X$. Therefore, the result is a consequence of Proposition 15-(ii) because $T : [-\mathbf{L}, \mathbf{L}] \to [-\mathbf{L}, \mathbf{L}]$ whenever $L \geq (1-\beta)^{-1} N$, with $|u(x,y)| \leq N$ ■

**Remark** Though we considered the bounded case, most of the properties that we established continue to hold in more general dynamic programming formulations.[13]

## 5.2 Existence and uniqueness

We can now establish the subhomogeneous counterparts of the order concave existence and uniqueness results for fixed points of Section 4. Throughout this subsection, $V$ denotes an Archimedean ordered vector space.

We begin with the counterpart of Theorem 10. Here $d$ is the Thompson metric.

[13] For instance (see [32]), we can define $T : V \to V$ as $T(v) = \max_{L\in\mathcal{L}} [u_L + Lv]$, where $\mathcal{L}$ is a set of linear operators mapping $V$ onto $V$ (the policies) and such that for each $L \in \mathcal{L}$, the inverse operator of $I - L$ exists and is positive. We do not pursue further this line because we will be interested in nonadditive temporal utilities.

**Theorem 18** *Let $T : [0, a] \to [0, a]$ be a monotone and strongly subhomogeneous self-map defined on a chain $\sigma$-complete interval of $V$. Assume that either $[0, a]$ is chain complete or $T$ is order continuous. If $T(x) \neq x$ for all $x \in \partial_\diamond [0, a]$, then $T$ has a unique fixed point $\bar{x}$. Moreover, for every initial point $x_0 \in [0, a]$ we have*

$$d\left(T^n(x_0), \bar{x}\right) \to 0 \tag{10}$$

*provided $x_0 \notin \partial_\diamond [0, a]$.*

Note that the order interval $[0, a]$ is star-shaped in $K$, a property that we need in the subhomogeneous case (so we cannot consider generic order intervals, as we did in the order concave case). The global attracting property (10) is remarkable. In particular, when $V$ is normed with a normal positive cone, then this property actually holds with respect to norm convergence (cf. Theorem 1).

Next we state the subhomogeneous counterpart of Theorem 11.

**Theorem 19** *Let $T : K \to K$ be monotone and strongly subhomogeneous and $e \in K$ an order unit of $V$. Then, $T$ has a unique fixed point on $K$ provided:*

*(i) $T(x) \neq x$ for each $x \in \partial_\diamond K$;*

*(ii) $T(\lambda e) \leq \lambda e$ for some $\lambda > 0$;*

*(iii) $[0, \lambda e]$ is chain complete or $T$ is order continuous.*

The proofs of these two results require a non-trivial analysis that we conduct later in the paper in Section A, after some applications are presented. Such analysis relies on a connection between subhomogeneity and the Thompson metric (Proposition 31), which is a main contribution of this paper.

# 6 Applications

## 6.1 Recursive utilities

Though the study of recursive utilities dates back to Koopmans [18], the idea of using intertemporal aggregators to generate recursive utilities is due to Lucas and Stokey [23] and [35]. Specifically, an *aggregator* is a function $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that satisfies the following properties:

(i) $W$ is positive and monotone;

(ii) the equation $W(c, \zeta) = \zeta$ has at least a positive solution for each $c \geq 0$.

A *recursive utility* $U : l_+^\infty \to \mathbb{R}$ generated by an aggregator $W$ is a solution $U : l_+^\infty \to \mathbb{R}$ to the *Koopmans equation*, i.e.,

$$U(c_0, c_1, c_2, ....) = W(c_0, U(c_1, c_2, ....)). \tag{11}$$

More concisely,

$$U({}_0c) = W(c_0, U({}_1c))$$

where ${}_0c = (c_0, c_1, c_2, ....)$ and ${}_1c = (c_1, c_2, ....)$ denotes the shift operator. For instance, standard time-additively separable $U$ are generated by the aggregators $W(c, \zeta) = u(c) + \beta\zeta$.

A key issue is whether an aggregator determines a unique recursive utility. The next proposition addresses this issue for a class of aggregators, introduced in [26] under the name of Thompson aggregators, that cannot be treated by the standard contraction methods employed by [23].

For simplicity, we restrict the analysis to utility functions $U(c)$ defined over bounded consumption streams $c = (c_0, c_1, c_2, ....) \in l_+^\infty$. The result is similar to the one proved in [26], the novelty being here the use of the techniques developed in this paper to prove it.[14]

[14] We refer to [26] for extensions to unbounded and to stochastic streams of consumption.

**Proposition 20** *Suppose the aggregator* $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *satisfies the conditions:*

*(a)* $\zeta \longmapsto W(c,\zeta)$ *is concave at* $0$*;*

*(b)* $W(c,0) > 0$ *for each* $c > 0$.

*Then,* $W$ *generates a unique recursive utility function on* $\operatorname{int} l_+^\infty$*. Moreover, we can replace* $\operatorname{int} l_+^\infty$ *with the entire cone* $l_+^\infty$ *if (b) is replaced by the stronger condition:*

*(b*)* $W(0,0) > 0$.

**Proof** Under assumptions (a) and (b), it easy to prove that the function $\zeta \longmapsto W(c,\zeta)/\zeta$ is strictly decreasing in $(0,\infty)$ for all $c > 0$.[15] This implies, in turn, that the equation $W(c,\zeta) = \zeta$ has a unique solution $\zeta_c$ for every $c > 0$ and, moreover, $W(c,\zeta) < \zeta$ if $\zeta > \zeta_c$.

Fix now any two numbers $0 < \varepsilon < L$ and consider the interval $[\varepsilon, L] \subseteq \operatorname{int} l_+^\infty$ where, with abuse of notation, we set $\varepsilon = (\varepsilon, \varepsilon, ...)$ and the same for $L$. Consider the space $B([\varepsilon, L])$ of the bounded functions on $[\varepsilon, L]$. Elements $U \in B_+([\varepsilon, L])$ are regarded as utility functions, and the recursive utilities will be fixed points of the operator $T : B_+([\varepsilon, L]) \to B_+([\varepsilon, L])$ defined by

$$T(U)({}_0c) = W(c_0, U({}_1c)).$$

By assumption (ii), there is a positive scalar $\zeta_L$ such that $W(L, \zeta_L) = \zeta_L$. If we take any arbitrary $u \geq \zeta_L$, the operator $T$ maps the interval $[\mathbf{0}, \mathbf{u}] \subseteq B_+([\varepsilon, L])$ into itself. Actually, if $0 \leq U(c) \leq u$, then by the monotonicity condition (i) we have:

$$0 \leq T(U)(c) = W(c_0, U({}_1c)) \leq W(L, u) \leq u.$$

Clearly $T$ is monotone and let us show that it is subconcave at 0. Actually, thanks to (a),

$$\begin{aligned} T(\alpha U)({}_0c) &= W(c_0, \alpha U({}_1c)) \geq \alpha W(c_0, U({}_1c)) + (1-\alpha) W(c_0, 0) \\ &\geq \alpha W(c_0, U({}_1c)) + (1-\alpha) W(\varepsilon, 0). \end{aligned}$$

As $W(\varepsilon, 0) > 0$, the constant function $W(\varepsilon, 0)$ is linked to the function 1 and so Proposition 15-(ii) implies that $T : B_+([\varepsilon, L]) \to B_+([\varepsilon, L])$ is strongly subhomogeneous.

In view of Theorem 19, to conclude that $T$ has a unique fixed point in $B_+([\varepsilon, L])$ it is sufficient to show that $\partial_\diamond B_+([\varepsilon, L])$ does not contain any recursive utility. Let $[U]_\infty = \inf_{c \in [\varepsilon, L]} U(c)$. If $U$ is recursive, then

$$[U]_\infty = \inf_{c \in [\varepsilon, L]} U(c) = \inf_{c \in [\varepsilon, L]} W(c_0, U({}_1c)) = \inf_{\varepsilon \leq c_0 \leq L} \inf_{{}_1c \in [\varepsilon, L]} W(c_0, U({}_1c))$$

$$\geq \inf_{\varepsilon \leq c_0 \leq L} W\left(c_0, \inf_{{}_1c \in [\varepsilon, L]} U({}_1c)\right) = \inf_{\varepsilon \leq c_0 \leq L} W(c_0, [U]_\infty) = W(\varepsilon, [U]_\infty) \geq W(\varepsilon, 0) > 0.$$

By Proposition 6, $U \notin \partial_\diamond B_+([\varepsilon, L])$. Thus, $T$ has a unique fixed point in $B_+([\varepsilon, L])$.

Since the two numbers $0 < \varepsilon < L$ are arbitrary, we conclude the existence and uniqueness of the recursive function defined on $\bigcup_{\varepsilon > 0, L > 0} [\varepsilon, L] = \operatorname{int} l_+^\infty$.

If assumption (b*) holds, we can then replicate the proof by replacing the interval $[\varepsilon, L]$ with $[0, L]$. In this case,

$$[U]_\infty \geq \inf_{0 \leq c_0 \leq L} W(c_0, [U]_\infty) = W(0, [U]_\infty) \geq W(0,0) > 0$$

The previous arguments now establish the uniqueness of the fixed point of $T : B_+\left(l_+^\infty\right) \to B_+\left(l_+^\infty\right)$. ■

[15] See [26, Lemma 1] for this and other related properties.

**Remarks** 1) By Tarski's Theorem, under the only assumptions (i) and (ii) the map $T : B_+\left(l_+^\infty\right) \to B_+\left(l_+^\infty\right)$ has fixed points. Specifically, there is the least $\underline{U}$ and the greatest $\overline{U}$ recursive functionals in $B\left(l_+^\infty\right)$ – see [26] and [7] for details. Therefore, our result implies that $\underline{U} = \overline{U}$ on $\operatorname{int} l_+^\infty$. That is, all the recursive preferences coincide over $\operatorname{int} l_+^\infty$.

2) Though the condition $W(0,0) = 0$ holds for several aggregators, condition (b$^*$) is still of some interest. On one hand, it encompasses aggregators of the type $\tilde{W}(c,\zeta) = W(u(c),\zeta)$ where $u(c)$ is the short-run utility of consumption: the hypothesis $u(0) \geq \eta > 0$ is then acceptable. On the other hand, aggregators satisfying condition (b$^*$) come up when we consider perturbed utility aggregators $W_\eta(c,\zeta) = W(c,\zeta) + \eta$ for small $\eta > 0$. This approach may be employed to study the behavior of the unique fixed point as $\eta \downarrow 0$ (see [7]).

3) If the aggregator $W$ is continuous, it is easy to show that the least recursive function $\underline{U}$ is lower semicontinuous in the relative product topology of $l_+^\infty$, while $\overline{U}$ is upper semicontinuous on the bounded sets of $l_+^\infty$ (see [26, Theorem 4]). Consequently, under (a) and (b) the unique recursive preference on $\operatorname{int} l_+^\infty$ are continuous in the relative product topology on the bounded sets of $\operatorname{int} l_+^\infty$.

Rather than solving directly the Koopmans equation (11), it is often useful to analyze an auxiliary "parametric" problem.[16] Specifically, for a given consumption plan $c = (c_t)_{t\geq 0} \in l_+^\infty$, we introduce the operator $T_c : l_+^\infty \to l_+^\infty$ defined, at every time $t$, by

$$T_c(v)_t = W(c_t, v_{t+1}) \qquad \forall v \in l_+^\infty. \tag{12}$$

Clearly, the sequence $v_t = U({}_tc)$ is a fixed point of $T_c$ if $U$ is a solution of (11). Conversely, the utility $U$ may be recovered by the fixed point of $T_c$ as the sequence $c$ varies.

The following is a uniqueness result that parallels Proposition 20.

**Proposition 21** *Let $W$ satisfy (a) and (b). The operator $T_c : l_+^\infty \to l_+^\infty$ has a unique fixed point provided* $\liminf_{t\to\infty} c_t > 0$.

**Proof** Suppose first that $c_t \geq \eta > 0$ for all $t$. In this case the proof follows the same lines of that of Proposition 20. The operator is strongly subhomogeneous and Theorem 11 provides the desired result.

In the more general case, we have $c_t \geq \eta > 0$ for all $t \geq N$. Let $\bar{v}$ and $\underline{v}$ be the greatest and the least fixed points of $T_c$, respectively. Obviously, we have ${}_N\bar{v} = {}_N\underline{v}$ because both are fixed points of $T_{{}_Nc}$, which has a unique fixed point thanks to the first part of the proof. By induction, we have

$$\bar{v}_{N-1} = W(c_{N-1}, \bar{v}_N) = W(c_{N-1}, \underline{v}_N) = \underline{v}_{N-1}$$

and so on. Therefore, $\bar{v} = \underline{v}$. ■

## 6.2 Bellman equations

We turn briefly to the Bellman equations associated with utilities that are recursively generated by aggregators. Now, the Bellman operator $T : B_+(X) \to B_+(X)$ is defined by

$$T(v)(x) = \sup_{y\in D(x)} W(u(x,y), v(y)). \tag{13}$$

We get back to (9) when $W(c,\zeta) = u(c) + \beta\zeta$. Under mild assumptions – e.g., the continuity of $W(c,\cdot)$ – for any recursive utility function $U : l_+^\infty \to \mathbb{R}_+$ generated by $W$, the associated value function $v \in B_+(X)$ is a fixed point of the Bellman operator (see [7]). In principle, its Bellman equation may admit multiple fixed points. Next we formulate a basic uniqueness result.

[16] This approach, used in [26], is especially useful in the stochastic case as well as in studying the associated Bellman equations.

**Proposition 22** *Let $W$ satisfy (a) and (b). If either (b$^*$) or $\bar{u} > 0$ holds (see Example 16), then the Bellman operator has a unique fixed point in $B_+(X)$.*

**Proof** Let $u(x,y) \leq N$. Then, by $(a)$ and $(b)$ there is a scalar $\zeta$ such that $W(N,\zeta) \leq \zeta$. This implies that $T([0,\zeta 1_X]) \subseteq [0,\zeta 1_X]$. Like in Proposition 17 we can easily show that $T$ is subconcave at 0. More specifically, $T(\alpha f) \geq \alpha T(f) + (1-\alpha)\varphi$ where $\varphi(x) = \inf_{y \in D(x)} W(u(x,y), 0)$.

Under (b$^*$), we have $\varphi(x) \geq W(0,0)$. While in the case $\bar{u} > 0$, we have $\varphi(x) \geq W(\bar{u},0) > 0$. In both cases we can invoke Proposition 15-(ii) and infer that $T$ is strongly subhomogeneous. For the same reason, if $v$ is a fixed point then either $v(x) \geq W(0,0)$ or $v(x) \geq W(\bar{u},0)$. Hence $v \notin \partial_\diamond B_+(X)$. Theorem 19 then implies the existence of a unique solution of the Bellman equation. ∎

Following [7], we can provide milder assumptions than those of the last proposition that still make the value function unique. Here we will use a method that fits well our theory and which differs partially from that of [7]. In the next result, besides standard convexity conditions, we add the crucial assumption on the existence of a feasible plan generating strictly positively utility along the path. Here $\bar{v}$ and $\underline{v}$ denote the greatest and the least fixed point of $T$, respectively (their existence is ensured by Tarski's Theorem).

**Proposition 23** *Suppose that:*

*(i) $X$ is a convex set, the correspondence $D : X \rightrightarrows X$ has a convex graph, and $u : \mathrm{Gr}\, D \to \mathbb{R}$ is bounded and concave;*

*(ii) $W$ satisfies (a) and (b) and $W(\cdot,\zeta)$ is concave.*

*If for every $x \in X$ the supremum in (13) is attained, then $\bar{v}(x) = \underline{v}(x)$ for all $x \in X$ such that either $D(x) = \{x\}$ or there is a feasible path $(x_t^*)_{t \geq 0}$ with $x_0^* = x$ and*

$$\liminf_{t \to \infty} u\left(x_t^*, x_{t+1}^*\right) > 0. \tag{14}$$

We refer to [7] for more comments about the interiority assumption (14). It is, however, fairly mild and so this result guarantees the uniqueness of the fixed point of (13) for standard problems, avoiding cake-like models. Actually, it suffices to postulate the existence of a sustainable state $x^*$, with strictly positive utility $u(x^*,x^*) > 0$, which may be reached from any state $x$ for which $D(x) \neq \{x\}$.

**Proof** The case $D(x) = \{x\}$ is trivial since $\bar{v}(x) = \underline{v}(x)$. Fix then an initial vector $x_0$ for which the condition (14) holds. By (iii), there exists a sequence $\bar{x} = (\bar{x}_t)$ such that

$$\bar{v}(\bar{x}_t) = W(u(\bar{x}_t, \bar{x}_{t+1}), \bar{v}(\bar{x}_{t+1}))$$

For sake of simplicity, by setting $\bar{v}(\bar{x}_t) = \bar{v}_t$ and $\bar{c}_t = u(\bar{x}_t, \bar{x}_{t+1})$, it becomes

$$\bar{v}_t = W(\bar{c}_t, \bar{v}_{t+1}). \tag{15}$$

Consider now the existing plan (14), $x^* = (x_t^*)$, with $x_0^* = \bar{x}_0 = x_0$ postulated by our hypothesis. Define the feasible perturbed plan $\underline{x} = (1-\alpha)\bar{x} + \alpha x^*$ with $\alpha \in (0,1)$. Since $\underline{v}$ is also a fixed point, we have

$$\underline{v}_t \geq W\left(\underline{c}_t, \underline{v}_{t+1}\right) \tag{16}$$

where, accordingly, we have set $\underline{v}_t = \underline{v}(\underline{x}_t)$ and $\underline{c}_t = u\left(\underline{x}_t, \underline{x}_{t+1}\right)$. Observe further that by construction

$$\liminf_{t \to \infty} \underline{c}_t > 0 \text{ if } \alpha > 0 \tag{17}$$

and that (16) is equivalent to the condition $T_{\underline{c}}\underline{v} \leq \underline{v}$ by using the auxiliary operator (12). Proposition 21 implies that $T_{\underline{c}}$ has a unique fixed point $w^*$ in $l_+^\infty$. More precisely, $w^* \in [0, \underline{v}]$.

Now by the concavity of $u$, we have $\underline{c}_t \geq (1-\alpha)\bar{c}_t + \alpha c_t^*$, where $c_t^* = u\left(x_t^*, x_{t+1}^*\right)$. So,

$$W\left(\underline{c}_t, \bar{v}_{t+1}\right) \geq (1-\alpha) W\left(\bar{c}_t, \bar{v}_{t+1}\right) + \alpha W\left(c_t^*, \bar{v}_{t+1}\right) \geq (1-\alpha) W\left(\bar{c}_t, \bar{v}_{t+1}\right).$$

In view of (15), it follows that $(1-\alpha)\bar{v}_t \leq W\left(\underline{c}_t, \bar{v}_{t+1}\right)$, namely, $T_{\underline{c}}\overline{v} \geq (1-\alpha)\overline{v}$.

Let us first assume that $\underline{c}_t \geq \eta > 0$ for every $t \geq 0$, in place of (17). Then $T_{\underline{c}}$ turns out to be $p$-subhomogeneous for some $p \in (0,1)$, i.e., $T_{\underline{c}}(\alpha w) \geq \alpha^p w$ if $\alpha \in [0,1]$.

Set $\mu_0 = (1-\alpha)^{1/(1-p)} < 1$. We have

$$T_{\underline{c}}\left(\mu_0 \overline{v}\right) \geq \mu_0^p T_{\underline{c}}\left(\overline{v}\right) \geq \mu_0^p (1-\alpha)\overline{v} = \mu_0 \overline{v}.$$

Consequently, $\mu_0 \overline{v} \leq w^* \leq \underline{v}$. As $\alpha \to 0^+$, we have that $\mu_0 \to 1$ and so $\overline{v} \leq \underline{v}$. That is, $\overline{v}(x_0) = \underline{v}(x_0)$.

If now it holds condition (17), then $\underline{c}_t \geq \eta > 0$ for every $t \geq N$. In this case we can use the above argument for the operator $T_{{}_N\underline{c}}$ and so concluding that $\overline{v}(x_N) = \underline{v}(x_N)$. Then

$$\overline{v}\left(\bar{x}_{N-1}\right) = W\left(u\left(\bar{x}_{N-1}, \bar{x}_N\right), \overline{v}\left(\bar{x}_N\right)\right) = W\left(u\left(\bar{x}_{N-1}, \bar{x}_N\right), \underline{v}\left(\bar{x}_N\right)\right) \leq \underline{v}\left(\bar{x}_{N-1}\right).$$

Therefore, $\overline{v}\left(\overline{x}_{N-1}\right) = \underline{v}\left(\bar{x}_{N-1}\right)$ and recursively we get again $\overline{v}(x_0) = \underline{v}(x_0)$. ∎

## 6.3 Implicit utilities and integral equations

**Implicit utilities** Dekel [13]'s implicit utilities are fixed points of problems like:

$$\int_0^1 k\left(x, y, \varphi(y)\right) dx = \varphi(y) \qquad \forall y \in [0,1]$$

where $\varphi \in B = B([0,1])$ is the unknown bounded function and the kernel $k : [0,1]^3 \to [0,1]$ is such that:
(i) $k(\cdot, y, z)$ is continuous on $[0,1]$ for every $y, z \geq 0$,
(ii) $k(x, y, \cdot)$ is increasing and concave for every $x, y \geq 0$.

Denote by $\mathbf{0}$ and $\mathbf{1}$ the functions constant to 0 and 1 for all $y \in [0,1]$, respectively. Let us consider the self-map $T : [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ defined by[17]

$$T(\varphi)(y) = \int_0^1 k\left(x, y, \varphi(y)\right) dx$$

Under our assumptions on the kernel $k$, the integral operator $T$ is monotone and order concave. By Proposition 6, we have

$$\partial_\diamond [\mathbf{0}, \mathbf{1}] = \left\{\varphi \in B : \inf_{y \in [0,1]} \varphi(y) = 0\right\}.$$

So, adding the assumption

$$\inf_{y \in [0,1]} \int_0^1 k(x, y, 0)\, dx = \eta > 0,$$

we have $T(\varphi) \neq \varphi$ for all $\varphi \in \partial_\diamond [\mathbf{0}, \mathbf{1}]$ because $T(\varphi) \geq T(\mathbf{0}) \geq \eta > \mathbf{0}$.

By Theorem 10, the self-map $T$ has then a unique fixed point. More is true, however. Observe that $T$ is subconcave at $\mathbf{0}$ and $T(\mathbf{0}) \sim \mathbf{1}$. Hence, by Proposition 15-(i), $T$ is $p$-subhomogeneous. The positive cone of $B([0,1])$, equipped with the supnorm, is normal. So, by Theorem 18 the iterates $T^n(\varphi_0)$ converge uniformly to the unique fixed point from any initial function $\varphi_0 \in [\mathbf{0}, \mathbf{1}]$ such that $\inf_{y \in [0,1]} \varphi(y) > 0$.

[17] The example can be generalized to integrals with respect to finitely additive Borel probability measures defined on the unit interval.

Finally, note that assumption (ii) can be replaced by the elasticity condition

$$\frac{D_3 k\left(x,y,t\right)t}{k\left(x,y,t\right)} \leq p < 1 \qquad \forall 0 \leq x, y \leq 1, \forall t \in (0,1]$$

provided $k\left(x,y,\cdot\right)$ is increasing and differentiable. Indeed, by Corollary 34 in Appendix the function $k\left(x,y,\cdot\right)$ is $p$-subhomogeneous. In turn, this implies that $T$ is $p$-subhomogeneous as well.

**Integral operators** An interesting class of integral operators is

$$\varphi\left(x\right) = \int k\left(x,y,\varphi\left(y\right)\right)\pi\left(dy \mid x\right) \qquad \forall x \in X \tag{18}$$

where $\varphi \in C\left(X\right)$, $\pi$ is a transition function on the compact metric space $X$ and $k : X \times X \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded (with other additional suitable conditions).

Equations like (18) arise in economics for instance in Markov equilibria (see for instance [35, Ch. 17]). They may be handled in different ways. For brevity, we just outline two possible routes. A first method circumvents the unpleasant fact that the space $C\left(X\right)$ is not a $\sigma$- Dedekind complete lattice. Therefore, the operator $T$ is replaced with the operator $T^* : B_+\left(X\right) \to B_+\left(X\right)$ defined by

$$T^*\left(f\right)\left(x\right) = \int^* k\left(x,y,f\left(y\right)\right)\pi\left(dy \mid x\right) \qquad \forall x \in X$$

where the symbol $\int^*$ denotes the outer integral of a function $\psi \in B_+\left(X\right)$ (see [6, Appendix A])

We can easily give conditions to apply Theorem 19 and to get a unique fixed point $f^*$ in $B_+\left(X\right)$. Note incidentally that $f^*$ attracts all the initial functions of $B_+\left(X\right)$ according to the supnorm (thanks to the normality of the positive cone; cf. Theorem 1). Then, under a Feller property that guarantees the continuity of $T^*\left(f\right)$ when $f$ is continuous, the fixed point $f^*$ will be continuous – being the uniform limit of continuous functions. Hence, $f^*$ solves (18).

A second method is to study directly the operator $T : C\left(X\right) \to C\left(X\right)$ which is well-defined under a Feller property. The space $C\left(X\right)$ is Banach space under the supnorm and its positive cone is normal. So, the component $\operatorname{int} C\left(X\right)$ is a complete metric space with respect to the Thompson metric (Theorem 1). Consequently, if $T$ is $p$-subhomogeneous we get the fixed point by the Banach fixed point principle.

## 6.4 Complementary problems and variational inequalities

Let $V$ be a vector lattice. The complementary problem, associated with a map $F : K \to V$, asks for a point $x^* \in K$ that satisfies the orthogonality condition[18]

$$F\left(x^*\right) \wedge x^* = 0.$$

For all $\lambda > 0$ we have:

$$\begin{aligned} F\left(x\right) \wedge x &= 0 \Longleftrightarrow F\left(x\right) \wedge \lambda x = 0 \Longleftrightarrow \left[F\left(x\right) - \lambda x\right] \wedge 0 = -\lambda x \\ &\Longleftrightarrow \left[\lambda x - F\left(x\right)\right] \vee 0 = \lambda x \Longleftrightarrow \lambda^{-1}\left[\lambda x - F\left(x\right)\right]^+ = x \end{aligned}$$

So, the complementary problem amounts to finding the fixed points of the self-map $T_\lambda : K \to K$ defined by

$$T_\lambda\left(x\right) = \lambda^{-1}\left[\lambda x - F\left(x\right)\right]^+ \tag{19}$$

where $\lambda > 0$ is an arbitrarily fixed parameter. The next result, which involves the upper perimeter, provides conditions that ensure the existence and uniqueness of complementary problem through the fixed point of $T_\lambda$.

[18] In $\mathbb{R}^n$ this complementary problem reduces to familiar problem of finding a vector $x^* \geq 0$ for which $F\left(x^*\right) \geq 0$ and $x^* \cdot F\left(x^*\right) = 0$ hold. This finite dimensional problem has two distinct extensions in infinite dimensional settings: the topological complementary problem and the order complementary problem. For the latter, we refer readers to [14] and [8].

**Proposition 24** *Let $V$ be a Dedekind complete vector lattice. Consider the following assumptions:*

*(i) $\lambda x - Fx$ is $\lambda$-weakly order-Lipschitz,*[19] *i.e., there exists $\lambda > 0$ such that*

$$F(x_2) - F(x_1) \leq \lambda (x_2 - x_1) \qquad \forall 0 \leq x_1 \leq x_2, \tag{20}$$

*(ii) $F$ is order concave on $K$,*

*(iii) there exists $\bar{x} \in K$ such that $F(\bar{x}) \geq 0$,*

*(iv) $T_\lambda(x) \neq x$ for all $x \in \partial^\diamond [0, \bar{x}]$.*

*Under (i) and (iii), there exists a vector $\xi \in [0, \bar{x}]$ such that $F(\xi) \wedge \xi = 0$. Under (i)-(iv), such a vector $\xi$ is unique. Moreover, (iv) holds if (iii) is replaced by the stronger assumption: $\exists \sigma > 0, F(\bar{x}) \geq \sigma \bar{x}$.*

The linear case $F(x) = Lx + q$, where $L : V \to V$ is a linear operator and $q \in V$, has been studied by [8]. The existence part of this theorem is, essentially, a nonlinear version of [8, Theorem 3.4] (note that (ii) trivially holds in such a case).

**Proof** Clearly, the operator $x \to x^+$ of $V$ into $V$ is monotone and convex. Therefore, by (i) it follows that $x \mapsto [\lambda x - F(x)]^+$ is monotone. Likewise, (i) and (ii) imply that $x \mapsto [\lambda x - F(x)]^+$ is order convex.

Note that the (iii) means that $\lambda \bar{x} - F(\bar{x}) \leq \lambda \bar{x}$. Hence, $T_\lambda(\bar{x}) = \lambda^{-1} [\lambda \bar{x} - F(\bar{x})] \vee 0 \leq \bar{x} \vee 0 = \bar{x}$. Namely, $T_\lambda(\bar{x}) \leq \bar{x}$. Consequently, the first claim follows from Tarski's Theorem applied to the self-map $T_\lambda : [0, \bar{x}] \to [0, \bar{x}]$. Theorem 10 provides the uniqueness result.

Regarding the last statement, if $F(\bar{x}) \geq \sigma \bar{x}$ then

$$\frac{\lambda \bar{x} - F(\bar{x})}{\lambda} \leq \left(1 - \frac{\sigma}{\lambda}\right) \bar{x}.$$

Therefore, for sufficiently large values of $\lambda$, we have $T_\lambda(\bar{x}) \leq (1 - \sigma/\lambda) \bar{x}$ with $0 < 1 - \sigma/\lambda < 1$. By monotonicity property of $T_\lambda$ we have that $T_\lambda [0, \bar{x}] \subseteq [0, \rho \bar{x}]$, where $\rho = 1 - \sigma/\lambda$. Hence, the fixed points of $T_\lambda$ lie into $[0, \rho \bar{x}]$. Let $T_\lambda(z) = z$. From the condition $0 \leq z \leq \rho \bar{x}$ it follows that

$$(1 - \rho) \bar{x} \leq \bar{x} - z \leq \bar{x}.$$

Consequently, $\bar{x} - z$ is linked to $\bar{x}$. In view of (3), we have that necessarily the fixed point $z \in [0, \bar{x}] \setminus \partial^\diamond [0, \bar{x}]$. Therefore, condition (iv) holds. ■

**Example 25** This example, an elaboration of [8, Example 3.10], shows inter alia that the condition on the upper perimeter is needed. Let $B = B([-1, 1])$ and $F : B \to B$ be given by

$$F(f) = i^+ f - i$$

where $i \in B$ is the identity function $i(t) = t$. In view of (19), the associated fixed point problem is

$$T_\lambda f = \frac{1}{\lambda} \left[\lambda f - i^+ f + i\right]^+.$$

Since $i^+ f \leq f$, the linear operator $f \longmapsto i^+ f$ is $\lambda$-weakly order-Lipschitz. Specifically, $T_\lambda$ is monotone for any $\lambda \geq 1$. Thus, set $\lambda = 1$. Namely, consider the monotone and convex operator $T : B_+ \to B_+$ given by

$$T(f) = \left[f - i^+ f + i\right]^+.$$

[19] See [29], who study the close relation with the notion of $Z$-map in [34] and of $\lambda I$ map in [8].

It is easy to check that $T$ maps the interval $[\mathbf{0}, \mathbf{L}]$ into itself, where $\mathbf{L}$ is the constant functions $\mathbf{L}(t) \equiv L \geq 1$. Moreover, $T(\mathbf{L}) < \mathbf{L}$. Nevertheless, the uniqueness part of Proposition 24 fails. Actually it is easy to see that $T$ has the continuum of fixed points

$$f_k(t) = \begin{cases} 0 & \text{if } t \in [-1, 0) \\ k & \text{if } t = 0 \\ 1 & \text{if } t \in (0, 1] \end{cases}$$

for each $k \geq 0$. Consequently, the fixed point $f_L$ belongs to $\partial^{\diamond}[\mathbf{0}, \mathbf{L}]$ for every interval $[\mathbf{0}, \mathbf{L}]$. ▲

Observe that to get the uniqueness of the fixed point in the last example, it is sufficient to consider the space $L^{\infty}([-1, 1], dx)$. Clearly, $\operatorname{essup}[f_k] = 1$ and so $[f_k] \notin \partial^{\diamond}[\mathbf{0}, \mathbf{L}]$ for $L > 1$.

**Example 26** A slight modification of the above example, in which we set

$$F(f) = i^{+}f - i + \varepsilon \tag{21}$$

with $\varepsilon > 0$, leads to uniquely solvable operators. Actually, if we consider the constant functions $\mathbf{L}(t) \equiv L \geq 1$, we have $F(\mathbf{L}) \geq \sigma \mathbf{L}$ for $0 < \sigma \leq \varepsilon / L$. Therefore, the complementary problem has a unique solution in $B([-1, 1])$ thanks to Proposition 24. ▲

The last example leads to some interesting considerations on how uniqueness may imply continuity. The fixed points of the maps

$$T_{\varepsilon}(f) = \left[f - i^{+}f + i - \varepsilon\right]^{+}$$

associated with (21), for $\lambda = 1$, are indeed continuous functions, i.e., $f_{\varepsilon} \in C([-1, 1])$. A direct check is hard because we cannot replace the space $B[-1, 1]$ with the incomplete vector lattice $C([-1, 1])$. However, as $T_{\varepsilon}^{n}(\mathbf{L}) \downarrow f_{\varepsilon}$, the function $f_{\varepsilon}$ is upper semicontinuous. By the uniqueness of the fixed point, $T_{\varepsilon}^{n}(\mathbf{0}) \uparrow f_{\varepsilon}$ holds as well, and so $f_{\varepsilon}$ is also lower semicontinuous.[20]

As well known, the complementary problem is closely related with the solvability of variational inequalities. Specifically, let $C$ be a nonempty closed and convex subset of an *Hilbert lattice* $H$ and let $F : C \to H$. The *variational problem* associated with the pair $(C, F)$ is to find an $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \qquad \forall x \in C. \tag{22}$$

The variational property of the *metric projection* $\pi_C : H \to C$ entails the equivalence of (22) with the fixed point problem

$$x^* = \pi_C(x^* - \lambda F(x^*))$$

where $\lambda > 0$ is a given parameter.

An order-theoretic approach to the variational problem (22) is studied by [21] and [29]. The solvability of (22) in such approach is based on the fact that $\pi_C$ is order-preserving if and only if $C$ is a sublattice of $H$ (see [29, Lemma 2.4]). Here, without any pretense to be exhaustive, we establish a uniqueness result in order to further illustrate our approach.

**Proposition 27** *Let $F : C = \{x \leq b\} \to H$ be $\lambda$-weakly order-Lipschitz and order convex. If*

*(i) there is $a < b$ such that $F(a) \leq 0$,*

*(ii) $x \neq \pi_C(x - \lambda F(x))$ for all $x \in \partial_{\diamond}[a, b]$,*

*then there is a unique vector $x^* \in [a, b]$ that solves the variational problem (22).*

[20] Note, in passing, that by Dini's Theorem the two sequences of continuous functions $T_{\varepsilon}^{n}(\mathbf{L})$ and $T_{\varepsilon}^{n}(\mathbf{0})$ approach the continuous function $f_{\varepsilon}$ uniformly.

**Proof** Define $\Phi_\lambda : C \to C$ by $\Phi_\lambda(x) = \pi_C(x - \lambda F(x))$. Then

$$\Phi_\lambda(x) = [x - \lambda F(x)] \wedge b = b - [\lambda F(x) + b - x]^+ . \tag{23}$$

The vector $x^*$ solves (22) if and only if $x^* = \Phi_\lambda(x^*)$. Consider now the interval $[a, b]$. Clearly, $\Phi_\lambda(b) \leq b$. Further, $a \leq a - \lambda F(a)$ that, in turn, implies $a = \pi_C(a) \leq \pi_C(a - \lambda F(a)) = \Phi_\lambda(a)$. Hence, $\Phi_\lambda([a, b]) \subseteq [a, b]$. By the, by now, usual arguments based on Theorem 10 we get the desired result because the self-map $\Phi_\lambda$ is monotone and, in view of (23), order concave. ■

The concavity or convexity of the projections is a key condition to obtain unique solutions to the variational inequality.[21] Unfortunately, this condition can be rather demanding.[22] So, we end with a positive result on cones. Here, $C^\circ$ is the *polar cone* of a cone $C$.

**Proposition 28** *Let $C \subseteq H$ be a closed and convex cone. The projection $\pi_C$ is convex (resp., concave) if either $C$ is a lattice and $C \supseteq K$ (resp., $C \supseteq -K$) or $C^\circ$ is a lattice and $C \subseteq K$ (resp., $C \subseteq -K$).*

**Proof** Assume that $C$ is a lattice with $C \supseteq K$. Let $x, y \in H$. The variational property of the metric projection for cones implies $x - \pi_C(x) \in C^\circ$ and $y - \pi_C(y) \in C^\circ$. Therefore, $x + y - \pi_C(x) - \pi_C(y) \in C^\circ \subseteq -K$. Hence, $x + y \leq \pi_C(x) + \pi_C(y)$. As $C$ is a lattice, $\pi_C$ is monotone. Consequently,

$$\pi_C(x + y) \leq \pi_C(\pi_C(x) + \pi_C(y)) = \pi_C(x) + \pi_C(y)$$

We conclude that $\pi_C$ is subadditive. As $\pi_C$ is positively homogeneous (see, e.g., [12, Proposition 5.6]), $\pi_C$ is convex on $H$. The concavity of $\pi_C$, provided the lattice $C \supseteq -K$ is proved similarly.

By the Moreau decomposition (see [12, Poposition 5.6]), for each $x \in H$ we have $x = \pi_C(x) + \pi_{C^\circ}(x)$. For instance, if $C^\circ$ be a lattice such that $C \subseteq K$, then $C^\circ \supseteq -K$. By what has been already proved, $\pi_{C^\circ}$ is concave. Hence, $\pi_C = I - \pi_{C^\circ}$ is convex. Similar argument holds when $C \subseteq -K$. ■

## 6.5 Operator equations

Define $T : \mathcal{L}_s^+(H) \to \mathcal{L}_s^+(H)$ by

$$T(X) = \sum_{k=1}^{m} \varphi_k((Xh_k, h_k))(X + A_k)^{\vartheta_k} \tag{24}$$

where $h_1, h_2, ..., h_m \in H,\ A_1, ..., A_m \in \mathcal{L}_s^+(H),\ \varphi_1, \varphi_2, ..., \varphi_m : \mathbb{R}_{++} \to \mathbb{R}_+$, and $\vartheta_k \in (0, 1)$. The resolution of the existence of fixed points of these operators is closely related to what studied in [16].

It is well known that the operator functions $X \longmapsto (X + A_k)^{\vartheta_k}$ are monotone and concave in $\mathcal{L}_s^+(H)$. Moreover, for $\lambda > 0$ large enough, $T$ maps the interval $[0, \lambda I]$ into itself (see [16] for more details). By the chain completeness of $[0, \lambda I]$, it follows the existence of fixed points in $[0, \lambda I]$.

Here, we are interested in the uniqueness of the fixed point. It is a consequence of Theorem 19 by taking the identity operator $I \in \mathcal{L}_s^+(H)$ as order unit. Formally:

**Proposition 29** *The operator $T : \mathcal{L}_s^+(H) \to \mathcal{L}_s^+(H)$ has a unique fixed point provided:*

*(i)* $A_{\bar{k}} \geq \sigma I$ *for some* $\bar{k}$ *and* $\sigma > 0$;

*(ii)* $\varphi_k$ *are increasing, concave and bounded and* $\varphi_k(0) > 0$ *for all* $k$;

*(iii)* $\vartheta_k < \varphi_k(0) / \varphi_k(\infty)$ *for all* $k$.

[21] The classical strict monotonicity condition $\langle x - y, F(x) - F(y) \rangle > 0$ for all $x, y \in C$ also guarantees a unique solution, but it is not related to order arguments.

[22] For instance, it easy to see that the projections on intervals $[a, b]$ are neither convex nor concave.

**Proof** Let us first show that $T$ is strongly subhomogeneous. Observe that by the same method employed to prove Proposition 15-(i) implies that the scalar functions $\varphi_k$ are strongly subhomogeneous, more specifically,

$$\varphi_k(\alpha t) \geq \alpha^{1-\frac{\varphi_k(0)}{\varphi_k(\infty)}} \varphi_k(t) \qquad \forall t \geq 0, \forall \alpha \in [0,1].$$

By setting $t = (Xh_k, h_k) \geq 0$, we have

$$\varphi_k((\alpha X h_k, h_k)) \geq \alpha^{1-\frac{\varphi_k(0)}{\varphi_k(\infty)}} \varphi_k((X h_k, h_k)).$$

Moreover, by monotonicity,

$$(\alpha X + A_k)^{\vartheta_k} \geq \alpha^{\vartheta_k} (X + A_k)^{\vartheta_k}.$$

Consequently,

$$\varphi_k((\alpha X h_k, h_k)) (\alpha X + A_k)^{\vartheta_k} \geq \alpha^{1-\eta_k} \varphi_k((X h_k, h_k)) (X + A_k)^{\vartheta_k}$$

holds where $\eta_k = \vartheta_k - \varphi_k(0) / \varphi_k(\infty)$ and, at last,

$$T(\alpha X) \geq \alpha^{1-\min \eta_k} T(X).$$

Under (iii), $\min \eta_k > 0$ and so $T$ is strongly subhomogeneous.

It remains to show that point (i) of Theorem 19 is fulfilled, that is, that no fixed point lies in $\partial_\diamond \mathcal{L}_s^+(H)$. In view of condition (i),

$$\sum_{k=1}^{m} \varphi_k((X h_k, h_k)) (X + A_k)^{\vartheta_k} \geq \varphi_{\bar{k}}(0) (X + A_{\bar{k}})^{\vartheta_{\bar{k}}} \geq \varphi_{\bar{k}}(0) \sigma^{\vartheta_{\bar{k}}} I.$$

Hence $T([0, \lambda I]) \subseteq [\sigma_1 I, \lambda I]$ where $\sigma_1 = \varphi_{\bar{k}}(0) \sigma^{\vartheta_{\bar{k}}}$. It follows that $Fix(T) \subseteq [\sigma_1 I, \lambda I]$. By Proposition 8, $Fix(T) \subseteq [0, \lambda I] \setminus \partial_\diamond [0, \lambda I]$. This concludes the proof. ■

We end by studying an extension of the discrete algebraic Riccati equation associated with the operator $R : \mathcal{L}_s^+(H) \to \mathcal{L}_s^+(H)$ defined by

$$R(X) = Y + \sum_{k=1}^{m} A_k^* \Phi(X) A_k \tag{25}$$

where $Y \in \mathcal{L}_s^+(H)$, $A_k \in \mathcal{L}(H)$ and $\Phi$ is a positive Loewner function defined on $\mathbb{R}_+$.

Fixed points of (25) have been studied – when $\dim H < \infty$ – in [33] via a Banach-like contraction result for posets. We provide here an alternative method based on our approach.

**Proposition 30** *The operator $R : \mathcal{L}_s^+(H) \to \mathcal{L}_s^+(H)$ has a unique fixed point $X^*$ in $\mathcal{L}_s^+(H)$ provided:*

*(i) $Y \geq \sigma I$ for some $\sigma > 0$,*

*(ii) $R(\bar{X}) \leq \bar{X}$ for some $\bar{X} \in \mathcal{L}_s^+(H)$.*

*Moreover, in this case the iterates $R^n(\mathbf{0})$ norm converge to the fixed point $X^*$.*

**Proof** Clearly, $R$ is monotone. Moreover, it is well known that the monotonicity of $X \longmapsto \Phi(X)$ implies that it is also concave, because $\Phi$ is positive. This implies, in turn, that $R$ is concave. As $R(0) \geq Y \geq \sigma I$, the operator $R$ is also subconcave at 0. Moreover, we have

$$\sigma I \leq Y \leq R(0) \leq R(\bar{X}) \leq \bar{X} \leq \|\bar{X}\| I$$

Hence $R(0)$ is linked to $\bar{X}$. By Proposition 15-(ii), the operator $R$ is strongly subhomogeneous in $\mathcal{L}_s^+(H)$. Theorem 19 implies the existence and the uniqueness of the fixed point in $\mathcal{L}_s^+(H)$, because there are no fixed point in $\partial_\diamond \mathcal{L}_s^+(H)$.

Note that the cone $\mathcal{L}_s^+(H)$ is normal, therefore the convergence in norm is a consequence of Proposition 36. ■

# 7 Related literature

The starting point of our analysis was a special case of our Theorem 10 proved in Baiocchi and Capelo [5] p. 224. The results that we proved here are more general, partly because – by leveraging on the notion of lower perimeter that we introduced – they are able to best exploit the interplay between order and vector structures. Earlier results on unique fixed points of concave and monotone self-maps can also be found in Amann [2] and [3]. They are, however, different from ours. For instance, also [2, p. 372] proves a version of Theorem 10 (see also [3, Theorem 24.4]). However, not relying upon the results of Kantorovich and Tarski,[23] it uses order notions of monotonicity and concavity on a hybrid structure (ordered topological vector spaces with weak units) that are stronger than the standard ones. Our analysis takes, in contrast, advantage of the Tarski-type theorems that enable us to use standard notions of order concavity and monotonicity through the notion of lower perimeter.

The results on subhomogeneous operators offer often a powerful alternative to those related to the order concavity. The results presented here are inspired to Krasnoselskii's seminal work, though the key connection with the Thompson metric that we develop in the Appendix is new.

Similar topological results can be found in [3]. More recently, the uniqueness part of the fixed point theorem established by [20] is closely related to Proposition 35, though adapted to spaces of functions (their existence result rests on Kantorovich's Theorem). We must also mention that many authors used related similar arguments to establish uniqueness results for equilibria of dynamic economies determined as fixed points (see [9] and [10]).

Finally, our analysis does not rely on any a priori given metric structure, so it is different from the recent fixed point literature that combines order and metric structures (see, e.g., [33], [15], and [28]). It is, instead, closely related to the papers that - like [20] and [7] – study the uniqueness of solutions of Bellman equations, as it was detailed in the paper.

# A Proofs of Section 5.2 and related analysis

## A.1 A key connection

There is a close connection between the subhomogeneity property introduced in Section 5 and the Thompson distance $d$, as well as with the logarithmic transformation of an operator. Next we explicit some results along this line. Throughout this section, $V$ denotes an Archimedean ordered vector space and $Q \in K/\sim$ is a component of $K$.

**Proposition 31** *Let $T : Q \to K$ be monotone.*

*(i) $T$ is subhomogeneous if and only if it is not expansive, i.e.,*

$$d\left(T\left(x\right), T\left(y\right)\right) \leq d(x, y) \qquad \forall x, y \in Q.$$

*(ii) $T$ is strongly subhomogeneous if and only if*

$$d\left(T\left(x\right), T\left(y\right)\right) < d(x, y) \qquad \forall x \neq y \in Q. \tag{26}$$

*(iii) $T$ is $p$-subhomogeneous if and only if it is a $p$-contraction, i.e.,*

$$d\left(T\left(x\right), T\left(y\right)\right) \leq pd(x, y) \qquad \forall x, y \in Q. \tag{27}$$

*Moreover, thanks to subhomogeneity, we have $T\left(Q\right) \subseteq Q$ if and only if $T\left(x_0\right) \in Q$ for some element $x_0 \in Q$.*

[23] These authors are not mentioned in Amann [3]. Tarski's Theorem is discussed in a later paper, [4], whose results are, however, not related to ours (it proves, inter alia, fixed point results a la Abian and Brown [1]).

**Proof** (iii) Let $T$ be $p$-subhomogeneous. By definition (2) and the fact that the infimum is a minimum, being $V$ Archimedean, then $e^{-d(x,y)}x \leq y \leq e^{d(x,y)}x$, for $x, y \in Q$. It follows that

$$e^{-pd(x,y)}T(x) \leq T(y) \leq e^{pd(x,y)}T(x).$$

In turn, this gives $d(T(x), T(y)) \leq pd(x, y)$. Conversely, assume that (27) holds. In particular, $d(T(x), T(\alpha x)) \leq pd(x, \alpha x)$ holds for $x \in K$ and $\alpha \in (0,1)$. Since $d(x, \alpha x) = -\log \alpha$, we get $d(T(x), T(\alpha x)) \leq -p \log \alpha$. Moreover,

$$T(x) \leq e^{d(T(x),T(\alpha x))}T(ax) \tag{28}$$

which provides $T(x) \leq e^{-\log \alpha^p}T(\alpha x) = \alpha^{-p}T(\alpha x)$, as desired.

(ii) Relation (28) can be interpreted as $T(\alpha x) \geq \varphi(x, \alpha) T(x)$, where $\varphi(x, \alpha) = e^{-d(T(x),T(\alpha x))}$. Under condition (26), we have $\varphi(x, \alpha) > \alpha$, whenever $\alpha \in (0,1)$. This proves (ii). The proof of (i) is similar. ■

A particularly elegant case is when $V$ is the space of bounded functions $B(X)$ endowed with the supnorm, and the component $Q$ is that containing a unit vector $e$, that is, $Q(e) = \text{int}B_+(X) = B_+(X) \setminus \partial_\diamond B_+(X)$.

**Proposition 32** *Let $B_+(X)$ be the cone of positive functions in $B(X)$. The Thompson metric $d$ on* $\text{int}\, B_+(X)$ *is*

$$d(f, g) = \sup_{x \in X} |\log f(x) - \log g(x)|$$

*Moreover, the map $\mathcal{L} : \text{int}\, B_+(X) \to B(X)$ defined by $\mathcal{L}(f)(x) = \log f(x)$ is an isometry of $(\text{int}\, B_+(X), d)$ onto $(B(X), \|\cdot\|)$.*

**Proof** Clearly, for two functions $f, g \in \text{int}\, B_+(X)$, we have

$$e^{-\lambda}g \leq f \leq e^{\lambda}g \Longleftrightarrow |\log f - \log g| \leq \lambda$$

which provides the desired result. Moreover, observe that the transformation $\mathcal{L} : f \longmapsto \log f$ is a bijection with inverse $\mathcal{L}^{-1} : f \longmapsto e^f$. Therefore,

$$d(f, g) = \|\log f - \log g\| = \|\mathcal{L}(f) - \mathcal{L}(g)\|$$

and so $\mathcal{L}$ is an isometry. ■

The logarithmic transformation is useful to solve the fixed problem $f = T(f)$ for operators $T : \text{int}\, B_+(X) \to \text{int}\, B_+(X)$.[24] If $\mathcal{L} : \text{int}B_+(X) \to B(X)$ denotes the log transformation $f \longmapsto \log f$, the conjugate operator is $\tilde{T} = \mathcal{L} \circ T \circ \mathcal{L}^{-1} : B(X) \to B(X)$. That is, $\tilde{T}(f) = \log T(e^f)$. Clearly, $f^*$ is a fixed point of $T$ if and only if $\mathcal{L}(f^*)$ is a fixed point of $\tilde{T}$.

The following corollaries are straightforward applications of Propositions 31 and 32.

**Corollary 33** *A monotone $T : \text{int}\, B_+(X) \to \text{int}\, B_+(X)$ is*

*(i) $p$-subhomogeneous if and only if $\tilde{T} = \mathcal{L} \circ T \circ \mathcal{L}^{-1}$ is a $p$-contraction on $(B(X), \|\cdot\|)$;*

*(ii) strongly subhomogeneous if and only if $\left\|\tilde{T}f - \tilde{T}g\right\| < \|f - g\|$ for all $f \neq g \in B(X)$.*

When $X$ is a singleton, $B(X) = \mathbb{R}$, and $\text{int}B_+(X) = (0, +\infty)$. The previous result has then the following useful consequence.

[24] The logarithmic transformation is often used in fixed points problems. See for instance [35, Sect. 17.2] and [20].

**Corollary 34** *A monotone and differentiable $f : (0, +\infty) \to (0, +\infty)$ is:*

*(i) $p$-subhomogeneous if $f'(x)\, x/f(x) \leq p < 1$ for all $x > 0$;*

*(ii) strongly subhomogeneous if $f'(x)\, x/f(x) < 1$ for all $x > 0$.*

By setting $\tilde{f}(t) = \log f(e^t)$, the derivative is $D\tilde{f}(t) = f'(e^t)\, e^t/f(e^t) = f'(x)\, x/f(x)$. So, $\left|D\tilde{f}(t)\right| \leq p$ implies that $\tilde{f}$ is a contraction.

An operator $T : K \to K$ is said to be $Q$-*monotone* on a given component $Q$ if

$$x < y \Longrightarrow T(x) + v(x, y) \leq T(y) \qquad \forall x, y \in Q$$

for some $v(x, y) \in Q$. The next proposition, which is closely related to [19, Theorems 6.3 and 6.4], deals with uniqueness (though not existence) of fixed points.[25]

**Proposition 35** *Let $T : K \to K$ be monotone and strongly subhomogeneous.*

*(i) There is at most a unique fixed point $\bar{x}$ on every component $Q$ of $K$; we have $T(y) < y$ for all $\bar{x} < y \in Q$.*

*(ii) The same result holds on every component $Q$ on which $T$ is strictly subhomogeneous and $Q$-monotone.*

**Proof** Recall that the Archimedean property of $V$ ensures the well known fact that the interval $\{\lambda \in \mathbb{R} : \lambda x \leq y\}$ has a greatest element for arbitrarily fixed vectors $x, y \in V$, as long as it is not empty.

(i) Let $T(x_1) = x_1$, $T(x_2) = x_2$ and $x_1 \neq x_2 \in Q$ where $Q \subseteq K$ is a component. It is not restrictive to assume $x_1 \nleq x_2$ (otherwise, replace $x_1$ by $x_2$). Since $x_1$ and $x_2$ are comparable, there is an $\alpha > 0$ such that $\alpha x_1 \leq x_2$. Pick the greatest $\bar{\alpha}$ for which $\alpha x_1 \leq x_2$ holds. Observe that $\bar{\alpha} < 1$, otherwise, with $\bar{\alpha} \geq 1$, we would have $x_1 \leq \bar{\alpha} x_1 \leq x_2$ which is a contradiction. Hence, $\bar{\alpha} x_1 \leq x_2$ with $0 < \bar{\alpha} < 1$. Then,

$$x_2 = T(x_2) \geq T(\bar{\alpha} x_1) \geq \varphi(\bar{\alpha}, x_1)\, T(x_1) = \varphi(\bar{\alpha}, x_1)\, x_1. \tag{29}$$

As $\varphi(\bar{\alpha}, x_1) > \bar{\alpha}$, this contradicts the hypothesis made on $\bar{\alpha}$. Hence $x_1 = x_2$.

Regarding the second point, observe that the chain of inequalities holds true by setting $x_2 = \bar{x}$ and $y = x_1$ and where $T(\bar{x}) = \bar{x}$ and $T(y) > y$. Therefore the three relations $T(\bar{x}) = \bar{x}$, $T(y) > y$ and $y \nleq \bar{x}$ must be inconsistent. Hence our claim follows.

(ii) As in point (i), assume the existence of the two fixed points $x_1 \neq x_2 \in Q$, with $x_1 \nleq x_2$. Let $\bar{\alpha} x_1 \leq x_2$, where $\bar{\alpha}$ enjoys the same property like in (i). If $\bar{\alpha} x_1 = x_2$, then

$$x_2 = T(x_2) = T(\bar{\alpha} x_1) > \bar{\alpha} x_1.$$

Hence, $\bar{\alpha} x_1 < x_2$. By $Q$-monotonicity,

$$\bar{\alpha} x_1 \leq T(\bar{\alpha} x_1) \leq T(x_2) - u = x_2 - u,$$

with $u \in Q$. Namely, $x_2 \geq \bar{\alpha} x_1 + u$. Since $u, x_1 \in Q$, we have $u \geq \lambda x_1$ for some $\lambda > 0$. It follows that $x_2 \geq (\bar{\alpha} + \lambda)\, x_1$, a contradiction. ■

Having dealt with uniqueness, next we consider global attractiveness.[26]

[25] Point (i) could be directly deduced from point (ii) of Proposition 31 because (26) implies the uniqueness of the fixed point of the operator.

[26] In the special case in which the operator is $p$-subhomogeneous, the result easily follows from the contraction property established in Proposition 31-(iii).

**Proposition 36** *Let $T : K \to K$ be monotone and strongly subhomogeneous. Let $\bar{x} \in Q$ be a fixed point of $T$. Then $T(Q) \subseteq Q$ and, given any initial condition $x_0 \in Q$, the iterates $T^n(x_0)$ order converge to $\bar{x}$, with*

$$d\left(T^n(x_0), \bar{x}\right) \to 0$$

*where $d$ is the Thompson metric defined on $Q$.*

Recall that Thompson convergence implies norm convergence when the space $V$ is endowed with a norm and the cone $K$ is normal (Theorem 1).

**Proof** We begin with the following claim.

**Claim** There exists $\alpha < \varphi(x, \alpha) < 1$, for $\alpha \in (0,1)$ and $x > 0$, such that (1) holds and $\varphi(x, \cdot)$ is continuous and monotone on $[0,1]$. Specifically, we can take

$$\varphi(x, \alpha) = e^{-d(T(x), T(\alpha x))} = \max\{\beta > 0 : T(\alpha x) \geq \beta T(x)\}. \tag{30}$$

**Proof of the Claim** The relation $\varphi(x, \alpha) = e^{-d(T(x), T(\alpha x))}$ has been already observed in the proof of Proposition 31. The last equality is easily obtained. Also the monotonicity property of $\varphi(x, \cdot)$ is easy. Let us show the continuity of $\varphi(x, \cdot)$. Let $(\alpha_n)$ be an increasing sequence in $(0,1)$ such that $\alpha_n \uparrow \alpha^*$. Then

$$T(\alpha_n x) = T\left(\frac{\alpha_n}{\alpha^*}\alpha^* x\right) \geq \frac{\alpha_n}{\alpha^*} T(\alpha^* x) \geq \frac{\alpha_n}{\alpha^*}\varphi(x, \alpha^*) T(x).$$

In view of (30), this implies

$$\varphi(x, \alpha_n) \geq \frac{\alpha_n}{\alpha^*}\varphi(x, \alpha^*)$$

By taking limit, this leads to $\lim_n \varphi(x, \alpha_n) \geq \varphi(x, \alpha^*)$. As $\varphi(x, \cdot)$ is increasing, we have also $\lim_n \varphi(x, \alpha_n) \leq \varphi(x, \alpha^*)$. The right limits are proved in the same way. □

We can now prove our proposition. The first statement is easy. Let now $x_0 \in Q(\bar{x})$. This implies that there is some $t_0 \in (0,1)$ for which

$$t_0 \bar{x} \leq x_0 \leq \frac{1}{t_0}\bar{x}. \tag{31}$$

As $T$ is strongly subhomogeneous, then there exists a continuous function $\varphi$ for which $T(t\bar{x}) \geq \varphi(t) T(\bar{x})$ holds for each $t \in [0,1]$ and $0 < \varphi(t) < 1$ if $t \in (0,1)$. By (31) we get

$$\varphi(t_0)\bar{x} \leq T(x_0) \leq \frac{1}{\varphi(t_0)}\bar{x}$$

and, iterating this procedure, we get the relation

$$t_n \bar{x} \leq T^n(x_0) \leq \frac{1}{t_n}\bar{x}$$

where $t_n = \varphi(t_{n-1})$. Thanks to the continuity of $\varphi$, the increasing trajectory $(t_n)$ must approach a fixed point of $\varphi$. Hence, $t_n \uparrow 1$. By definition of Thompson metric (2), it follows that

$$d\left(T^n(x_0), \bar{x}\right) \leq -\log t_n \longrightarrow 0.$$

On other hand,

$$-(1 - t_n)\bar{x} \leq T^n(x_0) - \bar{x} \leq \left(\frac{1}{t_n} - 1\right)\bar{x}$$

where $(1 - t_n) \downarrow 0$ and $1/t_n - 1 \downarrow 0$. Therefore, $T^n(x_0)$ order converges to $\bar{x}$. ■

## A.2 Proofs of Section 5.2

**Theorem 18** By hypothesis, the fixed points must lie in $[0,a] \backslash \partial_\diamond [0,a]$, which agrees with $Q(a)$, thanks to Proposition 3. By Proposition 35, the fixed point is unique whenever it exists. The rest of the proof is now obvious. ■

Inspection of this proof shows that the condition $T(x) \neq x$ for all $x \in \partial_\diamond [0,a]$ is enough to establish that $T$ has at most one fixed point.

**Theorem 19** If (i) holds, then the fixed points will be located in $K \backslash \partial_\diamond K$. By Proposition 4, $K \backslash \partial_\diamond K = Q(e)$. Therefore, Proposition 35 provides the first desired result. As to the last claim, under (ii) we have that $T$ maps monotonically $[0, \lambda e]$ into itself and thus (iii) implies the existence of fixed points. ■

Inspection of this proof shows that under (i) $T$ has at most one fixed point.

# References


[1] S. Abian and A. B. Brown, A theorem on partially ordered sets with applications to fixed point theorems, *Canadian Journal of Mathematics*, 13, 78-83, 1961.

[2] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, *Journal of Functional Analysis*, 11, 346-384, 1972.

[3] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Review*, 18, 620-709, 1976.

[4] H. Amann, Order structures and fixed points, *Seminario di Analisi Funzionale e Applicazioni*, Cosenza, 1977.

[5] C. Baiocchi and A. C. Capelo, *Variational and quasivariational inequalities: Applications to free boundary problems*, Wiley, New York, 1984.

[6] D. P. Bertsekas and S. E. Shreve, *Stochastic optimal control: the discrete-time case*, Athena Scientific, Belmont, 1978.

[7] G. Bloise and Y. Vailakis, Convex dynamic programming with (bounded) recursive utilities, mimeo, 2016.

[8] J. M. Borwein and A. H. Dempster, The linear order complementary problem, *Mathematics of Operations Research*, 14, 534-558, 1989.

[9] W. J. Coleman, Equilibrium in a production economy with an income tax, *Econometrica*, 59, 1091-1104, 1991.

[10] M. Datta, L. J. Mirman, O. F. Morand and K. L. Reffett, Monotone methods for Markovian equilibria in dynamic economies, *Annals of Operations Research*, 114, 117-144, 2002.

[11] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, Cambridge University Press, Cambridge, 2002.

[12] F. Deutsch, *Best approximation in inner product spaces*, Springer-Verlag, New York, 2001.

[13] E. Dekel, An axiomatic characterization of preferences under uncertainty: weakening the independence axiom, *Journal of Economic Theory*, 40, 304-318, 1986.

[14] T. Fujimoto, An extension of Tarski's fixed point theorem and its application to isotone complementary problems, *Mathematical Programming,* 28, 116-118, 1984.

[15] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis*, 65, 1379-1393, 2006.

[16] G. Herzog and C. Schmoeger, Suprema of chains of operators, *Positivity,* 15, 343-349*,* 2011.

[17] L. Kantorovich, The method of successive approximations for functional equations, *Acta Mathematica*, 71, 63-97, 1939.

[18] T. C. Koopmans, Stationary ordinal utility and impatience, *Econometrica*, 28, 287-309, 1960.

[19] M. A. Krasnoselskii, *Positive solutions of operator equations,* Noordhoff, Groningen, 1964.

[20] C. Le Van, L. Morhaim and Y. Vailakis, Monotone concave operators: an application to the existence and uniqueness of solutions to the Bellman equation, 2008, <hal-00294828>.

[21] J. Li and E. A. Ok, Optimal solutions of variational inequalities on Banach lattices, *Journal of Mathematical Analysis and Applications*, 388, 1157-1165, 2012.

[22] Z. D. Liang, W. X. Wang and S. J. Li, On concave operators, *Acta Mathematica Sinica,* 22, 577-582, 2006.

[23] R. E. Lucas and N. L. Stokey, Optimal growth with many consumers, *Journal of Economic Theory*, 32, 139-171, 1984.

[24] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces I*, North-Holland, Amsterdam, 1971.

[25] M. Marinacci and L. Montrucchio, Ultramodular functions*, Mathematics of Operations Research*, 30, 311-332, 2005.

[26] M. Marinacci and L. Montrucchio, Unique solutions for stochastic recursive utilities*, Journal of Economic Theory*, 145, 1776-1804, 2010.

[27] G. Markowsky, Chain-complete posets and directed sets with applications, *Algebra Universalis*, 6, 53-68, 1976.

[28] J. Nieto and R. Rodríguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Mathematica Sinica*, 23, 2205-2212, 2007.

[29] H. Nishimura and E. A. Ok, Solvability of variational inequalities on Hilbert lattices, *Mathematics of Operations Research*, 37, 608-625, 2012.

[30] R. D. Nussbaum, Hilbert's projective metric and iterated nonlinear maps, *Memoirs of the American Mathematical Society*, 391, Providence, 1988.

[31] E. A. Ok, Order theory and its applications, mimeo.

[32] M. L. Puterman and S. L. Brumelle, On the convergence of policy iteration in stationary dynamic programming, *Mathematics of Operations Research*, 4, 60-69, 1979.

[33] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proceedings of the American Mathematical Society*, 1435-1443, 2004.

[34] R. Riddell, Equivalence of nonlinear complementary problems and least element problems in Banach lattices, *Mathematics of Operations Research*, 6, 462-474, 1981.

[35] N. L. Stokey and R. E. Lucas, *Recursive Methods in Economic Dynamics,* Harvard University Press, Cambridge, 1989.

[36] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, *Pacific Journal of Mathematics*, 5, 285-309, 1955.

[37] A. C. Thompson, On certain contraction mappings in a partially ordered vector space, *Proceeding of American Mathematical Society,* 14, 438-443, 1963.

[38] D. M. Topkis, *Supermodularity and complementarity*, Princeton University Press, Princeton, 1998.