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Epistemic Foundations for Set-algebraic Representations of Knowledge*

Satoshi Fukuda†

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Abstract

This paper formalizes an informal idea that an agent's knowledge is characterized by a collection of sets such as a σ-algebra within the framework of a state space model of knowledge. The formalization is based on the agent's logical and introspective abilities and on the underlying structure of the state space. The agent is logical and introspective about what she knows if and only if her knowledge is summarized by a collection of events with the property that, for any event, the collection has the maximal event included in the original event. When the underlying space is a measurable space, the collection becomes a σ-algebra if and only if the agent is additionally introspective about what she does not know. The paper characterizes why the agent's knowledge takes (or does not take) such a set algebra as a σ-algebra or a topology, depending on the agent's logical and introspective abilities and on the underlying environment.

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1 Introduction

Economic agents base their decisions on their knowledge about uncertainty that they face. Where does their knowledge come from? Researchers often represent an agent's knowledge by a (sub-)σ-algebra on an underlying state space. For each element E of such σ-algebra, the agent is supposed to "know" whether the set E obtains at a realized state ω (i.e., ω lies in E) in an informal sense. This informal understanding of

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the σ-algebra as representing the agent's knowledge helps researchers to connect various measure- and probability-theoretical formal apparatus and their informal ideas behind economic problems at hand.

The main objective of this paper is to provide epistemic foundations for representations of an agent's knowledge by a collection of sets in terms of the agent's logical and introspective reasoning abilities and of the structure of an underlying state space. The framework of the paper is a state space model of knowledge. Conceptually, this paper fully characterizes the hidden assumptions on an agent's logical and introspective reasoning abilities when researchers use a particular form of a set algebra to represent the agent's knowledge. This paper also clarifies the formal sense in which a given collection of subsets of the underlying state space has an informational content. The paper also studies how the collection of sets that represents the agent's knowledge is generated from information sets. An information set associated with a given state is the set of states that the agent considers possible at that state, even though she cannot identify the realized state.

To fix an idea, consider an agent named Ashley. The underlying structure of uncertainty is represented as a pair of a set of states of the world and a collection of subsets of the state space (i.e., events) about which she reasons. Each event describes a certain aspect of the states of the world. With the underlying state space in mind, the knowledge of Ashley is given by her knowledge operator that maps each event E to the event that she knows E. While the specification of Ashely's knowledge by her knowledge operator is the most general way to describe her knowledge within a state space model, I also consider the case in which her knowledge is induced from her information sets. Ashley knows an event E at a state if her information set at that state implies (i.e., is included in) the event E.

So far, the analysts have the space of uncertainty (underlying states and events) and the list describing what Ashley knows about each event. If it is at all possible, under what conditions on the properties of Ashley's knowledge, can the analysts represent her knowledge by a set algebra such as a σ-algebra or a topology? The main result of the paper shows that one can represent Ashley's knowledge by a set algebra if her knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection. Truth Axiom means that Ashley can only know what is true (i.e., if she knows an event at a state, then the event is true at that state). Thus, by the representation of Ashley's knowledge, it means that one deals with her knowledge instead of her beliefs, which can be false. Monotonicity means that Ashley is a logical reasoner in that if she knows a certain event E and if E implies an event F, then she knows F as well. Positive Introspection means that if Ashley knows an event E then she knows that she knows E. Thus, Ashley is logical and introspective about her own knowledge. Roughly, the main result is stated as follows. If Ashley's knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection, then the collection of events E which she knows whenever E obtains (i.e., E is self-evident) turns out to be the collection of events such that, for any event E, the collection contains the maximal event included in E. Conversely, let $\mathcal{J}$ be the collection of events with this "maximality property." Then, one can construct a knowledge operator of Ashley which satisfies Truth Axiom, Monotonicity, and Positive Introspection. The knowledge operator satisfies the property that Ashley knows an event E at a state ω if and only if there is an event $F \in \mathcal{J}$ which is true at ω and which implies E (i.e., $\omega \in F \subseteq E$). This maximality property is originally studied in Samet (2010), which examines the relation between models of knowledge induced from an information partition and a knowledge operator on an algebra of sets.

The usefulness of this representation lies in the fact that an additional logical or introspective property of Ashley's knowledge is represented as a corresponding set-algebraic property of the collection of events. For example, she knows a tautology (the ambient set) if and only if the collection contains it. If Ashley is logical in the sense that she knows a tautology and that she knows any finite conjunction of her own knowledge, then her knowledge is characterized by a topology when the underlying space is sufficiently rich. Suppose that Ashley's knowledge satisfies Negative Introspection, i.e., if she does not know an event then she knows that she does not know it. Then, her knowledge is represented as a sub-algebra of the underling space. If it is a measurable space, then the collection of Ashley's knowledge is a sub-σ-algebra if and only if her knowledge additionally satisfies Negative Introspection.

If the collection of events that has the maximality property represents knowledge, does there exist the smallest such collection including any given collection of events? I provide the condition under which the given collection of events has an informational content in the sense that the smallest collection of events including the given collection and satisfying the maximality property exists.

I also study the relation between the collection of events that characterizes Ashley's knowledge and her information sets. Namely, if Ashley's knowledge is given by her information partition, then her knowledge (at least) satisfies Truth Axiom, Positive Introspection, Monotonicity, and Negative Introspection. Now, the collection of events that she knows whenever it obtains turns out to be the smallest collection including her information partition cells and satisfying the maximality property. This collection also forms a sub-algebra so that if an underlying structure is a measurable space then it is a sub-σ-algebra. If the information partition is countable, then the sub-σ-algebra is exactly the one generated by the information partition. The relation between information partitions and such collections are monotone in the sense that the finer an information partition is, the larger the collection of events is.

This paper is related to characterizations of knowledge and information by a collection of sets such as a σ-algebra not only in economics and game theory but also in such various fields as computer science and artificial intelligence, logic, mathematical psychology, and philosophy. While I discuss some of the related papers in Section 4 in more detail, here I briefly mention the literature.

First, while this paper focuses on epistemic foundations for knowledge representation by a set algebra, I start with the literature in economics that uses σ-algebras to represent agents' information. In general equilibrium theory, such papers as Radner (1968), Yannelis (1991), and Wilson (1978) employ (sub-σ-)algebras to represent agents' information. In Wilson (1978), an event E is in an agent's set algebra if and only if she knows whether the prevailing state is in E or E^c. Hérves-Beloso and Monteiro (2013) study this idea generally by information partitions. In game theory, the agent's knowledge is formalized as her knowledge operator. The knowledge operator is often induced by a possibility correspondence, which associates, with each state of the world, the set of states that she considers possible (e.g., Aumann (1976, 1999), Geanakoplos (1989), and Morris (1996)). When the possibility correspondence forms a partition on the state space as in Aumann (1976), the set of events that the agent always knows whenever they obtain forms a (σ-)algebra (e.g., Aumann (1999), Bacharach (1985), and Samet (2010)). I discuss in Section 4.4 that such set algebras formalize the notion of common knowledge (e.g., Aumann (1976) and Friedell (1969)).

The representation of agents' information by sub-σ-algebras enables one to study the family of sub-σ-algebras on a given measurable space. Allen (1983), building on Boylan (1971), introduces a topology on such family of agents' information (see also Cotter (1986) and Stinchcombe (1990)) that makes it possible to study the continuity between agents' information and economic variables. Applications to game and general equilibrium theory include Correia-da-Silva and Hervés-Beloso (2007) and Monderer and Samet (1996) (see also the references therein).

Second, this paper is most closely related to the literature studying the sense in which an agent's knowledge is summarized by a set algebra (especially a σ-algebra) and studying the relation between the agent's information sets and the set algebra. While I defer the detailed discussions to Section 4.1, this paper is related to Dubra and Echenique (2004), Hérves-Beloso and Monteiro (2013), and Lee (Forthcoming). This paper fully characterizes when and how the agent's knowledge is characterized by a σ-algebra in terms of (i) the agent's logical and introspective reasoning abilities, (ii) the structure of an underlying space, and (iii) a particular form of a σ-algebra that captures the agent's knowledge. This paper provides a unified understanding on some conflicting arguments among these papers as to the sense in which a σ-algebra captures an agent's knowledge and as to the relation between an information partition and a σ-algebra on a measurable space of uncertainty.

This paper is also closely related to Samet (2010), which studies the relation between an information partition and a knowledge operator. Samet (2010, Proposition 1) establishes that a knowledge operator on an algebra of sets is well defined when the collection of sets that represents knowledge satisfies his original maximality property. While Samet (2010) considers a fully introspective and perfectly logical agent, I identify the extent to which the maximality property characterizes an agent's knowledge (namely, Truth Axiom, Monotonicity, and Positive Introspection). Then, I establish various forms of set algebra that captures the agent's knowledge depending on her reasoning abilities on various underlying spaces.

Third, researchers in various fields also represent knowledge using a topology instead of a σ-algebra. One difference between a topology and a σ-algebra is the closure under complementation. Within the framework of this paper, this difference pertains to Negative Introspection. When an agent's knowledge violates Negative Introspection, for a self-evident event E, if E is false at a state ω, then she may not necessarily know the negation E^c. That is, while the agent can discern whether E is true or not under Negative Introspection, she can only know E is true whenever E obtains without Negative Introspection. Thus, this paper is related to the economics literature on non-partitional knowledge models that studies an agent who takes information at face value. This literature studies (i) its implications on interactive knowledge and solution concepts in games and (ii) what sort of agents' information processing leads to non-introspective knowledge that violates Negative Introspection.[1] Section 4.3 discusses other disciplines using topologies to represent agents' knowledge.

Fourth, I show that the characterization of knowledge under a particular setting reduces to a "knowledge structure" in the mathematical-psychology literature (Doignon and Falmagne, 1985, 2016; Falmagne and Doignon, 2011).[2]

The paper is organized as follows. Section 2 provides a framework. Section 3 presents the main results. Section 3.1 formalizes the equivalence between a knowledge operator and a collection of sets. Section 3.2 studies when a given collection has an informational content. Section 3.3 studies how the collection of sets that represents knowledge is generated from information sets. Section 4 discusses the main results in terms of how this paper fits into the diverse strands of literature on representations of knowledge and information. This section also provides concluding remarks. Proofs are relegated to Appendix A.

2 Framework

This section introduces a framework which represents agents' knowledge by knowledge operators on a "sample space." To that end, I start with three technical definitions on collections of sets. Fix a set Ω. First, for any infinite cardinal number κ, a subset $\mathcal{D}$ of the power set $\mathcal{P}(\Omega)$ is a *κ-complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union and intersection of any sub-collection with cardinality less than κ (i.e., *κ-union* and *κ-intersection*). I follow the conventions

[1]See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Samet (1990), and Shin (1993). Also, violation of Negative Introspection is related to a notion of unawareness in that an agent does not know an event and she does not know that she does not know it. For unawareness, see, for example, Dekel, Lipman, and Rustichini (1998), Fagin and Halpern (1987), Modica and Rustichini (1994, 1999), and Schipper (2015). Fukuda (2018b) studies a non-trivial form of unawareness exhibited by an agent who is logical and introspective about her own knowledge.

[2]This mathematical-psychology literature provides an alternative knowledge-belief representation referred to as a "surmise system." Fukuda (2018a) provides a generalization that encompasses a possibility correspondence model, the surmise system, and a "local reasoning" model of Fagin and Halpern (1987) in computer science and artificial intelligence.

that $\emptyset = \bigcup \emptyset \in \mathcal{D}$ and that, with Ω being an underlying set, $\Omega = \bigcap \emptyset \in \mathcal{D}$. For example, denoting by $\aleph_0$ the least infinite cardinal, an $\aleph_0$-complete algebra is an algebra of sets. Denoting by $\aleph_1$ the least uncountable cardinal, an $\aleph_1$-complete algebra is a σ-algebra.[3] Second, a subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is an (∞-)*complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union and intersection. Throughout the paper, κ denotes an infinite cardinal or the symbol ∞. If $\mathcal{D}$ is a κ-complete algebra on Ω, then I often call the pair $(\Omega, \mathcal{D})$ a κ-complete algebra as well. I discuss the meaning of κ in terms of "depths" of agents' reasoning in Section 2.1.

2.1 Knowledge Representation by a Knowledge Operator

Let I be a non-empty set of agents. A (κ-)*knowledge space* (of I) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i)_{i \in I} \rangle$ such that $(\Omega, \mathcal{D})$ is a κ-complete algebra and that $K_i : \mathcal{D} \to \mathcal{D}$ is agent i's *knowledge operator*. Each element ω of Ω is a *state* (of the world), $\mathcal{D}$ is the *domain*, and each element E of $\mathcal{D}$ is an *event*. Unless otherwise stated, I assume the following three properties on each K_i: (i) *Truth Axiom* ($K_i(E) \subseteq E$ for any $E \in \mathcal{D}$); (ii) *Monotonicity* ($E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$); and (iii) *Positive Introspection* ($K_i(\cdot) \subseteq K_i K_i(\cdot)$). An agent i knows an event E at a state ω if $\omega \in K_i(E)$. Thus, $K_i(E)$ denotes the event that (i.e., the set of states at which) agent i knows E.

The rest of this subsection discusses the components of the κ-knowledge space. I start with the role of the domain $\mathcal{D}$. While standard possibility correspondence models often take the power set of an underlying state space as its domain, a κ-knowledge space specifies the collection of events that are objects of agents' reasoning. Since I represent an agent's knowledge by a sub-collection of the domain $\mathcal{D}$, a choice of κ affects possible forms of a set algebra that represents the agent's knowledge.

I mention four reasons that the domain forms a κ-complete algebra. First, the specification of an underlying state space turns out to play an important role in the form of a set algebra that captures an agent's knowledge. In the literature looking at the sense in which σ-algebras capture agents' information, for example, Dubra and Echenique (2004) argue that a σ-algebra may fail to capture an informational content in that it may not be closed under arbitrary union. As they consider the power set algebra as an underlying space, Theorem 1 in Section 3.1 implies that the collection of events that captures an agent's knowledge is closed under arbitrary union if the underlying space is a complete algebra. If the underlying space is a measurable space, in contrast, the collection of events that captures an agent's knowledge may not necessarily be closed under arbitrary union.

Second, while the paper focuses on knowledge, one can additionally capture probabilistic beliefs. For example, Meier (2008) studies a model of knowledge and prob-

[3] As in Meier (2006, Remark 1), κ can be assumed to be an infinite regular cardinal. If an infinite cardinal κ is not regular then any κ-complete algebra is κ^+-complete, where the successor cardinal κ^+ is regular (supposing the axiom of choice). Note that $\aleph_0$ and $\aleph_1$ are regular.

abilistic beliefs where both are defined on a σ-algebra. In contrast, if agents' probabilistic beliefs are represented on a σ-algebra and if their knowledge is represented on the power set, then, without additional assumptions on knowledge, the event that an agent knows an E might not necessarily be measurable for a measurable event E. For example, it is assumed in Aumann (1976)'s partitional model that agents' partitions are at most countable.

Third, in the literature giving logical foundations to state space models of knowledge, events are generated by some logical system, and thus the domain may only form a certain algebra of sets (depending on the given logical system).[4] The specification of $\mathcal{D}$ as a κ-complete algebra is amenable to such logical foundations.

Fourth, specifying κ implicitly determines the possible depths of agents' reasoning. An $\aleph_0$-complete algebra (an algebra of sets) can accommodate agents' finite depths of reasoning. The analysts can define a finite chain of agents' knowledge in such form as "Alice knows that Bob knows that Carol knows that it is raining." An $\aleph_1$-complete algebra (a σ-algebra) can accommodate agents' countable depths of reasoning.[5]

Henceforth, since I almost always deal with a representation of a single agent's knowledge, I fix a generic agent i and drop the subscript unless otherwise stated. Now, I discuss the meaning of the assumptions imposed on the agent's knowledge. Truth Axiom says that the agent can only know what is true. It distinguishes knowledge from belief in that belief can be false while knowledge has to be true. It implies *Consistency*: $K(E) \subseteq (\neg K)(E^c)$ for any $E \in \mathcal{D}$. It means that, if the agent knows an event E then she considers E possible in that she does not know its negation E^c. In other words, she cannot know E and E^c at the same time. This notion of possibility is considered to be the dual of knowledge in a standard state space model. I define the *possibility operator* $L_K : \mathcal{D} \to \mathcal{D}$ by $L_K(E) := (\neg K)(E^c)$. The agent considers an event $E \in \mathcal{D}$ *possible* at a state ω if $\omega \in L_K(E)$. Consistency then means that knowledge implies possibility.

Monotonicity says that if the agent knows some event then she knows any of its logical consequences. Positive Introspection states that if the agent knows some event then she knows that she knows it. Thus, I consider the knowledge of the agent who is logical and introspective about her own knowledge in that her knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection.

Next, I introduce three additional logical and introspective properties of knowledge. First, *Necessitation* refers to $K(\Omega) = \Omega$. It means that the agent knows any tautology such as $E \cup E^c$. Second, *Non-empty λ-Conjunction* (where λ is an infinite

[4] A knowledge operator in Samet (2010) is defined on an ($\aleph_0$-complete) algebra. Set-theoretical (semantic) knowledge models where events are based on propositions include such papers as Aumann (1999), Bacharach (1985), Samet (1990), and Shin (1993).

[5] For example, the iterative definition of common knowledge of the form, Alice and Bob know that it is raining, they know that they know it is raining, and so forth *ad infinitum*, requires countable depths of reasoning. In a related but different context, rationalizability would call for transfinite levels of reasoning in the form of eliminations of never-best-replies. See, for instance, Chen, Luo, and Qu (2016) and Lipman (1994).

cardinal with $\lambda \le \kappa$) refers to $\bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. It states that the agent knows any non-empty conjunction of events with cardinality less than λ if she knows each. Non-empty (λ-)Conjunction and Necessitation are jointly equivalent to (λ-)Conjunction, by identifying Necessitation as the empty conjunction. Thus, *λ-Conjunction* refers jointly to Non-empty λ-Conjunction and Necessitation. I refer to (Non-empty) $\aleph_0$- and $\aleph_1$-Conjunction, respectively, as (*Non-empty*) *Finite* and *Countable Conjunction*. Also, *Arbitrary Conjunction* refers to a combination of Non-empty ∞-Conjunction and Necessitation.

Third, *Negative Introspection* refers to $(\neg K)(\cdot) \subseteq K(\neg K)(\cdot)$. It states that if the agent does not know some event then she knows that she does not know it. While the standard partitional knowledge model presupposes Negative Introspection, the literature on non-partitional knowledge models dispenses with Negative Introspection on the ground that it entails a strong form of rationality. Here, I remark on the implication of Negative Introspection on other properties of knowledge. Negative Introspection together with Truth Axiom imply Positive Introspection (see, for example, Aumann (1999, p. 270)). I show in Corollary 1 in Section 3.1 that Negative Introspection, Monotonicity, and Truth Axiom imply κ-Conjunction.

2.2 Knowledge Representation by Information Sets

Before I represent the agent's knowledge by a set algebra in Section 3, here I examine the condition under which her knowledge is derived from a possibility correspondence: it associates, with each state, the set of states that the agent considers possible. That is, I ask the condition under which a κ-knowledge space is identified as a possibility correspondence model of knowledge on a κ-complete algebra. For example, if $\kappa = \aleph_1$, then the condition reduces to the one under which the agent's knowledge operator is induced from her possibility correspondence on a measurable space.

Throughout the subsection, fix a κ-complete algebra $(\Omega, \mathcal{D})$ and an operator $K : \mathcal{D} \to \mathcal{D}$. While K is intended as the agent's knowledge operator, I do not assume any of Truth Axiom, Monotonicity, or Positive Introspection at this point.

I define a notion of possibility between states (i.e., a possibility relation). A state ω' is considered *possible* at a state ω if, for any event $E \in \mathcal{D}$ which the agent knows at ω, the event E is true at ω'. Denote by $b_K : \Omega \to \mathcal{P}(\Omega)$ the mapping b_K that associates, with each state ω, the set of states that the agent considers possible:

$$b_K(\omega) := \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in K(E)\} = \bigcap \{E \in \mathcal{D} \mid \omega \in K(E)\}.$$

Call each $b_K(\omega)$ the *information set* at ω. Note, however, that $b_K(\omega)$ may not necessarily be an event when $\mathcal{D}$ is not a complete algebra.

Now, K satisfies the *Kripke property* if $\omega \in K(E)$ if (and only if) $b_K(\omega) \subseteq E$ for all $(\omega, E) \in \Omega \times \mathcal{D}$. The Kripke property means that the agent knows an event E at a state ω if and only if (hereafter, often abbreviated as iff) E contains any state considered possible at ω. If K satisfies the Kripke property then it satisfies Monotonicity, Non-empty κ-Conjunction, and Necessitation.

When K satisfies the Kripke property, the logical and introspective properties of K can be stated as the corresponding properties of b_K as in the standard case where $\mathcal{D} = \mathcal{P}(\Omega)$. For example, under the Kripke property, K satisfies Truth Axiom and Positive Introspection iff b_K is reflexive (i.e., $\omega \in b_K(\omega)$) and transitive (i.e., $\omega' \in b_K(\omega)$ implies $b_K(\omega') \subseteq b_K(\omega)$). Likewise, K satisfies Truth Axiom, Positive Introspection, and Negative Introspection iff b_K yields a partition on Ω.

How can one characterize the Kripke property? On a complete algebra, it can be seen that the Kripke property is equivalent to Monotonicity and Arbitrary Conjunction (see, for example, Morris (1996) when $\mathcal{D} = \mathcal{P}(\Omega)$). On a general κ-complete algebra, however, the converse is not necessarily true. Samet (2010) provides specific examples where a knowledge operator on an ($\aleph_0$-complete) algebra satisfies the fully logical and introspective "S5" properties (i.e., Monotonicity, Finite Conjunction, Truth Axiom, Positive Introspection, and Negative Introspection) but is not derived from a partition (i.e., fails to satisfy the Kripke property).

The following proposition characterizes the Kripke property. It generalizes Samet (2010, Theorem)'s condition under which "S5" knowledge is derived from a partition in an $\aleph_0$-knowledge space. In the case of $\kappa = \aleph_1$, this proposition identifies the condition under which the analysts can introduce an agent's knowledge on a probabilistic-belief space from a possibility correspondence.[6]

Proposition 1. *Let $(\Omega, \mathcal{D})$ be a κ-complete algebra, and let $K : \mathcal{D} \to \mathcal{D}$ satisfy Monotonicity. The following characterizes the Kripke property. If $(\omega, F) \in \Omega \times \mathcal{D}$ satisfies $E \cap F \neq \emptyset$ for all $E \in \mathcal{D}$ with $\omega \in K(E)$, then (ω, F) satisfies $b_K(\omega) \cap F \neq \emptyset$.*

To conclude this subsection, I remark on how the possibility in terms of $\omega' \in b_K(\omega)$ relates to the possibility in the sense of $\omega \in L_K(\{\omega'\})$. For example, Morris (1996) introduces a possibility correspondence from the possibility operator L_K. If every singleton set $\{\omega'\}$ is an event and if i's knowledge satisfies Monotonicity, then b_K can be written as follows (see Remark A.1 in Appendix A for the proof):

$$b_K(\omega) = \{\omega' \in \Omega \mid \omega \in L_K(\{\omega'\})\}. \tag{1}$$

[6]On the one hand, a representation of knowledge by a possibility correspondence on a σ-algebra is non-trivial because the knowledge of an event is then written as the union of all information sets contained in the event. For example, Meier (2008, p. 56) puts: "We do not know of any natural, and not too restrictive condition on the partitions of the players that would guarantee that the derived knowledge operators send measurable sets to measurable sets." On the other hand, game-theoretical analyses would call for both knowledge and beliefs in, for example, dynamic games (see, for instance, Dekel and Gul (1997)).

3 Main Results

Having defined knowledge spaces and the properties of knowledge operators, here I provide a set-algebraic representation of knowledge. This embodies the intuition that the content of knowledge can be captured by a sub-collection of a given domain. To that end, I introduce the following two definitions. First, an event E is *self-evident* (to the agent) if $E \subseteq K(E)$. In words, E is self-evident if the agent knows E whenever E obtains. I denote the collection of self-evident events, which I call the *self-evident collection*, by $\mathcal{J}_K := \{E \in \mathcal{D} \mid E \subseteq K(E)\}$.

Second, a given sub-collection $\mathcal{J}$ of $\mathcal{D}$ satisfies the *maximality property* (with respect to (henceforth often abbreviated as w.r.t.) $\mathcal{D}$) if $\emptyset \in \mathcal{J}$ and for any $E \in \mathcal{D}$, the largest element of $\mathcal{J}$ included in E, $\max\{F \in \mathcal{D} \mid F \in \mathcal{J} \text{ and } F \subseteq E\}$, exists in $\mathcal{J}$. It can be seen that $\mathcal{J}$ satisfies the maximality property w.r.t. $\mathcal{D}$ iff $\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}$ for each $E \in \mathcal{D}$. The maximality property is extended from Samet (2010), which defines it for a sub-algebra $\mathcal{J}$ of an ($\aleph_0$-complete) algebra $\mathcal{D}$. As Theorem 1 and Corollary 1 demonstrate, this corresponds to the knowledge (defined on an algebra) which satisfies Truth Axiom, Monotonicity, (Positive Introspection, Finite Conjunction), and Negative Introspection.

With these two definitions in mind, I establish the following three strands of results. First, I show that properties of knowledge operators are equivalently expressed as set-algebraic properties of self-evident collections when Truth Axiom, Positive Introspection, and Monotonicity are imposed. Specifically, given a knowledge operator K which satisfies Truth Axiom, Positive Introspection, and Monotonicity, the event $K(E)$ is the largest (in the sense of set inclusion) self-evident event which is included in E. Other properties of K are translated into set-algebraic properties of $\mathcal{J}_K$. For example, Negative Introspection is translated as the closure under complementation. The self-evident collection $\mathcal{J}_K$, in turn, induces the knowledge operator through $K(E) = K_{\mathcal{J}_K}(E) := \max\{F \in \mathcal{J}_K \mid F \subseteq E\}$.

The second result is to identify when a given sub-collection $\mathcal{D}'$ of $\mathcal{D}$ has an informational content. I identify the condition on $\mathcal{D}'$ under which there exists the smallest sub-collection $\mathcal{J}(\mathcal{D}')$ including $\mathcal{D}'$ and satisfying the maximality property.

The third is to examine the relation between the self-evident collection and information sets. I show that, under the assumption that information sets are events, a knowledge operator satisfies the Kripke property if and only if the self-evident collection is the smallest collection including information sets and satisfying the maximality property. If the information sets form a countable partition, then the self-evident collection coincides with the σ-algebra generated by the partition.

3.1 Knowledge Representation by a Set Algebra

I provide the first main result on the equivalence between a knowledge operator and a self-evident collection. For any knowledge operator satisfying Truth Axiom, Monotonicity, and Positive Introspection, its self-evident collection satisfies the maximality property. Conversely, for any collection of events satisfying the maximality property, there is a knowledge operator whose self-evident collection coincides with the given collection.

Theorem 1. *Fix a κ-complete algebra $(\Omega, \mathcal{D})$.*

1. *For a given (knowledge) operator $K : \mathcal{D} \to \mathcal{D}$, the self-evident collection $\mathcal{J}_K = \{E \in \mathcal{D} \mid E \subseteq K(E)\}$ satisfies the following.*

   (a) *If K satisfies Truth Axiom, Monotonicity, and Positive Introspection, then $\mathcal{J}_K$ satisfies the maximality property in the sense that, for any $E \in \mathcal{D}$,*

   $$K(E) = \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_K \text{ and } F \subseteq E\} \qquad (2)$$
   $$(= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_K \text{ with } \omega \in F \subseteq E\}).$$

   (b) *If K satisfies Non-empty λ-Conjunction, then $\mathcal{J}_K$ is closed under non-empty λ-intersection.*

   (c) *If K satisfies Necessitation, then $\Omega \in \mathcal{J}_K$.*

   (d) *If K satisfies Truth Axiom and Negative Introspection, then $\mathcal{J}_K$ is closed under complementation.*

2. *Conversely, let $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfy the maximality property. The operator $K_{\mathcal{J}} : \mathcal{D} \to \mathcal{D}$ defined by $K_{\mathcal{J}}(E) := \max\{F \in \mathcal{J} \mid F \subseteq E\}$ satisfies the following.*

   (a) *$K_{\mathcal{J}}$ satisfies Truth Axiom, Monotonicity, and Positive Introspection.*

   (b) *If $\mathcal{J}$ is closed under non-empty λ-intersection, then $K_{\mathcal{J}}$ satisfies Non-empty λ-Conjunction.*

   (c) *If $\Omega \in \mathcal{J}$ then $K_{\mathcal{J}}$ satisfies Necessitation.*

   (d) *If $\mathcal{J}$ is closed under complementation then $K_{\mathcal{J}}$ satisfies Negative Introspection.*

3. *$K_{\mathcal{J}_K} = K$ and $\mathcal{J} = \mathcal{J}_{K_{\mathcal{J}}}$.*

4. *If $\kappa = \infty$, then $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfies the maximality property iff $\mathcal{J}$ is closed under arbitrary union. For an infinite cardinal κ, if $\mathcal{J}$ satisfies the maximality property, then it is closed under κ-union. Also, for any $\mathcal{E} \subseteq \mathcal{J}$, if $\bigcup \mathcal{E} \in \mathcal{D}$ then $\bigcup \mathcal{E} \in \mathcal{J}$.*

A corollary of Theorem 1 is that Negative Introspection, together with Truth Axiom and Monotonicity, imply all the other logical and introspective properties of knowledge postulated in Section 2.1.

Corollary 1. *Let $(\Omega, \mathcal{D})$ be a κ-complete algebra, and let $K : \mathcal{D} \to \mathcal{D}$ satisfy Truth Axiom, Monotonicity, and Negative Introspection. Then, K satisfies κ-Conjunction including Necessitation (as well as Consistency and Positive Introspection).*

Six additional remarks on Theorem 1 are in order. First, all of Truth Axiom, Positive Introspection, and Monotonicity are essential. If any one condition is absent, then there is a simple example (provided in Remark A.2 in Appendix A) where $K \neq K'$ and $\mathcal{J}_K = \mathcal{J}_{K'}$.

Second, the maximality property ensures that $K(E) = \bigcup\{F \in \mathcal{J}_K \mid F \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$, in spite of the fact that the self-evident collection (as well as the domain $\mathcal{D}$) may not necessarily be closed under arbitrary union.

Third, if K satisfies Truth Axiom and Positive Introspection, then $\mathcal{J}_K = \{K(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$. Thus, the self-evident collection comprises solely of the largest self-evident event $K(E)$ included in E for each $E \in \mathcal{D}$.

Fourth, while Section 4 discusses previous studies that have employed various forms of set algebras to represent knowledge, here I briefly introduce various forms of set algebras that represent different properties of knowledge. A self-evident collection in an ∞-knowledge space becomes a (sub-)topology under Truth Axiom, Monotonicity, Positive Introspection, and Finite Conjunction. Under Arbitrary Conjunction, such a topology is closed under arbitrary intersection. In this case, $b_K : \Omega \to \mathcal{D}$ turns out to be a reflexive and transitive (non-partitional) possibility correspondence. Moreover, if knowledge satisfies Truth Axiom, Monotonicity, and Negative Introspection (again, recall Corollary 1) in a κ-knowledge space, then the self-evident collection becomes a sub-κ-complete algebra. To restate, for $\kappa \in \{\aleph_0, \aleph_1, \infty\}$, knowledge takes a form of a sub-algebra, sub-σ-algebra, and sub-complete algebra, respectively.

Fifth, a self-evident collection may not necessarily be closed under complementation in the absence of Negative Introspection. If a self-evident event E satisfies $E^c \notin \mathcal{J}_K$, it is not the case that the agent knows E is false (E^c is true) whenever E is false. That is, the agent cannot conclude that E is false just by the fact that she does not observe E. The non-partitional knowledge models that dispense with Negative Introspection study agents who process information at face value.

In contrast, for any $E \in \mathcal{J}_K$ with $E^c \in \mathcal{J}_K$, the agent knows whether E is true or not at any state (i.e., $\Omega = K(E) \cup K(E^c)$). In other words, if $\mathcal{J}_K$ forms a sub-κ-complete algebra of $\mathcal{D}$, then $\mathcal{J}_K$ comprises of an event E such that the agent knows whether E is true or not at any state.[7] This formalizes the (common) sense in which knowledge is interpreted as a sub-algebra of a given domain (see, for example, Hérves-Beloso and Monteiro (2013), Lee (Forthcoming), Stinchcombe (1990), and Wilson (1978)). Indeed, a self-evident event turns out to coincide with an event

[7]Generally, the collection of events E such that the agent knows whether E is true or not at any state is $\{E \in \mathcal{D} \mid K(E) \cup K(E^c) = \Omega\}$. If the agent's knowledge satisfies κ-Conjunction, then this set algebra forms a sub-κ-complete algebra of $\mathcal{D}$. Lee (Forthcoming, Section 4) studies a generic equivalence between probability-one belief that may violate Truth Axiom and a sub-σ-algebra on a measurable space using the above collection of events.

which Hérves-Beloso and Monteiro (2013) call an informed set (with respect to a partition) in the following sense (see also Lee (Forthcoming, Lemma 3)). Suppose that the agent's knowledge is represented by a partition (i.e., her knowledge satisfies the Kripke property, Truth Axiom, and Negative Introspection). Then, an event is self-evident to her if and only if the event is an informed set with respect to her information partition (where technically I allow each partition cell $b_K(\omega)$ not to be an event). Section 3.3 studies the relation between information sets $(b_K(\omega))_{\omega\in\Omega}$ and the self-evident collection $\mathcal{J}_K$.

If I start with the possibility operator, then the corresponding dual collection $\overline{\mathcal{J}_K} := \{F \in \mathcal{D} \mid F^c \in \mathcal{J}_K\}$ satisfies the minimality property in the sense that $L_K(E) = \min\{F \in \overline{\mathcal{J}_K} \mid E \subseteq F\}$ for each $E \in \mathcal{D}$. That is, for any $E \in \mathcal{D}$, the dual collection $\overline{\mathcal{J}_K}$ always has the minimum event including E. Since $\mathcal{J}_K$ is not necessarily closed under complementation, however, $\mathcal{J}_K$ does not necessarily satisfy the minimality property in the form of $L_K(E) = \min\{F \in \mathcal{J}_K \mid E \subseteq F\}$.[8] Without Negative Introspection of K, these two minimality properties may not coincide.

Sixth, set-inclusion between self-evident collections naturally gives "knowledgeability" relation. Suppose that knowledge operators K_i and K_j of two agents i and j satisfy Truth Axiom, Positive Introspection, and Monotonicity. Then,

$$\mathcal{J}_{K_i} \subseteq \mathcal{J}_{K_j} \textit{ if and only if } K_i(\cdot) \subseteq K_j(\cdot). \tag{3}$$

The first condition implies the second through the maximality property. Conversely, the second condition implies the first because $E \subseteq K_i(E) \subseteq K_j(E)$ for any $E \in \mathcal{J}_{K_i}$.

3.2 Comparison of Knowledge

Moving on to the second result, I ask when a given collection $\mathcal{D}' \in \mathcal{P}(\mathcal{D})$ has an informational content. I examine the condition under which there exists the smallest collection including $\mathcal{D}'$ and satisfying the maximality property and possibly various other logical and introspective properties.

This observation provides a given sub-collection with an informational content in a well-defined manner in the following cases. First, for a given σ-algebra, one can consider the smallest σ-algebra including the original one and satisfying the maximality property. Such a σ-algebra can capture knowledge and probabilistic beliefs. Second, if an agent "learns" a certain collection of events, then one can obtain the smallest collection including the agent's original self-evident collection and learned events and satisfying the maximality property. Third, for a set of agents I, consider the intersection of self-evident collections (i.e., the publicly evident (Milgrom, 1981)

[8] As an example, consider $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{D} = \mathcal{P}(\Omega)$, and $\mathcal{J}_K = \{\emptyset, \{\omega_1\}\}$. I have $L_K(\{\omega_1\}) = \min\{F \in \overline{\mathcal{J}_K} \mid \{\omega_1\} \subseteq F\} = \Omega \neq \{\omega_1\} = \min\{F \in \mathcal{J}_K \mid E \subseteq F\}$. This happens because $\{\omega_1\}^c$ is not self-evident. This means that the maximality property of a self-evident collection is generally different from the "relative completeness" of a sub-Boolean algebra (Halmos, 1955, Section 4) (which could be seen as the minimality property) due to the failure of Negative Introspection.

events) $\bigcap_{i\in I}\mathcal{J}_i$. If there exists the smallest collection including $\bigcap_{i\in I}\mathcal{J}_i$ and satisfying the maximality property, then one can associate it with the knowledge of the most knowledgeable agent who is at least as less knowledgeable as any individual agent. As I briefly discuss the notion of common knowledge (e.g., Aumann (1976) and Friedell (1969)) in Section 4.4, knowledge of such hypothetical agent can be identified with common knowledge.

I formalize the sense in which a given sub-collection $\mathcal{D}'$ of $\mathcal{D}$ has an informational content. Namely, the sub-collection $\mathcal{D}'$ of $\mathcal{D}$ is an *information basis* if

$$K_{\mathcal{D}'}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{D}' \text{ such that } \omega \in F \subseteq E\} \in \mathcal{D} \text{ for each } E \in \mathcal{D}.$$

In words, the sub-collection $\mathcal{D}'$ is an information basis if, for each event E, the set of states at which E can be supported by some $F \in \mathcal{D}'$ forms an event. Note that the definition of $K_{\mathcal{D}'}$ is a slight weakening of $K_{\mathcal{J}}$ defined in Theorem 1 (2) in that $\mathcal{D}'$ may not satisfy the maximality property. If $\mathcal{D}'$ itself satisfies the maximality property then this definition coincides with the one in Theorem 1 (2).

If $\mathcal{D}'$ is an information basis, let $\mathcal{J}(\mathcal{D}') := \{K_{\mathcal{D}'}(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$. I remark that if $\mathcal{D}$ is a complete algebra then any sub-collection $\mathcal{D}'$ is an information basis because $K_{\mathcal{D}'}(E) = \bigcup\{F \in \mathcal{D} \mid F \in \mathcal{D}' \text{ and } F \subseteq E\} \in \mathcal{D}$. Especially, if $\mathcal{D}$ is finite then $(\Omega, \mathcal{D})$ is always a complete algebra, and thus any sub-collection $\mathcal{D}'$ is an information basis.

The next proposition establishes the following results. First, whenever $\mathcal{J}(\mathcal{D}')$ is well defined (i.e., when $\mathcal{D}'$ is an information basis), $\mathcal{J}(\mathcal{D}')$ is the smallest collection including $\mathcal{D}'$ and satisfying the maximality property. Second, for a given information collection $\mathcal{D}'$, I find the smallest collection including $\mathcal{D}$ and satisfying the maximality property and additional logical properties. Third, let $(\Omega, \mathcal{D})$ be a complete algebra. For any $\mathcal{D}'$, $\mathcal{J}(\mathcal{D}')$ is characterized as the smallest collection including $\mathcal{D}'$ and being closed under arbitrary union.

Proposition 2. *Let $\mathcal{D}'$ be an information basis on a κ-complete algebra $(\Omega, \mathcal{D})$.*

1. *The collection $\mathcal{J}(\mathcal{D}')$ is the smallest collection including $\mathcal{D}'$ and satisfying the maximality property. It can also be written as*

$$\mathcal{J}(\mathcal{D}') = \{E \in \mathcal{D} \mid \textit{if } \omega \in E \textit{ then there is } F \in \mathcal{D}' \textit{ with } \omega \in F \subseteq E\}. \quad (4)$$

   Moreover, $K_{\mathcal{D}'} = K_{\mathcal{J}(\mathcal{D}')}$. That is,

$$K_{\mathcal{D}'}(E) = \max\{F \in \mathcal{J}(\mathcal{D}') \mid F \subseteq E\} \textit{ for each } E \in \mathcal{D}. \quad (5)$$

2. *$\mathcal{D}' \cup \{\Omega\}$ is an information basis, and $\mathcal{J}(\mathcal{D}' \cup \{\Omega\}) = \mathcal{J}(\mathcal{D}') \cup \{\Omega\}$.*

3. *If $\hat{\mathcal{D}}' := \{\bigcap \mathcal{E} \in \mathcal{D} \mid \mathcal{E} \subseteq \mathcal{D}' \text{ with } 0 < |\mathcal{E}| < \lambda\}$ is an information basis, then $\mathcal{J}(\hat{\mathcal{D}}')$ is the smallest collection including $\mathcal{D}'$, satisfying the maximality property, and closed under non-empty λ-intersection. If $\mathcal{D}'$ is closed under non-empty λ-intersection, so is $\mathcal{J}(\mathcal{D}')$.*

4. *The collection $\mathcal{J}(\mathcal{D}')$ does not necessarily inherit the closure under complementation from $\mathcal{D}'$. Generally, the following are equivalent.*
   (a) *$\mathcal{J}(\mathcal{D}')$ is closed under complementation.*
   (b) *$\mathcal{J}(\mathcal{D}')$ is a sub-κ-complete algebra.*
   (c) *$K_{\mathcal{D}'}$ satisfies Negative Introspection.*

5. *Suppose that $|\mathcal{D}'| < \kappa$ (if $\kappa = \infty$ then there is no restriction on $\mathcal{D}'$). Then,*

$$\mathcal{J}(\mathcal{D}') = \bigcap \{\mathcal{J}' \in \mathcal{P}(\mathcal{D}) \mid \mathcal{D}' \subseteq \mathcal{J}' \text{ and } \mathcal{J}' \text{ is closed under } \kappa\text{-union}\}.$$

   Also, $\mathcal{J}(\mathcal{D}')$ is closed under complementation (if and) only if it is the smallest κ-complete algebra including $\mathcal{D}'$.

With regard to Proposition 2 (4), I remark that Proposition 3 in Section 3.3 shows that if $\mathcal{D}'$ is an information partition then $\mathcal{J}(\mathcal{D}')$ becomes a sub-κ-complete algebra of $\mathcal{D}$. Especially, if the partition $\mathcal{D}'$ is countable on a measurable space then $\mathcal{J}(\mathcal{D}')$ turns out to be the σ-algebra generated by $\mathcal{D}'$.

To conclude this subsection, I examine the sense in which $\mathcal{J}(\cdot)$ preserves collections of events. For information bases $\mathcal{E}$ and $\mathcal{E}'$, if $\mathcal{E} \subseteq \mathcal{E}'$ then $\mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$. Generally, the following statement characterizes $\mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$.

Corollary 2. *Let $\mathcal{E}$ and $\mathcal{E}'$ be information bases on a κ-complete algebra $(\Omega, \mathcal{D})$. Then, $\mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$ if and only if for any $E \in \mathcal{E}$ and $\omega \in E$, there is $E' \in \mathcal{E}'$ such that $\omega \in E' \subseteq E$.*

3.3 Information Sets

I examine the relation between a self-evident collection and information sets. First, I show that each information set $b_K(\omega)$ coincides with the intersection of self-evident events containing ω. Second, assuming that each $b_K(\omega)$ is an event, I show that K satisfies the Kripke property if and only if $\mathcal{J}(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$ is well defined.

Proposition 3. *Let $\overrightarrow{\Omega}$ be a κ-knowledge space.*

1. *$b_K(\omega) = \bigcap \{E \in \mathcal{J}_K \mid \omega \in E\}$ for each $\omega \in \Omega$.*

2. *Suppose that each $b_K(\omega)$ is an event (i.e., $b_K(\omega) \in \mathcal{D}$). The following are all equivalent.*
   (a) *The knowledge operator $K : \mathcal{D} \to \mathcal{D}$ satisfies the Kripke property.*
   (b) *$\mathcal{J}_K = \left\{E \in \mathcal{D} \mid E = \bigcup_{\omega \in F} b_K(\omega) \text{ for some } F \in \mathcal{P}(\Omega)\right\}$.*
   (c) *$\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$ is an information basis and $\mathcal{J}_K = \mathcal{J}(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$.*

3. *Suppose: (i) each $b_K(\omega)$ is an event; (ii) K satisfies the Kripke property; and (iii) $\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$ forms a partition (i.e., K additionally satisfies Negative Introspection). Then, $\mathcal{J}_K = \mathcal{J}(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$ is a sub-κ-complete algebra.*

Note that, in Proposition 3, the knowledge operator K satisfies Truth Axiom, Positive Introspection, and Monotonicity. Part (1) holds regardless of the Kripke property. This part means that each $b_K(\omega)$ is the "atom" of $\mathcal{J}_K$ at ω. Stinchcombe (1990) studies the relation between a sub-σ-algebra and its atoms on a measurable space.

Part (2b) shows that the self-evident collection $\mathcal{J}_K$ coincides with the collection of events that are formed by arbitrary unions of information sets. Note that a union of information sets that does not constitute an event is excluded. This part generalizes Noguchi (2018, Lemmas 15 and 16). Assuming the Kripke property (i.e., (2a)), this part holds without the assumption that each information set $b_K(\omega)$ is an event. In part (2c), assuming that each $b_K(\omega)$ is an event, the knowledge operator satisfies the Kripke property if and only if the self-evident collection is characterized as the smallest collection containing every $b_K(\omega)$ and satisfying the maximality property.

If $\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$ is a countable sub-collection of a σ-algebra $\mathcal{D}$ and if K additionally satisfies Negative Introspection, then it follows from Proposition 2 (5) that $\mathcal{J}_K$ is the smallest σ-algebra $\sigma(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$ including $\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$. See also Hérves-Beloso and Monteiro (2013, Proposition 4) and Lee (Forthcoming, Theorem 1), which show the equivalence of a countably generated σ-algebra and a countable measurable information partition. Part (3) shows that generally $\mathcal{J}_K = \mathcal{J}(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$ is a sub-κ-complete algebra. It generalizes Hérves-Beloso and Monteiro (2013, Theorem 1).

Now, I examine the sense in which $\mathcal{J}(\cdot)$ preserves information sets. A part of the arguments made by Dubra and Echenique (2004) that σ-algebras are not necessarily adequate for modeling information is that the operation of taking the smallest σ-algebra does not necessarily preserve set inclusion with respect to partitions (see also Hérves-Beloso and Monteiro (2013), Nielsen (1984), Stinchcombe (1990) and the references therein). Dubra and Echenique (2004, Theorem A) and Hérves-Beloso and Monteiro (2013, Propositions 1 and 5) study the operations that preserve an informational content (a partition) and establish Blackwell's theorem connecting the preferences over signals and information partitions.

Here, I show that $\mathcal{J}$ preserves information sets. Let i and j be agents whose knowledge operators are K_i and K_j, respectively. Suppose further that their knowledge operators satisfy the Kripke property, and let $b_{K_i}, b_{K_j} : \Omega \to \mathcal{D}$ be their possibility correspondences. Corollary 2 and Proposition 3 imply the following.

Corollary 3. *Let $\overrightarrow{\Omega}$ be a κ-knowledge space of $\{i, j\}$ such that K_i and K_j satisfy the Kripke property and that their possibility correspondences satisfy $b_{K_i}, b_{K_j} : \Omega \to \mathcal{D}$. The following are all equivalent.*

1. $b_{K_j}(\cdot) \subseteq b_{K_i}(\cdot)$. *2.* $K_i(\cdot) \subseteq K_j(\cdot)$. *3.* $\mathcal{J}_{K_i} \subseteq \mathcal{J}_{K_j}$.
4. For any $\omega \in \Omega$ *and* $\omega' \in b_{K_i}(\omega)$*, there is* $\omega'' \in \Omega$ *such that* $\omega' \in b_{K_j}(\omega'') \subseteq b_{K_i}(\omega)$.

Corollary 3, for example, generalizes Dubra and Echenique (2004, Theorem A) and Hérves-Beloso and Monteiro (2013, Proposition 1). Note that in (3), it follows from Proposition 3 that

$$\mathcal{J}_{K_i} = \{E \in \mathcal{D} \mid E = \bigcup_{\omega \in F} b_{K_i}(\omega) \text{ for some } F \in \mathcal{P}(\Omega)\} = \mathcal{J}(\{b_{K_i}(\omega) \in \mathcal{D} \mid \omega \in \Omega\}).$$

Thus, as in Dubra and Echenique (2004, Part 5 of Theorem A), (3) can be restated as follows: the collection of events that are formed by arbitrary unions of information sets $\{b_{K_i}(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$ is included in the collection of events that are formed by arbitrary unions of information sets $\{b_{K_j}(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$.

4 Discussions

4.1 The Sense in which a σ-algebra Represents Information

There is a strand of literature formally looking at the sense in which a σ-algebra represents information. Especially, the previous literature has reported an inconsistency between an information partition and a σ-algebra. See, for example, Billingsley (2012), Dubra and Echenique (2004), Hérves-Beloso and Monteiro (2013), Lee (Forthcoming), Nielsen (1984), and Stinchcombe (1990).

To discuss how this paper sheds light on the relation between an information partition and a σ-algebra, observe first that the agent's knowledge is summarized by a sub-κ-complete algebra if and only if she is fully introspective and logical. Next, the sub-κ-complete algebra has to have the maximality property with respect to an underlying set algebraic structure. Proposition 3 and Corollary 3 establish the correspondence between the information sets and the self-evident collection: $\mathcal{J}_K = \mathcal{J}(\{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$.

The sub-κ-complete algebra has a particular form of closure under union: if an arbitrary union of self-evident events forms an event, it is contained in the self-evident collection. Also, the sub-κ-complete algebra, by definition, depends on the underlying state space. On the one hand, these points confirm the argument by Dubra and Echenique (2004) in the following two ways. First, an arbitrary union of self-evident events (e.g., information partition cells) may be a self-evident event as long as it is a well-defined event. Thus, for example, if the agent has the finest partition consisting of singleton sets, then every event is self-evident. Second, the finding by Dubra and Echenique (2004) that a σ-algebra may not necessarily characterize information due to the failure of closure under arbitrary union is based on the assumption that an ambient structure is a complete algebra. In fact, their ambient space is the power set algebra $(\Omega, \mathcal{P}(\Omega))$. If an underlying structure forms a complete algebra, then the maximality property reduces to closure under arbitrary union. Hence, if an underlying structure is a complete algebra, this paper provides an epistemic foundation for the finding of Dubra and Echenique (2004) in that the self-evident collection is closed under arbitrary union if and only if the agent is logical and introspective about what she knows (Theorem 1 (4)). If the agent's knowledge is induced from her information sets, then her self-evident collection contains an event that is generated by an arbitrary union of her information sets. Hence, this paper points out the importance of the structure of an ambient space in this strand of literature.

On the other hand, as Hérves-Beloso and Monteiro (2013) argue, if the collection of information sets is countable, then the self-evident collection on a measurable space coincides with the σ-algebra generated by the information sets. For example, if each information set has a positive measure on a probability space as in a standard setting, then the self-evident collection coincides with the σ-algebra generated by the information sets. While this paper does not study probability-theoretical aspects of self-evident collections on a probability space, the paper helps to solve the conflicting views on the sense in which a σ-algebra represents knowledge.[9]

4.2 Relation to a Model of Knowledge in Psychology

Here, I state connections between a model of knowledge in psychology referred to as the "knowledge space theory" (Doignon and Falmagne, 1985, 2016; Falmagne and Doignon, 2011) and my framework. In a particular setting, a self-evident collection turns out to be mathematically equivalent to a "knowledge structure" in their knowledge space theory.[10]

Doignon and Falmagne (1985, 2016) and Falmagne and Doignon (2011) study the knowledge of a single agent regarding subsets of Ω, which, in their work, consists of questions. That is, an ambient set Ω represents a body of knowledge (say, a college-level chemistry), and each state ω is interpreted as a particular question in the body of knowledge Ω.

The knowledge of the agent (say, a college student) is modeled as a pair $(\Omega, \mathcal{J})$, where $\emptyset, \Omega \in \mathcal{J} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{J}$ is closed under arbitrary union. Thus, within the framework of this paper, the agent's knowledge is defined on $\mathcal{D} = \mathcal{P}(\Omega)$ and $\mathcal{J}$ satisfies the maximality property as well as Necessitation (recall Theorem 1). In their work, each set $E \in \mathcal{J}$ is interpreted as a set of questions that the agent is capable

[9] Particularly, if the collection of information sets is uncountable on a probability space, then some information sets have probability zero so that one has to take care of Bayesian updating on such a null set if the posterior belief is derived from a prior distribution. See, for example, Brandenburger and Dekel (1987), Lee (Forthcoming), and Noguchi (2018) for the use of regular conditional probability measures or Kajii and Morris (1997) and Nielsen (1984) for enriching the underlying state space.

[10] Fukuda (2018a) studies a model of knowledge and beliefs, which generalizes a possibility correspondence model, a "local reasoning" model of Fagin and Halpern (1987) in computer science and artificial intelligence, and another knowledge representation referred to as a "surmise system" in Doignon and Falmagne (1985, 2016) and Falmagne and Doignon (2011).

of solving. This means that while the closure under arbitrary union may look like a technical condition in the knowledge space theory,[11] it conceptually means: (i) the agent is logical in that she knows a tautology and any logical consequence of her own knowledge; and (ii) the agent is introspective about her own knowledge in that she knows what she knows.

Doignon and Falmagne (1985, 2016) and Falmagne and Doignon (2011) show one way to construct a knowledge structure is to define a collection $\mathcal{J}_{\preccurlyeq}$ from a pre-ordered set $(\Omega, \preccurlyeq)$. They call the pre-order a surmise relation, and it is interpreted as follows: a question ω' is surmised from a question ω (denoted $\omega \preccurlyeq \omega'$) if and only if it can be surmised that, from observing a correct response to question ω, a correct response would be given to question ω'. Thus, the question ω' is "at least as informative as" ω for assessing the agent's knowledge when $\omega \preccurlyeq \omega'$. The collection $\mathcal{J}_{\preccurlyeq}$ represents knowledge in that each $E \in \mathcal{J}_{\preccurlyeq}$ is an upper set with respect to $\preccurlyeq$ (i.e., $\omega' \in E$ for all $\omega \in E$ and $\omega \preccurlyeq \omega'$). It can be seen that $\mathcal{J}_{\preccurlyeq}$ is closed under arbitrary union and intersection (with $\emptyset, \Omega \in \mathcal{J}_{\preccurlyeq}$). Thus, knowledge derived from $\mathcal{J}_{\preccurlyeq}$ satisfies: Truth Axiom, Monotonicity, Positive Introspection, Necessitation, and Arbitrary Conjunction.

Observe that the above properties characterize the reflexive and transitive possibility correspondence models in ∞-knowledge spaces. Suppose that a pre-ordered set $(\Omega, \preccurlyeq)$ is given. Viewing Ω as a state space, each upper contour set of ω with respect to $\preccurlyeq$ defines the reflexive and transitive possibility correspondence $b_{\preccurlyeq}(\omega) = \{\omega' \in \Omega \mid \omega \preccurlyeq \omega'\}$. In relation to modal logic, the pre-ordered set $(\Omega, \preccurlyeq)$ is referred to as a reflexive and transitive Kripke frame (e.g., Chagrov and Zakharyaschev (1997)). An upper set $E \in \mathcal{J}_{\preccurlyeq}$ with respect to $\preccurlyeq$ is a self-evident event E in terms of $b_{\preccurlyeq}$: $b_{\preccurlyeq}(\omega) \subseteq E$ (i.e., $\omega \in K_{b_{\preccurlyeq}}(E)$) for all $\omega \in E$.

4.3 Information and Topology

While it has often been said in economics, finance, and probability theory that information takes a form of σ-algebras, it has also been said that information takes a form of topologies. For example, Shin (1993) studies knowledge by logical provability, where individual and common knowledge are represented as topologies. Stinchcombe (1990) introduces a topology which he calls a "Bayesian information topology" in his probability-theoretical setting (see also the references therein). Moreover, an agent's knowledge is represented by a topology in computer science, logic, mathematics, and philosophy.[12] Within the framework of this paper, recall that a knowledge operator satisfies: (i) Truth Axiom ($K(E) \subseteq E$), (ii) Monotonicity ($E \subseteq F$ implies $K(E) \subseteq K(F)$), and (iii) Positive Introspection ($K(E) \subseteq KK(E)$). Together with Necessita-

[11] Falmagne and Doignon (2011, p. 38) mention that the closure under arbitrary union "may not be convincing for an educator, *a priori*."

[12] See, for example, Barwise and Etchemendy (1990), Chagrov and Zakharyaschev (1997), Pacuit (2017), Vickers (1989), and the references therein.

tion ($K(\Omega) = \Omega$) and Non-empty Finite Conjunction ($K(E) \cap K(F) \subseteq K(E \cap F)$), the knowledge operator can be identified with an interior operator associated with its self-evident collection $\mathcal{J}_K$, which forms a topology on Ω.

Moreover, information is often represented by a topology which is closed under arbitrary intersection. Within the context of a possibility correspondence model, this is because the knowledge operator on a complete algebra induced by a possibility correspondence satisfies Arbitrary Conjunction ($\bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$ for any $\mathcal{E} \subseteq \mathcal{D}$). In mathematics, a topology which is closed under arbitrary intersection is often called an Alexsandroff topology (see, for example, Pacuit (2017), Vickers (1989), and the references therein).[13] Within the framework of this paper, the fact that a topology (which represents information) is closed under arbitrary intersection corresponds to the arbitrary conjunction ability.

4.4 Common Knowledge

While this paper focuses on representing a single agent's knowledge, one can analyze various forms of interactive knowledge such as common knowledge. To that end, consider interactive knowledge held by a group of agents I, and let K_i be agent i's knowledge operator defined on a κ-complete algebra. Call an event E *common knowledge* among the agents I at a state ω if there is a publicly evident event $F \in \bigcap_{i \in I} \mathcal{J}_{K_i}$ such that $\omega \in F \subseteq E$.[14] While each agent's knowledge is characterized by $\mathcal{J}_{K_i}$, the common knowledge is characterized by its intersection $\bigcap_{i \in I} \mathcal{J}_{K_i}$.

If E is common knowledge, then everybody knows it, everybody knows that everybody knows it, and so forth *ad infinitum*. If $\kappa \geq \aleph_1$ and if every agent's knowledge satisfies Countable Conjunction, the converse also holds. Common knowledge is then characterized as the chain of mutual knowledge.

Since common knowledge is characterized by the collection of publicly evident events $\bigcap_{i \in I} \mathcal{J}_{K_i}$, common knowledge is understood as the infimum of the individual knowledge. For example, Aumann (1976) represents common knowledge by the finest partition which is as coarse as every individual's partition. The common knowledge operator is well defined if the smallest collection including $\bigcap_{i \in I} \mathcal{J}_{K_i}$ and satisfying the

[13] While this paper does not aim at surveying a topology closed under arbitrary intersection, such a collection of sets is also termed as a complete ring (Birkhoff, 1987), a saturated topology (Lorrain, 1969), a principal topology (Steiner, 1966), and so on. Such a topology is also characterized as the one that has the minimal open neighborhood at each point (state) ω. Indeed, it is an information set $b_K(\omega)$ at ω. Also, such a topology has a one-to-one correspondence with a pre-order (a reflexive and transitive binary relation). The pre-ordered space is a reflexive and transitive Kripke frame in modal logic discussed in Section 4.2. The pre-order is often called a specialization pre-order (see, for example, Pacuit (2017), Vickers (1989), and the references therein).

[14] As to the characterization of common knowledge by the collection of publicly evident events, see, for example, Aumann (1999), Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger and Dekel (1987), Geanakoplos (1989), Milgrom (1981), Monderer and Samet (1989), Nielsen (1984), and Shin (1993).

maximality property exists. In this case, the common knowledge operator satisfies Truth Axiom, Positive Introspection, and Monotonicity.

4.5 Concluding Remarks

This paper clarified the way in which an agent's knowledge is represented by various different forms of set algebras depending on the properties of the agent's knowledge and the structure of an underlying state space. Thus, it gave a unified view on why the agent's knowledge takes or does not take a form of a σ-algebra depending on her properties of knowledge and on the underlying structure. The paper also formalized the sense in which a given sub-collection of events has an informational content. The paper then compared representations of knowledge by information sets and a self-evident collection.

One interesting avenue for future research is to study structures of the family of self-evident collections on a given state space. As discussed in the introduction, such pioneering papers as Allen (1983), Cotter (1986), and Stinchcombe (1990) introduce topological structures on the family of sub-σ-algebras on a given probability space in order to study the continuous dependence of economic variables on information. Another interesting avenue for future research is to study the extent to which qualitative or probability p-beliefs (Monderer and Samet, 1989) can be summarized by a set algebra. In doing so, as this paper characterized the failure of Negative Introspection as that of closure under complementation, it would be interesting to connect an agent's probabilistic sophistication and her logical or introspective ability (e.g., characterizations of non-additive beliefs by Ghirardato (2001) and Mukerji (1997) in terms of an agent's bounded perception).

A Appendix

Proof of Proposition 1. Assume the Kripke property. Take any $(\omega, F) \in \Omega \times \mathcal{D}$ such that $E \cap F \neq \emptyset$ for all $E \in \mathcal{D}$ with $\omega \in K(E)$. If $b_K(\omega) \cap F = \emptyset$, then $b_K(\omega) \subseteq F^c$ and thus $F^c \cap F \neq \emptyset$, a contradiction. Conversely, suppose to the contrary that there is $E' \in \mathcal{D}$ such that $b_K(\omega) \subseteq E'$ and $\omega \notin K(E')$. Since $b_K(\omega) \cap (E')^c = \emptyset$, there is $E \in \mathcal{D}$ such that $\omega \in K(E)$ and $E \cap (E')^c = \emptyset$. Since K satisfies Monotonicity, $E \subseteq E'$ implies $\omega \in K(E')$, a contradiction. □

Remark A.1. I prove Equation (1). If $\omega' \in b_K(\omega)$, then for any $E \in \mathcal{D}$ with $\omega \in K(E)$, I have $\omega' \in E$. Since $\omega' \notin \{\omega'\}^c$, it follows that $\omega \in (\neg K)(\{\omega'\}^c) = L_K(\{\omega'\})$. Conversely, suppose to the contrary that there is $E \in \mathcal{D}$ with $\omega' \notin E$ (i.e., $E \subseteq \{\omega'\}^c$) and $\omega \in K(E)$. Monotonicity implies $\omega \in K(\{\omega'\}^c)$, a contradiction.

Proof of Theorem 1. Part (1a). First, $\emptyset \subseteq K(\emptyset)$ implies $\emptyset \in \mathcal{J}_K$. Second, fix $E \in \mathcal{D}$, and I show that $K(E) = \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_K \text{ and } F \subseteq E\} \in \mathcal{J}_K$. For any $F \in \mathcal{J}_K$

with $F \subseteq E$, Monotonicity implies $F \subseteq K(F) \subseteq K(E)$. On the other hand, Positive Introspection and Truth Axiom imply $K(E) \in \{F \in \mathcal{J}_K \mid F \subseteq E\}$.

Part (1b). Assume Non-empty λ-Conjunction. Take $\mathcal{E} \in \mathcal{P}(\mathcal{J}_K) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. I have $\bigcap \mathcal{E} \subseteq \bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$, implying $\bigcap \mathcal{E} \in \mathcal{J}_K$.

Part (1c). Necessitation implies $\Omega \in \mathcal{J}_K$ since $K(\Omega) = \Omega$.

Part (1d). If $E \in \mathcal{J}_K$ then $E = K(E)$ by Truth Axiom. Negative Introspection yields $E^c = (\neg K)(E) \subseteq K(\neg K)(E) = K(E^c)$, i.e., $E^c \in \mathcal{J}_K$.

Part (2a). Truth Axiom of $K_{\mathcal{J}}$ follows because $K_{\mathcal{J}}(E) = \max\{F \in \mathcal{J} \mid F \subseteq E\} \subseteq E$. Next, $K_{\mathcal{J}}(E) \in \mathcal{J}$ implies $K_{\mathcal{J}}(E) \subseteq \max\{F \in \mathcal{J} \mid F \subseteq K_{\mathcal{J}}(E)\} = K_{\mathcal{J}} K_{\mathcal{J}}(E)$, establishing Positive Introspection. Finally, if $E \subseteq F$ then $K_{\mathcal{J}}(E)(\subseteq E) \subseteq F$. Thus, $K_{\mathcal{J}}(E) \subseteq \max\{F' \in \mathcal{J} \mid F' \subseteq F\} = K_{\mathcal{J}}(F)$, establishing Monotonicity.

Part (2b). Take $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. By (2a), $K_{\mathcal{J}}$ satisfies Truth Axiom and Monotonicity. By Truth Axiom, $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}}(E) \subseteq \bigcap \mathcal{E}$. Since $\mathcal{J}$ is closed under non-empty λ-intersection and since $K_{\mathcal{J}}$ satisfies Monotonicity, $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}}(E) \in \mathcal{J}$, and thus $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}}(E) \subseteq K_{\mathcal{J}}(\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}}(E)) \subseteq K_{\mathcal{J}}(\bigcap \mathcal{E})$. Hence, $K_{\mathcal{J}}$ satisfies Non-empty λ-Conjunction.

Part (2c). If $\Omega \in \mathcal{J}$, then $\Omega = \max\{E \in \mathcal{J} \mid E \subseteq \Omega\} = K_{\mathcal{J}}(\Omega)$.

Part (2d). Since $\mathcal{J}$ is closed under complementation, $K_{\mathcal{J}}(E) \in \mathcal{J}$ implies $(\neg K_{\mathcal{J}})(E) \in \mathcal{J}$. Thus, $(\neg K_{\mathcal{J}})(E) \subseteq \max\{F \in \mathcal{J} \mid F \subseteq (\neg K_{\mathcal{J}})(E)\} = K_{\mathcal{J}}(\neg K_{\mathcal{J}})(E)$.

Part (3). First, $K = K_{\mathcal{J}_K}$ follows from Equation (2). Next, I show $\mathcal{J} = \mathcal{J}_{K_{\mathcal{J}}}$. If $E \in \mathcal{J}$, then $E \subseteq \max\{F \in \mathcal{J} \mid F \subseteq E\} = K_{\mathcal{J}}(E)$ and thus $E \in \mathcal{J}_{K_{\mathcal{J}}}$. Conversely, if $E \in \mathcal{J}_{K_{\mathcal{J}}}$, then $E = K_{\mathcal{J}}(E) = \max\{F \in \mathcal{J} \mid F \subseteq E\} \in \mathcal{J}$ by the maximality property of $\mathcal{J}$.

Part (4). First, I show that the maximality property implies that $\mathcal{J}_K$ is closed under κ-union, where ∞-union refers to arbitrary union. The maximality property, by definition, implies $\emptyset \in \mathcal{J}_K$, i.e., $\mathcal{J}_{K_i}$ is closed under the empty union. Let $\mathcal{E} \in \mathcal{P}(\mathcal{J}_K) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. For any $E \in \mathcal{E}$, I have $E \subseteq K(E) \subseteq K(\bigcup \mathcal{E})$, and thus $\bigcup \mathcal{E} \subseteq K(\bigcup \mathcal{E})$, establishing $\bigcup \mathcal{E} \in \mathcal{J}_K$. Indeed, for any $\mathcal{E} \subseteq \mathcal{J}_K$, if $\bigcup \mathcal{E} \in \mathcal{D}$ then $\bigcup \mathcal{E} \in \mathcal{J}_K$.

Second, I establish the converse when $\kappa = \infty$. Suppose that $\mathcal{J}_K$ is closed under arbitrary union. For each $E \in \mathcal{D}$,

$$\max\{F \in \mathcal{D} \mid F \in \mathcal{J}_K \text{ and } F \subseteq E\} = \bigcup \{F \in \mathcal{D} \mid F \in \mathcal{J}_K \text{ and } F \subseteq E\} \in \mathcal{J}_K.$$

Finally, I show that if $\mathcal{E} \subseteq \mathcal{J}$ satisfies $\bigcup \mathcal{E} \in \mathcal{D}$ then $\bigcup \mathcal{E} \in \mathcal{J}$. Since $E \subseteq \max\{F \in \mathcal{J} \mid F \subseteq \bigcup \mathcal{E}\}$ for all $E \in \mathcal{E}$, it follows that $\bigcup \mathcal{E} = \max\{F \in \mathcal{J} \mid F \subseteq \bigcup \mathcal{E}\} \in \mathcal{J}$. □

Remark A.2. I provide three counterexamples to Theorem 1 in which one of Truth Axiom, Monotonicity, and Positive Introspection fails. Throughout the examples, let $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{D} = \mathcal{P}(\Omega)$.

First, define K and K' as follows: $K(\emptyset) = K(\{\omega_2\}) = \emptyset$ and $K(\{\omega_1\}) = K(\Omega) = \{\omega_1\}$; and $K'(\emptyset) = \emptyset$ and $K'(E) = \{\omega_1\}$ for any $E \in \mathcal{P}(\Omega) \setminus \{\emptyset\}$. Then, K' violates Truth Axiom, $K \neq K'$, and $\mathcal{J}_K = \mathcal{J}_{K'}$.

Second, let K and K' be defined as follows: $K(\{\omega_1\}) = K(\Omega) = \{\omega_1\}$ and $K(\emptyset) = K(\{\omega_2\}) = \emptyset$; and $K'(\{\omega_1\}) = \{\omega_1\}$ and $K'(E) = \emptyset$ for any $E \in \mathcal{P}(\Omega) \setminus \{\{\omega_1\}\}$. Then, K' violates Monotonicity, $K \neq K'$, and $\mathcal{J}_K = \mathcal{J}_{K'}$.

Third, define K and K' as follows: $K(\cdot) = \emptyset$; and $K'(\Omega) = \{\omega_1\}$ and $K'(E) = \emptyset$ for any $E \in \mathcal{D} \setminus \{\Omega\}$. Then, K' violates Positive Introspection, $K \neq K'$, and $\mathcal{J}_K = \mathcal{J}_{K'}$.

Proof of Corollary 1. Truth Axiom and Negative Introspection imply Positive Introspection as $K(E) = (\neg K)(\neg K)(E) = K(\neg K)(\neg K)(E) = KK(E)$ (e.g., Aumann (1999, p. 270)). By Theorem 1, K satisfies κ-Conjunction (including Necessitation) iff $\mathcal{J}_K$ is closed under κ-intersection. Now, by Theorem 1, $\mathcal{J}_K$ is closed under κ-union and complementation. Thus, it is closed under κ-intersection. □

Remark A.3. I provide a direct alternative proof of Corollary 1. Necessitation follows from Negative Introspection and $\emptyset = K(\emptyset)$ (which follows from Truth Axiom) because $\Omega = (\neg K)(\emptyset) \subseteq K(\neg K)(\emptyset) = K(\Omega)$. Thus, I establish Non-empty κ-Conjunction. Take $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$.

Recall (again) that Truth Axiom and Negative Introspection yield Positive Introspection. By Positive Introspection and Monotonicity, $\bigcup_{E \in \mathcal{E}} K(E) \subseteq \bigcup_{E \in \mathcal{E}} KK(E) \subseteq K(\bigcup_{E \in \mathcal{E}} K(E))$. Negative Introspection and Truth Axiom imply $(\neg K)(\neg K)(E) = K(E) \subseteq E$.

Now, it follows from Monotonicity, Negative Introspection, and Truth Axiom that

$$(\neg K)\left(\neg \bigcup_{E \in \mathcal{E}} K(\neg K)(E)\right) \subseteq (\neg K)(\neg K)\left(\bigcup_{E \in \mathcal{E}} K(\neg K)(E)\right) \subseteq \bigcup_{E \in \mathcal{E}} K(\neg K)(E).$$

Thus, taking the complementation yields

$$\bigcap_{E \in \mathcal{E}} K(E) = \bigcap_{E \in \mathcal{E}} (\neg K)^2(E) \subseteq K\left(\neg \bigcup_{E \in \mathcal{E}} K(\neg K)(E)\right) = K\left(\bigcap_{E \in \mathcal{E}} (\neg K)^2(E)\right) \subseteq K\left(\bigcap \mathcal{E}\right).$$

Proof of Proposition 2. 1. First, $\mathcal{D}' \subseteq \mathcal{J}(\mathcal{D}')$ because $K_{\mathcal{D}'}(E) = E$ for any $E \in \mathcal{D}'$. Second, I show that $\mathcal{J}(\mathcal{D}')$ satisfies the maximality property. By definition, $\emptyset = K_{\mathcal{D}'}(\emptyset) \in \mathcal{J}(\mathcal{D}')$. For each $E \in \mathcal{D}$,

$$\max\{F \in \mathcal{J}(\mathcal{D}') \mid F \subseteq E\} = \{\omega \in \Omega \mid \text{there is } F' \in \mathcal{D}' \text{ with } \omega \in F' \subseteq E\} \in \mathcal{J}(\mathcal{D}').$$

Also, Expression (5) holds.

Third, take any collection $\mathcal{J}'$ including $\mathcal{D}'$ and satisfying the maximality property. I show $\mathcal{J}(\mathcal{D}') \subseteq \mathcal{J}'$. Take any $E \in \mathcal{J}(\mathcal{D}')$. For any $\omega \in E$, there is $F \in \mathcal{D}'$ with $\omega \in F \subseteq E$. Since $\mathcal{D}' \subseteq \mathcal{J}'$, I have $F \in \mathcal{J}'$. Hence,

$$E = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}' \text{ such that } \omega \in F \subseteq E\} = \max\{F \in \mathcal{J}' \mid F \subseteq E\} \in \mathcal{J}'.$$

Fourth, I establish Expression (4). Let $E \in \mathcal{D}$ be such that if $\omega \in E$ then there is $F \in \mathcal{D}'$ with $\omega \in F \subseteq E$. Then, $E = K_{\mathcal{D}'}(E) \in \mathcal{J}(\mathcal{D}')$. Conversely, if $\omega \in K_{\mathcal{D}'}(E)$ then there is $F \in \mathcal{D}'$ such that $\omega \in F \subseteq E$. Then, $\omega \in F = K_{\mathcal{D}'}(F) \subseteq K_{\mathcal{D}'}(E)$.

2. First, $\mathcal{J}(\mathcal{D}') \cup \{\Omega\}$ is an information basis including $\mathcal{D}' \cup \{\Omega\}$. Thus, $\mathcal{J}(\mathcal{D}' \cup \{\Omega\}) \subseteq \mathcal{J}(\mathcal{D}') \cup \{\Omega\}$. Second, $\mathcal{J}(\mathcal{D}') \cup \{\Omega\} \subseteq \mathcal{J}(\mathcal{D}' \cup \{\Omega\})$.

3. It is enough to prove the first assertion. First, by construction, $\mathcal{J}(\hat{\mathcal{D}}')$ includes $\mathcal{D}'$ and satisfies the maximality property. Second, I show that $\mathcal{J}(\hat{\mathcal{D}}')$ is closed under non-empty λ-intersection. Take $\mathcal{E} \subseteq \mathcal{J}(\hat{\mathcal{D}}')$ with $0 < |\mathcal{E}| < \lambda$. If $\omega \in \bigcap \mathcal{E}$, then, for each $E \in \mathcal{E}$, there is $F_E \in \hat{\mathcal{D}}'$ such that $\omega \in F_E \subseteq E$. Then, $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap \mathcal{E}$ and $\bigcap_{E \in \mathcal{E}} F_E \in \hat{\mathcal{D}}'$. Third, for any collection $\mathcal{J}'$ including $\mathcal{D}'$, satisfying the maximality property, and being closed under non-empty λ-intersection, I have $\hat{\mathcal{D}}' \subseteq \mathcal{J}'$, establishing $\mathcal{J}(\hat{\mathcal{D}}') \subseteq \mathcal{J}'$.

4. For the first statement, the following counterexample shows that $\mathcal{J}(\mathcal{D}')$ is not necessarily closed under complementation even if $\mathcal{D}'$ is. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. If $\mathcal{D}' = \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{w_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\}$ then $\mathcal{J}(\mathcal{D}') = \mathcal{P}(\Omega) \setminus \{\{\omega_2\}\}$.

   Next, $\mathcal{J}(\mathcal{D}')$ satisfies the maximality property, and $K_{\mathcal{D}'} = K_{\mathcal{J}(\mathcal{D}')}$ holds by part (1) of this proposition. Thus, the second statement follows from Theorem 1.

5. The second assertion follows from the first, as it implies that $\mathcal{J}(\mathcal{D}')$ is closed under κ-union. Thus, I show the first assertion. Observe first that

$$\bigcap\{\mathcal{J}' \in \mathcal{P}(\mathcal{D}) \mid \mathcal{D}' \subseteq \mathcal{J}' \text{ and } \mathcal{J}' \text{ is closed under } \kappa\text{-union}\} = \{\bigcup \mathcal{E} \in \mathcal{D} \mid \mathcal{E} \subseteq \mathcal{D}'\}.$$

   Next, it can be seen that this collection satisfies the maximality property and includes $\mathcal{D}'$. Thus, this collection includes $\mathcal{J}(\mathcal{D}')$. To get the converse set inclusion, observe that $\mathcal{J}(\mathcal{D}')$ includes $\mathcal{D}'$ and that $\mathcal{J}(\mathcal{D}')$ is closed under κ-union (Theorem 1 (4)).

□

Proof of Corollary 2. The first condition implies $\mathcal{E} \subseteq \mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$, and thus the second condition follows. Conversely, if $F \in \mathcal{J}(\mathcal{E})$ and $\omega \in F$ then it follows from Equation (4) that there is $E \in \mathcal{E}$ with $\omega \in E \subseteq F$. By the second condition, there is $E' \in \mathcal{E}'$ such that $\omega \in E' \subseteq E \subseteq F$. Hence, $F \in \mathcal{J}(\mathcal{E}')$. □

Proof of Proposition 3. 1. Recall that $E = K(E)$ for any $E \in \mathcal{J}_K$. Thus, $b_K(\omega) = \bigcap\{E \in \mathcal{D} \mid \omega \in K(E)\} \subseteq \bigcap\{E \in \mathcal{J}_K \mid \omega \in E\}$. Conversely, since $K(E) \in \mathcal{J}_K$ for all $E \in \mathcal{D}$, it follows that $\bigcap\{E \in \mathcal{J}_K \mid \omega \in E\} \subseteq K(F) \subseteq F$ for each $F \in \mathcal{D}$ with $\omega \in K(F)$. Thus, $\bigcap\{E \in \mathcal{J}_K \mid \omega \in E\} \subseteq \bigcap\{E \in \mathcal{D} \mid \omega \in K(E)\} = b_K(\omega)$.

2. For ease of notation, let $\mathcal{J} := \{E \in \mathcal{D} \mid E = \bigcup_{\omega \in F} b_K(\omega)$ for some $F \in \mathcal{P}(\Omega)\}$ and $\mathcal{D}' := \{b_K(\omega) \in \mathcal{D} \mid \omega \in \Omega\}$. First, I show that (2a) implies (2b). Take $E \in \mathcal{J}_K$. For any $\omega \in E = K(E)$, Truth Axiom implies $\omega \in b_K(\omega) \subseteq E$. Thus, $E = \bigcup_{\omega \in E} b_K(\omega) \in \mathcal{J}$. Conversely, take $E = \bigcup_{\omega \in F} b_K(\omega) \in \mathcal{J}$. If $\omega \in E$ then there is $\omega' \in F$ such that $\omega \in b_K(\omega')$. By Positive Introspection, $b_K(\omega) \subseteq b_K(\omega') \subseteq E$. By the Kripke property, $\omega \in K(E)$. Thus, $E \in \mathcal{J}_K$. I remark that in this part of the proof I have not used the assumption that each $b_K(\omega)$ is an event.

   Second, I show that (2b) implies (2c). I start with showing that $\mathcal{D}'$ is an information basis. For any $E \in \mathcal{D}$,

   $$\{\omega \in \Omega \mid \text{there is } \omega' \in \Omega \text{ such that } \omega \in b_K(\omega') \subseteq E\} = \bigcup_{\omega' \in \Omega : b_K(\omega') \subseteq E} b_K(\omega') \in \mathcal{J}_K.$$

   Next, since $\mathcal{D}' \subseteq \mathcal{J}_K$, it follows $\mathcal{J}(\mathcal{D}') \subseteq \mathcal{J}_K$. For the converse set inclusion, it follows from (2b) and Theorem 1 (4) that $\mathcal{J}_K = \mathcal{J} \subseteq \mathcal{J}(\mathcal{D}')$.

   Third, I show that (2c) implies (2a). Since $b_K(\omega) \in \mathcal{J}_K$, I have $b_K(\omega) \subseteq K(b_K(\omega))$. Truth Axiom implies $\omega \in b_K(\omega) \subseteq K(b_K(\omega))$. Thus, by Monotonicity, $\omega \in K(E)$ if (and only if) $b_K(\omega) \subseteq E$.

3. Since K satisfies Negative Introspection as well as Truth Axiom and Monotonicity, Theorem 1 and Corollary 1 imply that $\mathcal{J}_K$ is a sub-κ-complete algebra.

□

Proof of Corollary 3. First, it can be seen that (1) and (2) are equivalent. Second, as discussed in the main text (recall Expression (3)), (2) and (3) are equivalent. Third, it follows from Proposition 3 that (3) and (4) are equivalent. □

References

[1] B. Allen. "Neighboring Information and Distributions of Agents' Characteristics under Uncertainty". J. Math. Econ. 12 (1983), 63–101.

[2] R. J. Aumann. "Agreeing to Disagree". Ann. Statist. 4 (1976), 1236–1239.

[3] R. J. Aumann. "Interactive Epistemology I, II". Int. J. Game Theory 28 (1999), 263–300, 301–314.

[4] M. Bacharach. "Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge". J. Econ. Theory 37 (1985), 167–190.

[5] J. Barwise and J. Etchemendy. “Information, Infons, and Inference”. Situation Theory and Its Applications. Ed. by R. Cooper, K. Mukai, and J. Perry. Vol. 1. CSLI, 1990, 33–78.

[6] P. Billingsley. Probability and Measure. Anniversary Edition. Wiley, 2012.

[7] K. Binmore and A. Brandenburger. “Common Knowledge and Game Theory”. Essays on the Foundations of Game Theory. Ed. by K. Binmore. Basil Blackwell, 1990, 105–150.

[8] G. Birkhoff. “Rings of Sets”. Duke Math. J. 3 (1987), 443–454.

[9] E. S. Boylan. “Equiconvergence of Martingales”. Ann. Math. Stat. 42 (1971), 552–559.

[10] A. Brandenburger and E. Dekel. “Common Knowledge with Probability 1”. J. Math. Econ. 16 (1987), 237–245.

[11] A. Brandenburger, E. Dekel, and J. Geanakoplos. “Correlated Equilibrium with Generalized Information Structures”. Games Econ. Behav. 4 (1992), 182–201.

[12] A. Chagrov and M. Zakharyaschev. Modal Logic. Oxford University Press, 1997.

[13] Y.-C. Chen, X. Luo, and C. Qu. “Rationalizability in General Situations”. Econ. Theory 61 (2016), 147–167.

[14] J. Correia-da-Silva and C. Hervés-Beloso. “Private Information: Similarity and Compatibility”. Econ. Theory 30 (2007), 395–407.

[15] K. D. Cotter. “Similarity of Information and Behavior with a Pointwise Convergence Topology”. J. Math. Econ. (1986).

[16] E. Dekel and F. Gul. “Rationality and Knowledge in Game Theory”. Advances in Economics and Econometrics: Theory and Applications, Seventh World Congress. Ed. by D. M. Kreps and K. F. Wallis. Vol. 1. Econometric Society Monograph. Cambridge University Press, 1997, 87–172.

[17] E. Dekel, B. L. Lipman, and A. Rustichini. “Standard State-Space Models Preclude Unawareness”. Econometrica 66 (1998), 159–173.

[18] J.-P. Doignon and J.-C. Falmagne. “Spaces for the Assessment of Knowledge”. Int. J. Man Mach. Stud. 23 (1985), 175–196.

[19] J.-P. Doignon and J.-C. Falmagne. “Knowledge Spaces and Learning Spaces”. New Handbook of Mathematical Psychology. Cambridge University Press, 2016, 274–321.

[20] J. Dubra and F. Echenique. “Information is not about Measurability”. Math. Soc. Sci. 47 (2004), 177–185.

[21] R. Fagin and J. Y. Halpern. “Belief, Awareness, and Limited Reasoning”. Art. Intell. 34 (1987), 39–76.

[22] J.-C. Falmagne and J.-P. Doignon. Learning Spaces: Interdisciplinary Applied Mathematics. Springer, 2011.

[23] M. F. Friedell. "On the Structure of Shared Awareness". Behav. Sci. 14 (1969), 28–39.

[24] S. Fukuda. "An Information Correspondence Approach to Bridging Knowledge-Belief Representations in Economics and Mathematical Psychology". 2018.

[25] S. Fukuda. "Representing Unawareness on State Spaces". 2018.

[26] J. Geanakoplos. Games Theory without Partitions, and Applications to Speculation and Consensus. Cowles Foundation Discussion Paper No. 914, Yale University. 1989.

[27] P. Ghirardato. "Coping with Ignorance: Unforeseen Contingencies and Non-additive Uncertainty". Econ. Theory 17 (2001), 247–276.

[28] P. R. Halmos. "Algebraic Logic I. Monadic Boolean Algebras." Compos. Math. 12 (1955), 217–249.

[29] C. Hérves-Beloso and P. K. Monteiro. "Information and σ-algebras". Econ. Theory 54 (2013), 405–418.

[30] A. Kajii and S. Morris. "Common p-Belief: The General Case". Games Econ. Behav. 18 (1997), 73–82.

[31] J. J. Lee. "Formalization of Information: Knowledge and Belief". Econ. Theory (Forthcoming).

[32] B. L. Lipman. "A Note on the Implications of Common Knowledge of Rationality". Games Econ. Behav. 6 (1994), 114 –129.

[33] F. Lorrain. "Notes on Topological Spaces with Minimum Neighborhoods". Am. Math. Mon. 76 (1969), 616–627.

[34] M. Meier. "Finitely Additive Beliefs and Universal Type Spaces". Ann. Probab. 34 (2006), 386–422.

[35] M. Meier. "Universal Knowledge-Belief Structures". Games Econ. Behav. 62 (2008), 53–66.

[36] P. Milgrom. "An Axiomatic Characterization of Common Knowledge". Econometrica 49 (1981), 219–222.

[37] S. Modica and A. Rustichini. "Awareness and Partitional Information Structures". Theory Decis. 37 (1994), 107–124.

[38] S. Modica and A. Rustichini. "Unawareness and Partitional Information Structures". Games Econ. Behav. 27 (1999), 265–298.

[39] D. Monderer and D. Samet. "Approximating Common Knowledge with Common Beliefs". Games Econ. Behav. 1 (1989), 170–190.

[40] D. Monderer and D. Samet. “Proximity of Information in Games with Incomplete Information”. Math. Oper. Res. 21 (1996), 707–725.

[41] S. Morris. “The Logic of Belief and Belief Change: A Decision Theoretic Approach”. J. Econ. Theory 69 (1996), 1–23.

[42] S. Mukerji. “Understanding the Nonadditive Probability Decision Model”. Econ. Theory 9 (1997), 23–46.

[43] L. T. Nielsen. “Common Knowledge, Communication, and Convergence of Beliefs”. Math. Soc. Sci. 8 (1984), 1–14.

[44] M. Noguchi. “Alpha Cores of Games with Nonatomic Asymmetric Information”. J. Math. Econ. 75 (2018), 1–12.

[45] E. Pacuit. Neighborhood Semantics for Modal Logic. Springer, 2017.

[46] R. Radner. “Competitive Equilibrium under Uncertainty”. Econometrica 36 (1968), 31–58.

[47] D. Samet. “Ignoring Ignorance and Agreeing to Disagree”. J. Econ. Theory 52 (1990), 190–207.

[48] D. Samet. “S5 Knowledge without Partitions”. Synthese 172 (2010), 145–155.

[49] B. C. Schipper. “Awareness”. Handbook of Epistemic Logic. Ed. by H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. P. Kooi. College Publications, 2015, 77–146.

[50] H. S. Shin. “Logical Structure of Common Knowledge”. J. Econ. Theory 60 (1993), 1–13.

[51] A. K. Steiner. “The Lattice of Topologies: Structure and Complementation”. Trans. Am. Math. Soc. 122 (1966), 379–398.

[52] M. B. Stinchcombe. “Bayesian Information Topologies”. J. Math. Econ. 19 (1990), 233–253.

[53] S. Vickers. Topology Via Logic. Cambridge University Press, 1989.

[54] R. Wilson. “Information, Efficiency, and the Core of an Economy”. Econometrica 46 (1978), 807–816.

[55] N. C. Yannelis. “The Core of an Economy with Differential Information”. Econ. Theory 1 (1991), 183–197.