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Satoshi Fukuda

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IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
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Representing Unawareness on State Spaces*

Satoshi Fukuda†

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Abstract

I study unawareness by the lack of knowledge on a generalized state space. In order to understand and contrast properties of unawareness in a non-partitional standard state space model and a partitional generalized state space model, I provide a generalized framework that accommodates both models. I ask: when and how a generalized (in particular, standard) state space model has a sensible form of unawareness; and how unawareness relates to ignorance and possibility. First, unawareness can only take two forms: an agent is ignorant of knowing that she does not know an event; and the agent is ignorant of knowing an event. In either case, unawareness is also associated with the ignorance of the possibility of knowing an event. Second, the agent, who is unaware of an event, is ignorant (but not necessarily unaware) of being unaware of it. Third, the agent, facing infinitely many objects of knowledge, may know that there is an event of which she is unaware, while she cannot know that she is unaware of any particular event. Fourth, getting more information can cause the agent to become unaware of some event.

JEL Classification: C70, D83
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1 Introduction

A state space model of knowledge, since Aumann (1976, 1999), has been developed to model rational agents who reason interactively with each other. One of the subsequent

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†Department of Decision Sciences and IGIER, Bocconi University, Milan 20136, Italy.

research agendas has been to accommodate interactive knowledge among "boundedly rational" agents who lack logical or introspective reasoning abilities. Especially, unawareness has been actively investigated in economics since Modica and Rustichini (1994, 1999): an agent is unaware of a statement if she does not know it and she does not know that she does not know it.[1]

Dekel, Lipman, and Rustichini (1998), however, establish the following negative result. No state space model of knowledge can capture a sensible form of unawareness if a logical agent satisfies the following three axioms: Plausibility, KU Introspection, and AU Introspection. Plausibility states that if the agent is unaware of an event, then she does not know it and she does not know that she does not know it. Under KU Introspection, the agent does not know that she is unaware of any particular event. AU Introspection states that if the agent is unaware of an event, then she is unaware of being unaware of that event. Modica and Rustichini (1994, Theorem) demonstrate another negative result when unawareness is symmetric (i.e., the unawareness of an event entails that of its negation). Since then, in order to represent non-trivial forms of unawareness satisfying desirable features including these three axioms, the research focus on unawareness has shifted to scrutinizing concepts of unawareness from the lack of knowledge to the lack of conception on an enhanced structure (e.g., a generalized state space consisting of multiple sub-spaces).[2]

This paper, going back to the original idea of capturing unawareness as the lack of knowledge, studies how state space models can (and cannot) capture a sensible form of unawareness. I provide a generalized model of unawareness on an enriched state space which also nests a standard state space model of non-introspective knowledge. My framework enables one to directly compare unawareness represented on a generalized state space such as Heifetz, Meier, and Schipper (2006) on the one hand and a standard "non-partitional" state space on the other.

I impose the following conditions on agents' knowledge and unawareness. First, agents are assumed to be logical and introspective about their own knowledge. Namely, each agent's knowledge satisfies at least the following three properties: (i) Truth Axiom (the agent can only know what is true), (ii) Positive Introspection (if the agent knows an event then she knows that she knows it), and (iii) Monotonicity (the agent knows any logical consequence of what she knows). The first property distinguishes knowledge from belief. While I drop Negative Introspection (if the agent does not know an event then she knows that she does not know it) as such agent is never unaware of any event, I assume that agents are "rational" to the extent that they are introspective about their own knowledge. These properties are assumed in standard non-partitional (reflexive and transitive) state space models involving boundedly ra-

[1]Other pioneering papers include Fagin and Halpern (1987) and Pires (1994).

[2]Pioneering attempts along this line of research include: Board and Chung (2007, 2008), Board, Chung, and Schipper (2011), Galanis (2011, 2013), Halpern (2001), Halpern and Rêgo (2008, 2009), Heifetz, Meier, and Schipper (2006, 2008, 2013), Heinsalu (2014), and Li (2009). See Schipper (2015) for an overview.

tional agents as well as generalized state space models.[3] I do not require Necessitation, i.e., the knowledge of a tautology.

Second, I define unawareness solely in terms of (the lack of) knowledge. An agent is k^n-*unaware* of an event if she does not know it, she does not know that she does not know it, and so forth n times, including the case of $n = \infty$. Thus, notions of unawareness are derived from given properties of knowledge and a level of the lack of knowledge.

I ask the following three strands of questions. First, I ask conditions on knowledge under which the derived notions of unawareness have a non-trivial (or trivial) form in state space models. In particular, I aim to answer the question raised by Dekel, Lipman, and Rustichini (1998) and subsequently studied by Chen, Ely, and Luo (2012): which of the previous three axioms is to be retained to represent an interesting form of unawareness in a standard state space model?

Second, I scrutinize how different notions of unawareness as the lack of knowledge and as the lack of conception (as in Heifetz, Meier, and Schipper (2006, 2008)) relate with each other. I ask when these notions of unawareness coincide. I also study distinct implications of unawareness based on these different notions.

Third, I relate the derived notions of unawareness to such notions derived from knowledge as possibility and ignorance. Following Modica and Rustichini (1999), an agent considers an event E possible if she does not know its negation $\neg E$ and if she is aware (not unaware) of E. Thus, unlike a standard state space model, the notion of possibility depends also on that of (un)awareness. The agent is ignorant of an event E if she does not know E and she does not know its negation $\neg E$ (e.g., Lehrer and Samet (2011)). That is, she is ignorant of E if shes does not know "whether" E is true (e.g., Hintikka (1962) and Hart, Heifetz, and Samet (1996)).

The results are as follows. First, three levels of lack of knowledge imply any higher level. Thus, the derived notions of unawareness reduce to the two forms: either two levels of lack of knowledge or infinitely many levels of lack of knowledge. I characterize each form of unawareness in terms of ignorance and possibility (Proposition 1). I show that k^2-unawareness is equivalent to the ignorance of own knowledge: an agent is k^2-unaware of an event E if and only if (hereafter, iff) she is ignorant of (not) knowing E. Next, the agent is k^∞-unaware of an event E iff she is ignorant of (not) knowing that she does not know E. Indeed, the agent is k^∞-unaware of an event E iff she is k^2-unaware of not knowing E. In either case, I also show that the agent is (k^n-)unaware of an event E iff she is ignorant of the possibility that she knows E. Thus, unawareness is a particular form of ignorance. I use this result to show that k^2-unawareness coincides with the lack of conception iff it does with k^∞-unawareness. In fact, these two forms of k^n-unawareness coincide iff k^2-unawareness satisfies AU Introspection (or still equivalently, Symmetry). Thus, if these two forms of k^n-unawareness coincide

[3]Previous studies on such non-partitional models include: Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Rubinstein and Wolinsky (1990), Samet (1990, 1992) and Shin (1993).

in a standard state space model, then unawareness becomes rather degenerate.

Next, I characterize (in Proposition 2) when a state space model is non-trivial. For example, any properly non-partitional standard state space model can capture a non-trivial form of k^2-unawareness.

What properties of unawareness do state space models satisfy? I show (in Proposition 3) that unawareness satisfies such properties as Plausibility, KU Introspection, the converse of AU Introspection, and the property which I call JU Introspection ("J" in JU Introspection refers to the knowing-whether operator in Hart, Heifetz, and Samet (1996)). The converse of AU introspection states that an agent, who is unaware of being unaware of an event E, is unaware of E. Under JU Introspection, an agent, who is unaware of an event E, is ignorant of being unaware of E.

To restate, if agents are logical and introspective and notions of unawareness are defined in terms of the lack of knowledge, then any state space model satisfies Plausibility, KU Introspection, and JU Introspection (instead of AU Introspection). I also examine (in Proposition 4) properties of unawareness (e.g., AU Introspection and Symmetry) which lead to a degenerate form in a standard state space model.

Finally, I study the following two properties of unawareness that hinge on AU Introspection, that is, two properties that would differ between a non-partitional standard state space model and a partitional generalized state space model. First, recall that, under KU Introspection, there is no state at which an agent knows that she is unaware of a particular event. Can the agent know her own unawareness? I define an event that captures whether there is an event of which the agent is unaware in a simple manner.[4] I show (in Proposition 5) that if a given state space model has an infinite number of objects of knowledge and if the agent's awareness fails AU Introspection, then it can be the case that she knows there is an event of which she is unaware (while she does not know that she is unaware of any particular event). If the given model has finitely many objects of knowledge or if the agent's unawareness satisfies AU Introspection, then the agent does not know that there is an event of which she is unaware.

Second, while unawareness as the lack of conception is monotonic in knowledgeability, unawareness as the lack of knowledge may be non-monotonic in knowledgeability. Specifically, I show by example that getting more information can cause an agent to become unaware of some event in the absence of AU Introspection. I also study (in Proposition 6 and Corollary 1) possible forms of monotonicity of unawareness.

The paper is organized as follows. Section 2 provides the generalized-state-space-based framework. Section 3 studies properties of unawareness. Section 3.1 restates unawareness in terms of ignorance and possibility. Section 3.2 characterizes non-trivial unawareness. Section 3.3 investigates the existing axioms of unawareness. Section 4 studies knowledge of self-unawareness (Section 4.1) and (non-)monotonicity of un-

[4]See also Board and Chung (2007), Halpern and Rêgo (2009), Schipper (2015, Section 3.5), and the references therein for representing the event that an agent is unaware of *something* using the first-order logic.

awareness in knowledgeability (Section 4.2). Section 5 provides concluding remarks. Proofs are relegated to Appendix A.

2 Information Structures

This section presents the framework, which I call an information structure. The information structure represents agents' knowledge and unawareness by their knowledge operators on a generalized state space. On the one hand, the information structure generalizes the unawareness structure of Heifetz, Meier, and Schipper (2006) in which agents' knowledge operators are induced from generalized possibility correspondences. On the other hand, the information structure generalizes a non-partitional possibility correspondence model on a standard state space.

Throughout the paper, let I denote a non-empty set of *agents*. I introduce an underlying generalized state space of Heifetz, Meier, and Schipper (2006) in an abstract way.[5] A *generalized state space* $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle$ consists of the following three primitives. First, $(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}$ is a non-empty collection of complete algebras of sets with the following properties. Each $S_\alpha \in \Sigma := \{S_\alpha\}_{\alpha\in\mathcal{A}}$ is non-empty and referred to as a *subspace*. The collection Σ is assumed to be disjoint. Each S_α is endowed with a collection $\mathcal{D}_\alpha$ of subsets of S_α (i.e., $\mathcal{D}_\alpha$ is a subset of the power set $\mathcal{P}(S_\alpha)$) that is closed under arbitrary union, arbitrary intersection, and complementation. I follow the conventions that $\bigcup \emptyset = \emptyset \in \mathcal{D}_\alpha$ and $\bigcap \emptyset = S_\alpha \in \mathcal{D}_\alpha$. The set of *states of the world* is the entire union $\Omega := \bigcup_{\alpha\in\mathcal{A}} S_\alpha$.

Second, $\langle \Sigma, \succeq\rangle$ is a complete lattice. The partial order $\succeq$ ranks subspaces Σ by amounts of "concepts" or "expressive power." Third, $r := (r_S^{S'})_{S'\succeq S}$ is a collection of surjective projections $r_S^{S'} : (S', \mathcal{D}') \to (S, \mathcal{D})$ for each pair $(S, S') \in \Sigma^2$ with $S' \succeq S$. I assume that: (i) $(r_S^{S'})^{-1}(B) \in \mathcal{D}'$ for all $B \in \mathcal{D}$; (ii) each r_S^S is the identity mapping; and (iii) $S'' \succeq S' \succeq S$ implies $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. The generalized state space is *standard* if Σ is a singleton, i.e., $\Sigma = \{\Omega\}$.

Events are objects of agents' knowledge and unawareness. Formally, an *event* is a pair $(B^\uparrow, S) \in \mathcal{P}(\Omega) \times \Sigma$ with $B^\uparrow := \bigcup\{(r_S^{S'})^{-1}(B) \in \mathcal{P}(\Omega) \mid S' \succeq S \text{ for some } S' \in \Sigma\}$ and $B \in \mathcal{D}$. Define the *domain* $\mathcal{E}$ as the collection of events. The domain on a standard state space is identified with the underlying complete algebra on Ω.

Fix an event $(B^\uparrow, S_\alpha)$. Call S_α the *base space* of $(B^\uparrow, S_\alpha)$ (or $B^\uparrow$), and denote $S(B^\uparrow, S_\alpha) = S_\alpha$ (or $S(B^\uparrow) = S_\alpha$). Call B the *basis* of $B^\uparrow$. For an event (E, S), denote by $B(E)$ the basis of E (i.e., $E = B^\uparrow(E) := (B(E))^\uparrow$). Denote $\overline{B}^\uparrow := (B^\uparrow, S(B^\uparrow)) \in \mathcal{E}$ with the convention to denote by $\overline{\emptyset}^S$ the event $(\emptyset^S, S) := (\emptyset^\uparrow, S) = (\emptyset, S) \in \mathcal{E}$.

I introduce the following four operations on $\mathcal{E}$. The first is a partial order $\leqslant$ on $\mathcal{E}$: $\overline{E} \leqslant \overline{F}$ iff $E \subseteq F$ and $S(E) \succeq S(F)$. The greatest element is $(\Omega, \inf \Sigma) =$

[5]The unawareness structure of Heifetz, Meier, and Schipper (2006) is also related to that of Board and Chung (2008) (see also Board, Chung, and Schipper (2011)) and that of Fagin and Halpern (1987) (see also Halpern and Rêgo (2008)).

$((\inf \Sigma)^{\uparrow}, \inf \Sigma)$ while the least element is $(\emptyset, \sup \Sigma)$.

Second, for any collection of events $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$, define its *conjunction* as

$$\bigwedge_{\lambda \in \Lambda} (B_{\lambda}^{\uparrow}, S_{\lambda}) := \left(\bigcap_{\lambda \in \Lambda} B_{\lambda}^{\uparrow}, \sup_{\lambda \in \Lambda} S_{\lambda} \right) = \left(\left(\bigcap_{\lambda \in \Lambda} (r_{S_{\lambda}}^{\sup_{\lambda \in \Lambda} S_{\lambda}})^{-1}(B_{\lambda}) \right)^{\uparrow}, \sup_{\lambda \in \Lambda} S_{\lambda} \right) \in \mathcal{E}.$$

Since $\bigwedge_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda})$ is indeed the infimum of events $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ in a partially ordered set $\langle \mathcal{E}, \leqslant \rangle$, it forms a complete lattice.

Third, define the *negation* of an event $(B^{\uparrow}, S)$ by $\neg(B^{\uparrow}, S) := (\neg B^{\uparrow}, S) := ((S \setminus B)^{\uparrow}, S) \in \mathcal{E}$. I have $\neg\neg(B^{\uparrow}, S) = (B^{\uparrow}, S)$. By letting $\neg \emptyset^{S} := S^{\uparrow}$ and $\neg S^{\uparrow} := \emptyset^{S}$, I can unambiguously write $\neg\neg B^{\uparrow} = B^{\uparrow}$ for any $(B^{\uparrow}, S) \in \mathcal{E}$. As mentioned in Heifetz, Meier, and Schipper (2006), if $S(E) = S(F)$, then $E \subseteq F$ iff $\neg F \subseteq \neg E$.

Fourth, define the *disjunction* of $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ as $\bigvee_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda}) := \neg(\bigwedge_{\lambda \in \Lambda} \neg(B_{\lambda}^{\uparrow}, S_{\lambda})) \in \mathcal{E}$. Note that the disjunction $\bigvee_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda})$ is generally different from the supremum of $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ in $\langle \mathcal{E}, \leqslant \rangle$. As in Heifetz, Meier, and Schipper (2006), the following can be verified: (i) if $S = S_{\lambda}$ for all $\lambda \in \Lambda$, then $\bigvee_{\lambda \in \Lambda} B_{\lambda}^{\uparrow} = \bigcup_{\lambda \in \Lambda} B_{\lambda}^{\uparrow}$; and (ii) $(B^{\uparrow}, S) \vee (\neg B^{\uparrow}, S) = (S^{\uparrow}, S)$.

With these definitions in mind, an *information structure* (of I) is a tuple $\mathcal{S} := \langle \langle (S_{\alpha}, \mathcal{D}_{\alpha})_{\alpha \in \mathcal{A}}, \succeq, r \rangle, (K_i, U_i)_{i \in I} \rangle$ with the following ingredients. First, $\langle (S_{\alpha}, \mathcal{D}_{\alpha})_{\alpha \in \mathcal{A}}, \succeq, r \rangle$ is a generalized state space. Let $\mathcal{E}$ be the domain. Second, $\overline{K}_i : \mathcal{E} \to \mathcal{E}$ is an agent i's *knowledge operator* satisfying (at least) the following: (i) $S(\overline{K}_i(\overline{E})) = S(\overline{E})$ for any $\overline{E} \in \mathcal{E}$; (ii) Truth Axiom: $\overline{K}_i(\overline{E}) \leqslant \overline{E}$ (for all $\overline{E} \in \mathcal{E}$); (iii) Positive Introspection: $\overline{K}_i(\cdot) \leqslant \overline{K}_i \overline{K}_i(\cdot)$; and (iv) Monotonicity: $\overline{E} \leqslant \overline{F}$ implies $\overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\overline{F})$. Third, $\overline{U}_i : \mathcal{E} \to \mathcal{E}$ is i's *unawareness operator* with the property that $S(\overline{U}_i(\overline{E})) = S(\overline{E})$ for any $\overline{E} \in \mathcal{E}$. Before discussing the assumptions, for ease of notation, I often identify $\mathcal{S} := \langle \langle (S_{\alpha}, \mathcal{D}_{\alpha})_{\alpha \in \mathcal{A}}, \succeq, r \rangle, (K_i, U_i)_{i \in I} \rangle$ with $\mathcal{S} = \langle \mathcal{E}, (K_i, U_i)_{i \in I} \rangle$.

Fix an event $(E, S(E))$. The pair $(K_i(E), S(E)) := \overline{K}_i(E, S(E))$ is the event that i knows $(E, S(E))$. The set $K_i(E)$ is interpreted as the set of states at which i knows $(E, S(E))$. Likewise, $(U_i(E), S(E)) := \overline{U}_i(E, S(E))$ is the event that i is unaware of $(E, S(E))$, and $U_i(E)$ is the set of states at which i is unaware of $(E, S(E))$. For both knowledge and unawareness operators, the knowledge and unawareness of the event $(E, S(E))$ reside in the same subspace $S(E)$. In Heifetz, Meier, and Schipper (2006), this condition is ensured by the assumptions on their possibility correspondences (see also Grant et al. (2015)). If the underlying state space is standard, I simply denote the knowledge and unawareness operators by K_i and U_i, respectively. The same notational convention applies to other operators on the domain.

Truth Axiom distinguishes knowledge from beliefs in that knowledge is truthful. Positive Introspection allows the agent to know what she knows. Monotonicity renders the agent a logical inference ability.

I introduce further properties of knowledge. First, $\overline{K}_i$ satisfies *Necessitation* if $\overline{K}_i(\Omega, \inf \Sigma) = (\Omega, \inf \Sigma)$. It states that the agent knows a tautology $(\Omega, \inf \Sigma)$. Second, $\overline{K}_i$ satisfies *Non-empty Conjunction* if $\bigwedge_{\lambda \in \Lambda} \overline{K}_i(E_{\lambda}, S_{\lambda}) \leqslant \overline{K}_i(\bigwedge_{\lambda \in \Lambda}(E_{\lambda}, S_{\lambda}))$

for any non-empty index set Λ. *Finite Conjunction* and *Countable Conjunction*, respectively, refer to the case in which a non-empty Λ is finite and countable. Note that Non-empty Conjunction and Necessitation can be jointly regarded as *Arbitrary Conjunction* because Necessitation corresponds to the case with $\Lambda = \emptyset$. These conjunction properties mean the closure of knowledge under conjunction. Third, I define *Negative Introspection* of $\overline{K}_i$ to be $(\neg\overline{K}_i)(\cdot) \leqslant \overline{K}_i(\neg\overline{K}_i)(\cdot)$. It means that if the agent does not know an event then she knows that she does not know it.

Next, I state some of joint postulates on knowledge and unawareness. First, $(\overline{K}_i, \overline{U}_i)$ is *plausible* if $\overline{U}_i(\cdot) \leqslant (\neg\overline{K}_i)(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$. Plausibility says that if the agent is unaware of an event then she does not know it and she does not know that she does not know it. If every $(\overline{K}_i, \overline{U}_i)$ is plausible, $\mathcal{S}$ is *plausible*. Second, $(\overline{K}_i, \overline{U}_i)$ satisfies *KU Introspection* if $\overline{K}_i\overline{U}_i(E, S) = \overline{\emptyset}^S$. KU Introspection means that, for any event, there is no state at which the agent knows that she is unaware of it. Third, $(\overline{K}_i, \overline{U}_i)$ satisfies *AU Introspection* if $\overline{U}_i(\cdot) \leqslant \overline{U}_i\overline{U}_i(\cdot)$. Under AU Introspection, if the agent is unaware of an event then she is unaware of being unaware of it. These three properties are proposed in Dekel, Lipman, and Rustichini (1998). Section 3.3 examines other properties.

To conclude the exposition of the framework, consider a standard state space. A knowledge operator K_i satisfying Monotonicity and Arbitrary Conjunction can equivalently be induced from a possibility correspondence. See, for example, Morris (1996) when the domain is the power set of an underlying standard state space. If K_i also satisfies Truth Axiom and Positive Introspection, then the possibility correspondence is reflexive and transitive (see Footnote 3 for the literature). If K_i additionally satisfies Negative Introspection, then the possibility correspondence forms a partition (e.g., Aumann (1976, 1999)). If (K_i, U_i) is plausible, however, then $U_i(\cdot) = \emptyset$.

2.1 Associated Concepts

I define associated concepts derived from knowledge (and unawareness). In this subsection, fix an information structure $\mathcal{S}$ of I.

Defining Unawareness from Knowledge. Define unawareness operators from the given knowledge operator as follows. Let $n \in \mathbb{N}_2^\infty := \{n \in \mathbb{N} \mid n \geq 2\} \cup \{\infty\}$. Define the k^n-unawareness operator $\overline{U}_i^{(n)}(\cdot) := \bigwedge_{r=1}^n (\neg\overline{K}_i)^r(\cdot)$. Agent i is (k^n-)*unaware* of an event (E, S) at a state ω if $\omega \in U_i^{(n)}(E) = \bigcap_{r=1}^n (\neg K_i)^r(E)$. Modica and Rustichini (1994) define unawareness by $\overline{U}_i^{(2)}(\cdot) = (\neg\overline{K}_i)(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$ while Dekel, Lipman, and Rustichini (1998) consider $\overline{U}_i^{(\infty)}(\cdot) = \bigwedge_{r \in \mathbb{N}} (\neg\overline{K}_i)^r(\cdot)$.

Derived Operators. I define the following four operators on $\mathcal{E}$. Fix $\overline{E} \in \mathcal{E}$ and $n \in \mathbb{N}_2^\infty$. First, define the *awareness* operator by $\overline{A}_i(\cdot) := (\neg\overline{U}_i)(\cdot)$ (Modica and Rustichini, 1994, 1999). In particular, define the k^n-awareness operator by $\overline{A}_i^{(n)}(\cdot) :=$

$(\neg\overline{U}_i^{(n)})(\cdot)$. Second, define the *possibility* operator by $\overline{L}_i(\overline{E}) := (\neg\overline{K}_i)(\neg\overline{E}) \wedge \overline{A}_i(\overline{E})$ (Modica and Rustichini, 1999). Also, let $\overline{L}_i^{(n)}(\overline{E}) := (\neg\overline{K})(\neg\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{E})$. Third, define the *ignorance* operator by $\overline{\partial}_i(\overline{E}) := (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E})$ (e.g., Lehrer and Samet (2011)). Fourth, define the *knowing-whether* operator by $\overline{J}_i(\overline{E}) := (\neg\overline{\partial}_i)(\overline{E})(= \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E}))$ (e.g., Hintikka (1962) and Hart, Heifetz, and Samet (1996)).

First, $\overline{A}_i(\overline{E})$ is the event that i is aware of $\overline{E}$ in that i is not unaware of $\overline{E}$. Second, $\overline{L}_i(\overline{E})$ is the event that i considers $\overline{E}$ possible in that i does not know its negation $\neg\overline{E}$ and i is aware of $\overline{E}$. Unlike the standard notion of possibility (e.g., Hintikka (1962)) stating that i considers an event possible when she does not know its negation, I follow Modica and Rustichini (1999) so that, for agent i to consider an event possible, she has to be aware of the event itself.[6] Third, $\overline{\partial}_i(\overline{E})$ is the event that i is ignorant of $\overline{E}$ in that i does not know $\overline{E}$ nor $\neg\overline{E}$.[7] Fourth, $\overline{J}_i(\overline{E})$ is the event that i knows whether $\overline{E}$ obtains (or not) in that either i knows $\overline{E}$ or she knows its negation $\neg\overline{E}$.

Self-evident Collection. Call an event $\overline{E} \in \mathcal{E}$ *self-evident* to an agent i if $\overline{E} \leqslant \overline{K}_i(\overline{E})$ (i.e., $E \subseteq K_i(E)$). Call $\mathcal{J}_i := \{\overline{E} \in \mathcal{E} \mid \overline{E} \leqslant \overline{K}_i(\overline{E})\}$ agent i's *self-evident collection*. The self-evident collection summarizes or recovers $\overline{K}_i$ in the sense that $\overline{K}_i(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \in \mathcal{J}_i \text{ and } \overline{F} \leqslant \overline{E}\}$, where the supremum is taken on $\langle \mathcal{E}, \leqslant \rangle$.[8] Denoting $\mathcal{J}_i^* := \{\overline{E} \in \mathcal{E} \mid \neg\overline{E} \in \mathcal{J}_i\}$, it can be seen that $\overline{K}_i$ satisfies Negative Introspection iff $\mathcal{J}_i = \mathcal{J}_i^*$ (i.e., $\mathcal{J}_i$ is closed under negation). Also, $\overline{K}_i$ satisfies Necessitation iff $(\Omega, \inf\Sigma) \in \mathcal{J}_i$. Moreover, $\overline{K}_i$ satisfies Non-empty (or Finite/Countable) Conjunction iff $\mathcal{J}_i$ is closed under non-empty (or non-empty finite/countable) conjunction. Call an event $\overline{E} \in \mathcal{E}$ *publicly evident* among agents I if it is self-evident to every $i \in I$ (Milgrom, 1981). The collection of publicly-evident events is $\bigcap_{i \in I} \mathcal{J}_i$.

Knowledgeability. Denote by $\mathrm{IK}_i(\omega) := \{\overline{E} \in \mathcal{E} \mid \omega \in K_i(E)\}$ the collection of events that an agent i knows at a state ω.[9] Agent i is *at least as knowledgeable as* agent j *at* a state ω if $\mathrm{IK}_j(\omega) \subseteq \mathrm{IK}_i(\omega)$. Agents i and j are *equally knowledgeable at* ω if $\mathrm{IK}_j(\omega) = \mathrm{IK}_i(\omega)$. Likewise, i is *at least as knowledgeable as* j if $\mathrm{IK}_j(\cdot) \subseteq \mathrm{IK}_i(\cdot)$. Agents i and j are *equally knowledgeable* if $\mathrm{IK}_j = \mathrm{IK}_i$. Note that i is at least as knowledgeable as j iff $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ iff $\mathcal{J}_j \subseteq \mathcal{J}_i$.

Common Knowledge. I define common knowledge (e.g., Aumann (1976) and Friedell (1969)) among agents I as the knowledge that would be possessed by the most knowledgeable agent who is at least as less knowledgeable as every agent within I. In a

[6]Note that "L" does not mean the implicit knowledge operator as in Fagin and Halpern (1987).

[7]I use the symbol "∂" of the boundary operator on a topological space in the sense that K_i satisfies a part of the properties of the interior operator.

[8]Fukuda (2018) studies the sense in which an agent's knowledge is represented by a set algebra such as a σ-algebra or a topology on a standard state space using a self-evident collection.

[9]In a standard state space model, this mapping IK_i is called a neighborhood system (see, for example, Pacuit (2017)).

partitional standard state space model, Aumann (1976) defines common knowledge from the finest partition which is coarser than every agent's partition.

Define the *common knowledge operator* $\overline{C}_I : \mathcal{E} \to \mathcal{E}$ by $\overline{C}_I(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \in \bigcap_{i \in I} \mathcal{J}_i \text{ and } \overline{F} \leqslant \overline{E}\}$. Since $\overline{\emptyset}^{S(E)} \leqslant \overline{C}_I(\overline{E}) \leqslant \overline{E}$, it follows that $S(\overline{C}_I(\overline{E})) = S(\overline{E})$. Letting $\overline{C}_I(\overline{E}) := (C_I(E), S(E))$, an event $\overline{E} \in \mathcal{E}$ is *commonly known* (or *common knowledge*) among I at a state ω if $\omega \in C_I(E)$. By construction, $\overline{C}_I$ satisfies Truth Axiom, Positive Introspection, and Monotonicity. Thus, I can treat $\overline{C}_I$ as the knowledge operator of a (hypothetical) agent. Since $\overline{C}_I(\cdot) \leqslant \overline{C}_I\overline{C}_I(\cdot) \leqslant \bigwedge_{i \in I} \overline{K}_i\overline{C}_I(\cdot)$, it follows $\overline{C}_I(\cdot) \in \bigcap_{i \in I} \mathcal{J}_i$. Then, an event $\overline{E} \in \mathcal{E}$ is common knowledge among I at ω (i.e., $\omega \in C_I(E)$) iff there is $\overline{F} \in \bigcap_{i \in I} \mathcal{J}_i$ with $\omega \in F$ and $\overline{F} \leqslant \overline{E}$. It can be seen that $\overline{C}_I$ inherits each of Necessitation, conjunction properties, and Negative Introspection from the agents' knowledge operators if every agent's knowledge satisfies it.

Common knowledge implies mutual knowledge: $\overline{C}_I(\cdot) \leqslant \overline{K}_I(\cdot) := \bigwedge_{i \in I} \overline{K}_i(\cdot)$, where $\overline{K}_I(\overline{E})$ is the event that everyone in I knows $\overline{E}$. Together with Positive Introspection and Monotonicity, if $\overline{E}$ is commonly known among I at ω, then everyone in I knows $\overline{E}$ at ω, everyone knows that everyone knows $\overline{E}$ at ω, and so forth *ad infinitum*. Conversely, if every agent's knowledge satisfies Countable Conjunction, then $\overline{C}_I(\cdot) = \bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\cdot)$. This is because $\bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\overline{E})$ is the maximal publicly evident event satisfying $\bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\overline{E}) \leqslant \overline{E}$ (observe $\bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\overline{E}) \leqslant \bigwedge_{n \in \mathbb{N}} \overline{K}_I^{n+1}(\overline{E}) \leqslant \overline{K}_I(\bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\overline{E}))$). The formalization of common knowledge here extends, for example, those of Galanis (2013) and Heifetz, Meier, and Schipper (2006) on their generalized state space models while requiring weaker assumptions on agents' knowledge. The formalization of common knowledge also nests Monderer and Samet (1989) on a standard state space.

2.2 A Standard-State-Space Example

Information structures accommodate both non-partitional standard state space models and unawareness structures on generalized state spaces by Heifetz, Meier, and Schipper (2006). This subsection provides an example of an information structure on a standard state space. The example will be used to examine (non-)trivial forms and properties of unawareness on a standard state space.

Example 1. Let $I = \{i_1, i_2, i_3, i_4\}$, and consider a standard state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Identify the domain with $\mathcal{E} = \mathcal{P}(\Omega)$. Let each K_i be as in Table 1. Agent i_1's knowledge coincides with Dekel, Lipman, and Rustichini (1998, Example 1). Each K_i satisfies Non-empty Conjunction as well as Truth Axiom, Positive Introspection, and Monotonicity. Knowledge operators of i_1 and i_3 also satisfy Necessitation. Each pair $(K_i, U_i^{(n)})$ satisfies Plausibility and KU Introspection.

Additional remarks are in order. First, $\mathcal{J}_{i_1} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}, \Omega\}$, $\mathcal{J}_{i_2} = \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\}$, $\mathcal{J}_{i_3} = \{\emptyset, \{\omega_1\}, \Omega\}$, and $\mathcal{J}_{i_4} = \{\emptyset, \{\omega_1\}\} = \bigcap_{i \in I} \mathcal{J}_i$. Thus, agent i_1 is at least as knowledgeable as $j \in \{i_2, i_3, i_4\}$, and agent i_4's knowledge coincides with common knowledge (i.e., $C_I = K_{i_4}$).

E	K_{i_1}	$(\neg K_{i_1})$	$(\neg K_{i_1})^2$	$(\neg K_{i_1})^3$	$(\neg K_{i_1})^4$	∂_{i_1}	$U_{i_1}^{(2)}$	$U_{i_1}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_3\}$	$\{\omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	K_{i_2}	$(\neg K_{i_2})$	$(\neg K_{i_2})^2$	$(\neg K_{i_2})^3$	$(\neg K_{i_2})^4$	∂_{i_2}	$U_{i_2}^{(2)}$	$U_{i_2}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
Ω	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$

E	K_{i_3}	$(\neg K_{i_3})$	$(\neg K_{i_3})^2$	$(\neg K_{i_3})^3$	$(\neg K_{i_3})^4$	∂_{i_3}	$U_{i_3}^{(2)}$	$U_{i_3}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	$C_I = K_{i_4}$	$(\neg K_{i_4})$	$(\neg K_{i_4})^2$	$(\neg K_{i_4})^3$	$(\neg K_{i_4})^4$	∂_{i_4}	$U_{i_4}^{(2)}$	$U_{i_4}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$

Table 1: Agents' Knowledge and Unawareness in Example 1 ($n \geq 3$ or $n = \infty$)

Second, while $(K_i, U_i^{(n)})$ satisfies AU Introspection for each $i \in \{i_2, i_4\}$, the other pairs $(K_j, U_j^{(n)})$ $(j \in \{i_1, i_3\})$ do not necessarily satisfy AU introspection. Such j's information structure $\langle \mathcal{E}, (K_j, U_j^{(n)}) \rangle$ $(n \in \mathbb{N}_2^{\infty})$ is considered to be a reflexive and transitive possibility correspondence model. Indeed, j's knowledge can be induced from the possibility correspondence $b_j : \Omega \to \mathcal{D}$, where $b_{i_1}(\omega_1) = \{\omega_1\}$, $b_{i_1}(\omega_2) = \{\omega_2\}$, and $b_{i_1}(\omega_3) = \Omega$; and $b_{i_3}(\omega_1) = \{\omega_1\}$ and $b_{i_3}(\omega_2) = b_{i_3}(\omega_3) = \Omega$. The fact that

$(K_j, U_j^{(n)})$ does not satisfy AU Introspection is consistent with Dekel, Lipman, and Rustichini (1998) that there is no non-trivial possibility correspondence model which satisfies all of Plausibility, KU Introspection, and AU Introspection.

3 Unawareness as Lack of Knowledge

Having defined the basic framework, I now proceed with the main analyses.

3.1 Equivalent Representations

I relate the concepts of ignorance, knowing-whether, and possibility to that of unawareness. I also study the relations among k^n-unawareness. Throughout the subsection, fix an information structure $\mathcal{S} = \langle \mathcal{E}, (\overline{K}, \overline{U}) \rangle$ of a single agent.

The first benchmark result is that, under Truth Axiom, Positive Introspection, and Monotonicity, $(\neg\overline{K})^2 = (\neg\overline{K})^{2n}$ for all $n \in \mathbb{N}$ (documented in Lemma A.1 in Appendix A).[10] It implies that $\overline{U}^{(\infty)} = \overline{U}^{(n)}$ for all $n \geq 3$. In other words, if the chain of the lack of knowledge holds repeatedly three times, then the chain continues without an end. Hence, as long as the notions of unawareness are derived from the lack of knowledge, I can restrict attention to $\overline{U}^{(n)}$ with $n \in \{2, \infty\}$, and I can replace $\overline{U}^{(\infty)}$ with $\overline{U}^{(3)}$. Note that $\overline{U}^{(2)}$ and $\overline{U}^{(\infty)} (= \overline{U}^{(3)})$ are generally different (e.g., agents i_1 and i_3 in Example 1). I will characterize in Proposition 4 when $\overline{U}^{(2)} = \overline{U}^{(\infty)}$ holds. This preliminary result leads to the following restatement of unawareness.

Proposition 1. *Fix $\overline{E} \in \mathcal{E}$.*

1. $\overline{U}^{(\infty)}(\overline{E}) = \overline{U}^{(2)}(\neg\overline{K})(\overline{E})$. *Equivalently,* $\overline{A}^{(\infty)}(\overline{E}) = \overline{A}^{(2)}(\neg\overline{K})(\overline{E})$.
2. $\overline{U}^{(2)}(\overline{E}) = \overline{\partial K}(\overline{E}) \leqslant \overline{\partial}(\overline{E})$. *Also,* $\overline{U}^{(2)}(\overline{E}) = \overline{\partial L}^{(2)}\overline{K}(\overline{E})$.
3. $\overline{U}^{(\infty)}(\overline{E}) = \overline{\partial K}(\neg\overline{K})(\overline{E})$. *Also,* $\overline{U}^{(\infty)}(\overline{E}) = \overline{\partial L}^{(\infty)}\overline{K}(\overline{E})$.

Part (1) of Proposition 1 relates $U^{(\infty)}$ and $U^{(2)}$ in that the agent is k^∞-unaware of an event $\overline{E}$ iff she is k^2-unaware of not knowing $\overline{E}$. Parts (2) and (3) characterize k^2-unawareness and k^∞-unawareness by ignorance and possibility, respectively. The agent is k^2-unaware of an event $\overline{E}$ iff she does not know whether she knows $\overline{E}$ (i.e., she is ignorant of (not) knowing $\overline{E}$). Likewise, the agent is k^∞-unaware of $\overline{E}$ iff she does not know whether she knows that she does not know $\overline{E}$ (i.e., she is ignorant of knowing that she does not know $\overline{E}$).

[10] Mathematically, this property is related to the notion of regularly open/closed sets in general topology (e.g., Willard (2004)) in the sense that the assumptions on K are related to a part of the properties of the interior operator on a topological space.

In either case, the agent is k^n-unaware of $\overline{E}$ iff she is ignorant of the possibility that she knows $\overline{E}$. Also, k^n-unawareness implies ignorance: if the agent knows whether $\overline{E}$ is true, then she is k^n-aware of $\overline{E}$.

3.2 Characterization of Non-triviality

An information structure $\mathcal{S} = \langle \mathcal{E}, (\overline{K}, \overline{U}) \rangle$ represents a *non-trivial form of unawareness* (or $\mathcal{S}$ is *non-trivial*) if $\overline{U}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$ (i.e., $U(E) \neq \emptyset$) for some $\overline{E} \in \mathcal{E}$. The information structure $\mathcal{S}$ is *trivial* otherwise. Dekel, Lipman, and Rustichini (1998, Theorem 1) show that any standard possibility correspondence model is trivial under Plausibility, KU Introspection, and AU Introspection. Modica and Rustichini (1994) show that $\mathcal{S}$ is trivial under Symmetry ($\overline{U}(\overline{E}) = \overline{U}(\neg\overline{E})$).

The following two questions naturally arise. First, when does a state space model (especially, a standard one) represent a non-trivial form of unawareness? Second, what properties have to be retained in order to represent a non-trivial form of unawareness in a standard state space model (Dekel, Lipman, and Rustichini, 1998, p. 166)?

I examine the above first question by providing a necessary and sufficient condition for an information structure $\mathcal{S}^{(n)} := \langle \mathcal{E}, (\overline{K}, \overline{U}^{(n)}) \rangle$ (with $n \in \{2, \infty\}$) to be non-trivial. The characterization implies that $\mathcal{S}^{(2)}$ on a standard state space is generically non-trivial even when it is induced from a reflexive and transitive possibility correspondence.

Proposition 2. *1. (a) For any $\overline{E} \in \mathcal{E}$, $\overline{U}^{(2)}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$ iff $\overline{K}(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$.*

(b) $\mathcal{S}^{(2)}$ is non-trivial iff $\mathcal{J} \setminus \mathcal{J}^* \neq \emptyset$ iff $\mathcal{J}^* \setminus \mathcal{J} \neq \emptyset$ iff $\mathcal{J}^* \triangle \mathcal{J} \neq \emptyset$.

2. (a) For any $\overline{E} \in \mathcal{E}$, $\overline{U}^{(\infty)}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$ iff $\overline{K}(\neg\overline{K})(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$.

(b) $\mathcal{S}^{(\infty)}$ is non-trivial iff $\{\overline{F} \in \mathcal{J} \setminus \mathcal{J}^* \mid \overline{K}(\neg\overline{F}) \in \mathcal{J} \setminus \mathcal{J}^*\} \neq \emptyset$.

For the rest of this subsection, consider standard state space models. By Proposition 2, the triviality of a standard partitional information structure follows from $\mathcal{J} = \mathcal{J}^*$. Recall that Negative Introspection is equivalent to this condition. In contrast, $\mathcal{S}^{(2)}$ on a standard state space is non-trivial iff $\mathcal{J}$ is not closed under negation. It follows that any standard reflexive and transitive possibility correspondence model is non-trivial as long as it is not partitional. An underlying intuition for this particular case is very simple: $\mathcal{S}^{(2)}$ is non-trivial iff Negative Introspection fails. Thus, any (properly) non-partitional model of knowledge represents a non-trivial form of unawareness.

Consider an information structure $\mathcal{S}^{(n)}$ on a standard state space which satisfies Necessitation. For example, the agent i_1 in Example 1 entails a non-trivial form of unawareness because, for example, $\{\omega_2\} \in \mathcal{J}_{i_1} \setminus \mathcal{J}^*_{i_1}$ satisfies $K(\neg\{\omega_2\}) = \{\omega_1\} \in \mathcal{J}_{i_1} \setminus \mathcal{J}^*_{i_1}$. Noting that $K(E) = \emptyset$ implies $(\neg K)^2(E) = \emptyset$, if an agent is ($k^n$-)unaware of an event E at some state ω, then she does not know E at ω and she knows E at another state ω'. If she does not know E at any state, then she knows that she does not know E at any state, and thus she is not unaware of E at any state. In contrast, the failure of Necessitation in a standard state space implies the non-triviality because $\emptyset \neq (\neg K)(\Omega) \subseteq U^{(\infty)}(E) \subseteq U^{(2)}(E)$ for all E. At any state $\omega \in (\neg K)(\Omega)$, an agent does not know anything and she is unaware of everything. The next subsection (Proposition 4) shows that AU Introspection is equivalent to $\overline{A}^{(n)}(E,S) = \overline{K}(\overline{S}^{\uparrow})$ in a generalized state space.

3.3 Properties of Unawareness

Consider $\mathcal{S}^{(n)}$ with $n \in \{2, \infty\}$. I first examine properties of $\overline{U}^{(n)}$ that hold. Next, I study properties that lead to a degenerate form of unawareness in a standard state space.

First, any $\mathcal{S}^{(n)}$ satisfies Plausibility by definition. Also, $\mathcal{S}^{(\infty)}$ satisfies Strong Plausibility: $\overline{U}^{(\infty)}(\cdot) \leqslant \bigwedge_{r \in \mathbb{N}} (\neg \overline{K})^r(\cdot)$ with equality (Heifetz, Meier, and Schipper, 2006, 2008).[11] One could regard $\overline{U}^{(n)}(\cdot) \leqslant \overline{\partial}(\cdot)$ (i.e., unawareness implies ignorance) as a plausibility condition.

Second, any $\mathcal{S}^{(n)}$ satisfies KU Introspection as Dekel, Lipman, and Rustichini (1998, Footnote 10) note that Truth Axiom, Monotonicity, and Plausibility yield KU Introspection. Now, I provide other properties of $\mathcal{S}^{(n)}$.

Proposition 3. *Any $\mathcal{S}^{(n)}$ satisfies the following. Let $\overline{E} = (E, S) \in \mathcal{E}$.*

1. $\overline{A}^{(n)}(E,S) \leqslant \overline{K}(\overline{S}^{\uparrow})$. *Also,* $\overline{A}^{(n)}\overline{U}^{(n)}(E,S) = \overline{K}(\overline{S}^{\uparrow})$.
2. *Reverse AU Introspection:* $\overline{U}^{(n)}\overline{U}^{(n)}(\overline{E}) \leqslant \overline{U}^{(n)}(\overline{E})$. *Also,* $\overline{U}^{(n)}\overline{U}^{(n)}\overline{U}^{(n)}(\overline{E}) = \overline{U}^{(n)}\overline{U}^{(n)}(\overline{E})$.
3. *JU Introspection:* $\overline{U}^{(n)}(\overline{E}) = \overline{\partial U}^{(n)}(\overline{E})$. *Equivalently,* $\overline{A}^{(n)}(\overline{E}) = \overline{JA}^{(n)}(\overline{E})$.
4. *Weak A-Negative Introspection for $n = 2$:* $(\neg\overline{K})(\overline{E}) \wedge \overline{A}^{(2)}(\overline{E}) = \overline{K}(\neg\overline{K})(\overline{E})$.
5. *AK Self-Reflection:* $\overline{A}^{(n)}(\overline{E}) = \overline{A}^{(n)}\overline{K}(\overline{E})$. *Equivalently,* $\overline{U}^{(n)}(\overline{E}) = \overline{U}^{(n)}\overline{K}(\overline{E})$.
6. *A-Introspection:* $\overline{A}^{(n)}(\overline{E}) = \overline{KA}^{(n)}(\overline{E})$. *Equivalently,* $\overline{U}^{(n)}(\overline{E}) = (\neg\overline{K})\overline{A}^{(n)}(\overline{E})$.
7. *Weak AA Self-Reflection:* $\overline{A}^{(n)}(\overline{E}) \leqslant \overline{A}^{(n)}\overline{A}^{(n)}(\overline{E})$ *with equality if $n = 2$. Also,* $\overline{A}^{(n)}\overline{A}^{(n)}(\overline{E}) = \overline{A}^{(n)}\overline{A}^{(n)}\overline{A}^{(n)}(\overline{E})$.

[11] I also use other terminologies coined by Heifetz, Meier, and Schipper (2006, 2008) and Schipper (2015) postulated in Proposition 3 (specifically, Properties 5, 6, and 9).

8. $\overline{L}^{(n)}\overline{A}^{(n)}(\overline{E}) = \overline{A}^{(n)}\overline{A}^{(n)}(\overline{E})$.

If $\overline{K}$ satisfies Finite Conjunction, then so does $\overline{A}^{(n)}$:

9. *A-Conjunction:* $\overline{A}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}(\overline{F}) \leqslant \overline{A}^{(n)}(\overline{E} \wedge \overline{F})$.

Property 1 is a part of Weak Necessitation ($\overline{A}^{(n)}(E, S) = \overline{K}(\overline{S}^{\uparrow})$) originally coined by Dekel, Lipman, and Rustichini (1998). This part follows from Monotonicity of $\overline{K}$. In a standard state space, Property 1 reduces to $A^{(n)}(E) \subseteq K(\Omega)$. Proposition 4 studies the implication of Weak Necessitation. While the entire part of Weak Necessitation may not necessarily hold, Monotonicity of $\overline{K}$ and KU Introspection yield $\overline{A}^{(n)}\overline{U}^{(n)}(E, S) = \overline{K}(\overline{S}^{\uparrow})$, i.e., $\overline{U}^{(n)}\overline{U}^{(n)}(E, S) = (\neg\overline{K})(\overline{S}^{\uparrow})$.

Property 2 is the "converse" of AU Introspection. Reverse AU Introspection follows from Property 1 (a part of Weak Necessitation). Similarly, the ignorance operator satisfies $\overline{\partial}\overline{\partial}(\cdot) \leqslant \overline{\partial}(\cdot)$.

JU Introspection (Property 3) states that if the agent is unaware of an event then she is *ignorant* of being unaware of it. Indeed, JU Introspection holds with equality. Reverse AU Introspection is also seen as a consequence of JU Introspection. As a remark, I provide an alternative proof of Reverse AU Introspection from JU Introspection (Property 3) together with $\overline{U}^{(n)}(\cdot) \leqslant \overline{\partial}(\cdot)$ (Proposition 1): $\overline{U}^{(n)}\overline{U}^{(n)}(\cdot) \leqslant \overline{\partial}\overline{U}^{(n)}(\cdot) = \overline{U}^{(n)}(\cdot)$.

Weak A-Negative Introspection (Property 4) is proposed by Li (2009).[12] If $n = \infty$, this is equivalent to Weak Negative Introspection for $n = 2$: $(\neg\overline{K})(\overline{E}) \wedge \overline{A}^{(2)}(\neg\overline{K})(\overline{E}) = \overline{K}(\neg\overline{K})(\overline{E})$ (Fagin and Halpern (1987) and Halpern (2001)). Weak A-Negative Introspection for $n = \infty$, however, may not hold (i.e., the "$\leqslant$" part may fail). Proposition 4 studies the implication of Weak A-Negative Introspection for $n = \infty$. As a particular example, consider Example 1: $(\neg K_{i_1})(\{\omega_1, \omega_2\}) \cap A_{i_1}^{(\infty)}(\{\omega_1, \omega_2\}) = \{\omega_3\} \not\subseteq \emptyset = K_{i_1}(\neg K_{i_1})(\{\omega_1, \omega_2\})$. Moreover, $(\neg K_{i_1})(\{\omega_1, \omega_2\}) \cap A_{i_1}^{(\infty)}(\neg K_{i_1})(\{\omega_1, \omega_2\}) = \{\omega_3\}$.

Properties 5,6, and 9 are proposed by Modica and Rustichini (1994, 1999). AK Self-Reflection (Property 5) is equivalent to $\overline{U}^{(n)} = \overline{U}^{(n)}\overline{K}$: the agent is unaware of $\overline{E}$ iff she is unaware of knowing $\overline{E}$. A-Introspection (Property 6) is equivalent to $\overline{U}^{(n)} = (\neg\overline{K})\overline{A}^{(n)}$: the agent is unaware iff she does not know that she is aware.

Property 7 (Weak AA Self-Reflection) is based on AA Self-Reflection (Modica and Rustichini, 1994, 1999): $\overline{A}^{(n)} = \overline{A}^{(n)}\overline{A}^{(n)}$. While AA Self-Reflection holds when $n = 2$, only this weak form is true when $n = \infty$. For instance, in Example 1, $A_{i_1}^{(\infty)}(\{\omega_1\}) = \{\omega_1, \omega_2\} \neq \Omega = A_{i_1}^{(\infty)}A_{i_1}^{(\infty)}(\{\omega_1\})$.

Property 8 states that the agent is aware of being aware of an event iff she considers it possible that she is aware of the event.

[12] Given an information structure $\langle \mathcal{E}, (\overline{K}, \overline{U})\rangle$, Plausibility and Weak A-Negative Introspection induce $\overline{U} = \overline{U}^{(2)}$. It would be interesting to ask whether there are other "non-trivial" combinations of axioms that yield $\overline{U} = \overline{U}^{(2)}$ given a pair $(\overline{K}, \overline{U})$.

Property 9 (A-Conjunction) is studied by Modica and Rustichini (1994, 1999). The converse of A-Conjunction (i.e., $\overline{A}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}(\overline{F}) \geqslant \overline{A}^{(n)}(\overline{E} \wedge \overline{F})$), studied by Fagin and Halpern (1987), Halpern (2001) and Modica and Rustichini (1999), may not necessarily hold. In Example 1, $A_{i_1}^{(n)}(\{\omega_1, \omega_3\}) \cap A_{i_1}^{(n)}(\{\omega_2, \omega_3\}) = \{\omega_1, \omega_2\} \subsetneq \Omega = A_{i_1}^{(n)}(\{\omega_3\})$. Proposition 4 studies the converse of A-Conjunction.

I return to the question raised by Dekel, Lipman, and Rustichini (1998, p. 166), which of their three axioms is to be retained in a standard possibility correspondence model so as to capture a non-trivial form of unawareness. One implication of Proposition 3 is that any information structure satisfies Plausibility (by definition), KU Introspection, Reverse AU Introspection, and JU Introspection (instead of AU Introspection). Thus, practically, one could analyze agents' unawareness involving these properties in standard state spaces.

I remark on how KU Introspection and AU Introspection lead to a trivial form of unawareness in a standard state space model with Necessitation. Monotonicity and Necessitation of K and KU Introspection yield $U^{(n)}U^{(n)}(E) = (\neg K)(\Omega) = \emptyset$. Intuitively, since KU introspection implies that the agent does not know that she is unaware of an event E at any state, the statement that the agent does not know that she is unaware of an event E is a tautology. Hence, the agent knows that she does not know that she is unaware of an event E at any state. Thus, there is no state at which she is unaware of being unaware of E. Now, AU introspection implies that there is no state at which the agent is unaware of an event.

Next, I turn to examining other properties which lead to a degenerate form of unawareness in a standard state space. More generally, I characterize Weak Necessitation. On the one hand, these properties lead to trivial unawareness under Necessitation in a standard state space. This is because Weak Necessitation reduces to $U^{(n)}(\cdot) = (\neg K)(\Omega)$ in a standard state space. On the other hand, in a generalized state space model of Heifetz, Meier, and Schipper (2006, 2008), Weak Necessitation connects notions of unawareness as the lack of knowledge and the lack of concept.

Proposition 4. *Let $\mathcal{S}^{(n)}$ be an information structure.*

1. *Let $n = 2$. (a)-(i) are all equivalent to Weak Necessitation: $\overline{U}^{(2)}(E, S) = (\neg\overline{K})(\overline{S}^{\uparrow})$ (for all $(E, S) \in \mathcal{E}$). Under Finite Conjunction of $\overline{K}$, it is also equivalent to (k).*

2. *Let $n = \infty$. (g)-(j) are all equivalent to Weak Necessitation: $\overline{U}^{(\infty)}(E, S) = (\neg\overline{K})(\overline{S}^{\uparrow})$. Under Finite Conjunction of $\overline{K}$, it is also equivalent to (k).*

(a) Subjective Negative Introspection: $K(\overline{S}^{\uparrow}) \wedge (\neg\overline{K})(E, S) \leqslant \overline{K}(\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg\overline{K})(E, S))$.

(b) If $(E, S) \in \mathcal{J}$ *then* $K(\overline{S}^{\uparrow}) \wedge (\neg(E, S)) \in \mathcal{J}$.

(c) Negative Non-Introspection: $(\neg\overline{K})(\cdot) \wedge (\neg\overline{K})^2(\cdot) \leqslant (\neg\overline{K})^3(\cdot)$.

(d) $\overline{U}^{(2)} = \overline{U}^{(\infty)}$ *(i.e., Strong Plausibility of* $\overline{U}^{(2)}$*).*

(e) *Weak A-Negative Introspection for* $n = \infty$*:* $(\neg\overline{K})(\cdot) \wedge \overline{A}^{(\infty)}(\cdot) = \overline{K}(\neg\overline{K})(\cdot)$.

(f) *Symmetry of* $\overline{U}^{(2)}$*:* $\overline{U}^{(2)}(\overline{E}) = \overline{U}^{(2)}(\neg\overline{E})$.

(g) *AU introspection of* $\overline{U}^{(n)}$.

(h) *LU introspection of* $\overline{U}^{(n)}$*:* $\overline{L}^{(n)}\overline{U}^{(n)}(E, S) = \overline{\emptyset}^{S}$.

(i) *Monotonicity of* $\overline{A}^{(n)}$*: if* $\overline{E} \leqslant \overline{F}$ *then* $\overline{A}^{(n)}(\overline{E}) \leqslant \overline{A}^{(n)}(\overline{F})$.

(j) *AA Self Reflection of* $\overline{A}^{(\infty)}$*:* $\overline{A}^{(\infty)} = \overline{A}^{(\infty)}\overline{A}^{(\infty)}$.

(k) $\overline{A}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}(\overline{F}) \geqslant \overline{A}^{(n)}(\overline{E} \wedge \overline{F})$ *(i.e., A-Conjunction with equality).*

Remarks on Proposition 4 are in order. First, assume Necessitation and Weak Necessitation. In a generalized state space, while $\overline{U}^{(n)}(\cdot, \inf\Sigma) = \overline{\emptyset}^{\inf\Sigma}$, it is not necessarily the case that $\overline{U}^{(n)}(\cdot, S) = \overline{\emptyset}^{S}$ for $S \neq \inf\Sigma$. Thus, each property in Proposition 4 distinguishes standard and generalized state space models especially under Necessitation. If the analysts would like to impose any of the above properties of unawareness on the agent together with Necessitation, then generalized state space models could describe a non-trivial form of unawareness.

Second, Proposition 4 suggests that Weak Necessitation holds when awareness satisfies logical properties such as Symmetry (i.e., (f)) and Monotonicity (i.e., (i)).

Third, compare (b) with Proposition 2 (1a). On the one hand, (b) states that $K(\overline{S}^{\uparrow}) \wedge (\neg(E, S)) \in \mathcal{J}$ (for all $(E, S) \in \mathcal{J}$) characterizes Weak Necessitation. On the other, Proposition 2 (1a) states that a given information structure is trivial when $\neg(E, S) \in \mathcal{J}$ (for all $(E, S) \in \mathcal{J}$).

Fourth, Negative Non-Introspection is coined by Schipper (2015). LU Introspection states that there is no state at which the agent considers it possible that she is unaware of any particular event. The equivalence of LU Introspection and AU Introspection follows from A-Introspection (Proposition 3 (6)).

Fifth, the equivalence between Subjective Negative Introspection and AU Introspection is closely related to Chen, Ely, and Luo (2012), which show the equivalence of Negative Introspection and AU Introspection in a standard state space with Necessitation. Here, Subjective Negative Introspection reduces to Negative Introspection under Necessitation.

Sixth, Modica and Rustichini (1994, Theorem) show k^2-unawareness is trivial under Symmetry. While Symmetry of $\overline{U}^{(2)}$ yields a rather degenerate form of unawareness, Symmetry of $\overline{U}^{(\infty)}$ does not necessarily imply Symmetry of $\overline{U}^{(2)}$ even in a standard state space (e.g., agent i_1 in Example 1). Similarly, while Weak Necessitation of $\overline{U}^{(2)}$ implies that of $\overline{U}^{(\infty)}$, the converse does not necessarily hold (e.g., agent i_3 in Example 1).

4 Further Properties of Unawareness

In this section, I study properties of unawareness that hinge on whether AU Introspection (or equivalently Weak Necessitation) is satisfied.

4.1 Knowledge of Self-awareness

I ask the knowledge of self-unawareness, i.e., I ask whether there exists a state in which the agent knows that she is unaware of *something*, even though KU Introspection requires that she do not know that she is unaware of any *particular* event. I define the event that the agent is unaware of something in an information structure in a "reduced-form" manner.[13]

Throughout this subsection, fix $\mathcal{S}^{(n)}$ with $n \in \{2, \infty\}$. I formalize the event that the agent is unaware of something by

$$\overline{\mathbb{U}}^{(n)} := (\mathbb{U}^{(n)}, \sup \Sigma) := \left(\bigcup_{\overline{E} \in \mathcal{E}} (r^{\sup \Sigma}_{S(E)})^{-1}(B(\overline{U}^{(n)}(\overline{E}))), \sup \Sigma \right) \in \mathcal{E}.$$

By definition, $\overline{\mathbb{U}}^{(n)} = \bigvee_{\overline{E} \in \mathcal{E}} \overline{U}^{(n)}(\overline{E})$ and $\mathbb{U}^{(n)} = \{\omega \in \sup \Sigma \mid \text{there is } \overline{E} \in \mathcal{E} \text{ such that } \omega \in U^{(n)}(E)\}$. Thus, $\mathcal{S}^{(n)}$ is non-trivial iff $\mathbb{U}^{(n)} \neq \emptyset$. I study the agent's knowledge and awareness of the event $\overline{\mathbb{U}}^{(n)}$.

Proposition 5. *1. (a) If $\mathcal{S}^{(n)}$ additionally satisfies AU Introspection, then $\overline{\mathbb{U}}^{(n)} = \overline{U}^{(n)}(\overline{\sup \Sigma})$, $\overline{K}(\overline{\mathbb{U}}^{(n)}) = \overline{\emptyset}^{\sup \Sigma}$, and $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) = \overline{K}(\overline{\sup \Sigma})$.*

(b) Assume Finite Conjunction on $\overline{K}$. If $\mathcal{E}$ is finite, $\overline{K}(\overline{\mathbb{U}}^{(n)}) = \overline{\emptyset}^{\sup \Sigma}$ and $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) = \overline{K}(\overline{\sup \Sigma})$.

(c) If $\mathcal{E}$ is infinite, it is possible that $\overline{K}(\overline{\mathbb{U}}^{(n)}) \neq \overline{\emptyset}^{\sup \Sigma}$.

2. $\overline{A}(\overline{\mathbb{U}}^{(n)}) \neq \overline{\emptyset}^{\sup \Sigma}$ iff $\overline{K}(\overline{\sup \Sigma}) \neq \overline{\emptyset}^{\sup \Sigma}$.

The first part of Proposition 5 states the following. First, under AU Introspection, the agent can never know that she is unaware of something. Second, under Finite

[13]This subsection asks how one can semantically define such an event. For first-order-logic approaches to self-awareness, see Board and Chung (2007), Halpern and Rêgo (2009), Schipper (2015, Section 3.5), and the references therein. I leave it open whether and how the first-order-logic approaches to self-awareness can be translated into semantic (set-theoretical) forms.

Conjunction, if the domain is finite then the agent never knows that she is unaware of something. Third, however, if a given domain is infinite and if AU Introspection fails, it is possible (even in a standard state space) that the agent knows that she is unaware of something, even though she never knows that she is unaware of any particular event.

The second part of the proposition implies that there is a state at which the agent is *aware* of being unaware of something (i.e., $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) \neq \overline{\emptyset}^{\sup \Sigma}$) iff her knowledge is not degenerate in the subspace $\sup \Sigma$ (i.e., $\overline{K}(E, \sup \Sigma) \neq \overline{\emptyset}^{\sup \Sigma}$ for some $(E, \sup \Sigma) \in \mathcal{E}$).

4.2 (Non-)monotonicity of Unawareness in Knowledge

Under Weak Necessitation, awareness is monotonic in knowledgeability . As Proposition 2 shows that non-trivial unawareness hinges on the qualitative feature of knowledge (e.g., whether the lack of knowledge is self-evident), however, generally awareness may not be monotone in knowledgeability. That is, the fact that $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ does not necessarily imply $\overline{U}_i^{(n)}(\cdot) \leqslant \overline{U}_j^{(n)}(\cdot)$. In Example 1, while agents $(i, j) = (i_1, i_4)$ satisfy this relation, agents $(i, j) = (i_1, i_3)$ do not.

The intuition behind non-monotonicity is that, while an increase in knowledge enhances awareness through knowledge itself, a decrease in knowledge also enhances awareness through the knowledge of the lack of knowledge. In an extreme case, consider an agent i^* with her self-evident collection $\mathcal{J}_{i^*} = \{\emptyset, \Omega\}$ on a standard state space (i.e., $K_{i^*}(E) = \emptyset$ for all $E \in \mathcal{E} \setminus \{\Omega\}$ and $K_{i^*}(\Omega) = \Omega$). She is aware of every event (recall Proposition 2). This is because she always knows that she does not know any non-tautological event. Such agent knows her own ignorance.

One can compare agents' knowledge and unawareness as one agent's knowledge and unawareness over time. Let a state space be given by $\Omega = \{\omega_1, \omega_2, \omega_3\}$ as in Example 1. Denote an agent i's knowledge at time t by $i(t)$. Specifically, let $i(0) = i_4$, $i(1) = i_1$, and $i(2) = i^*$. At time 1, getting more information causes agent i to get aware of some event at each realized state. At time 2, on the other hand, she "forgets" some events, and this may make her aware of some events at some states.

Also, the entire discussion applies to common knowledge. It is possible that if some event is not commonly known then it is commonly known that this is not common knowledge. When each agent receives some events, on the contrary, it may become possible that it is not common knowledge that this is not common knowledge.

Now, I move on to examining possible forms of monotonicity of unawareness in knowledgeability. The key observation is that the knowledge and ignorance operators are monotonic in knowledgeability. Let j be at least as knowledgeable as i. Then, KU Introspection of $(\overline{K}_j, \overline{U}_j^{(n)})$ yields $\overline{K}_i \overline{U}_j^{(n)}(\overline{E}) = \overline{\emptyset}^{S(E)}$. Ignorance is "decreasing" in knowledge because ignorance of an event $\overline{E}$ is expressed in terms of the lack of knowledge of $\overline{E}$ and its negation $\neg\overline{E}$. Thus, for any event $\overline{E}$, if j is ignorant of $\overline{E}$ then so is i. Monotonicity of these operators in knowledgeability implies the following.

Proposition 6. *Let j be at least as knowledgeable as i. Fix $n \in \{2, \infty\}$ and $\overline{E} \in \mathcal{E}$.*

1. (a) $\overline{\partial}_j \overline{K}_i(\overline{E}) \leqslant \overline{U}_i^{(2)}(\overline{E})$. *Equivalently,* $\overline{A}_i^{(2)}(\overline{E}) \leqslant \overline{J}_j \overline{K}_i(\overline{E})$.
   (b) $\overline{\partial}_j \overline{K}_i(\neg \overline{K}_i)(\overline{E}) \leqslant \overline{U}_i^{(\infty)}(\overline{E})$. *Equivalently,* $\overline{A}_i^{(\infty)}(\overline{E}) \leqslant \overline{J}_j \overline{K}_i(\neg \overline{K}_i)(\overline{E})$.
   (c) $\overline{\partial}_j \overline{L}_i^{(n)} \overline{K}_i(\overline{E}) \leqslant \overline{U}_i^{(n)}(\overline{E})$. *Equivalently,* $\overline{A}_i^{(n)}(E) \leqslant \overline{J}_j \overline{L}_i^{(n)} \overline{K}_i(\overline{E})$.
   (d) $\overline{U}_j^{(n)} \overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{\partial}_j \overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{U}_i^{(n)}(\overline{E})$.

2. (a) $\overline{U}_j^{(2)}(E) \leqslant \overline{\partial}_i \overline{K}_j(\overline{E})$. *Equivalently,* $\overline{J}_i \overline{K}_j(\overline{E}) \leqslant \overline{A}_j^{(2)}(\overline{E})$.
   (b) $\overline{U}_j^{(\infty)}(\overline{E}) \leqslant \overline{\partial}_i \overline{K}_j(\neg \overline{K}_j)(\overline{E})$. *Equivalently,* $\overline{J}_i \overline{K}_j(\neg \overline{K}_j)(\overline{E}) \leqslant \overline{A}_j^{(\infty)}(\overline{E})$.
   (c) $\overline{U}_j^{(n)}(E) \leqslant \overline{\partial}_i \overline{L}_j^{(n)} \overline{K}_j(\overline{E})$. *Equivalently,* $\overline{J}_i \overline{L}_j^{(n)} \overline{K}_j(\overline{E}) \leqslant \overline{A}_j^{(n)}(\overline{E})$.
   (d) $\overline{U}_j^{(n)}(\overline{E}) \leqslant \overline{\partial}_i \overline{U}_j^{(n)}(\overline{E})$

3. $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_j \overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_j^{(n)} \overline{A}_i^{(n)}(\overline{E})$.

Suppose that agent j is at least as knowledgeable as i. The first two statements are comparative statics of unawareness with respect to ignorance. First, (1a) states that if j is ignorant of i knowing $\overline{E}$ then i is (k^2-)unaware of $\overline{E}$. Second, on the contrary, (2a) states that if j is (k^2-)unaware of $\overline{E}$ then i is ignorant of j knowing $\overline{E}$. The similar relation holds between the other parts (e.g., (1b) and (2b)).

The third statement says that i's awareness of an event is self-evident to j and that i's awareness of an event $\overline{E}$ implies j's awareness of i's awareness of $\overline{E}$.

Observing that any event $\overline{E}$ that is commonly known among a group of agents is known by any agent in the group, Proposition 6 applies to an individual agent's knowledge and common knowledge operators. I examine the implications of the first two statements of Proposition 6.

Corollary 1. *Fix $i \in I$ and $\overline{E} \in \mathcal{E}$.*

1. (a) $\overline{\partial}_i \overline{C}_I(\overline{E}) \leqslant (\neg \overline{C}_I)(\overline{E}) \wedge (\neg \overline{C}_I)^2(\overline{E})$.
   (b) $\overline{\partial}_i \overline{C}_I(\neg \overline{C}_I)(\overline{E}) \leqslant \bigwedge_{r=1}^{\infty} (\neg \overline{C}_I)^r(E)$.

2. (a) $\overline{U}_i^{(2)}(\overline{E}) \leqslant (\neg \overline{C}_I) \overline{K}_i(\overline{E}) \wedge (\neg \overline{C}_I)(\neg \overline{K}_i)(\overline{E})$.
   (b) $\overline{U}_i^{(\infty)}(\overline{E}) \leqslant (\neg \overline{C}_I) \overline{K}_i(\neg \overline{K}_i)(\overline{E}) \wedge (\neg \overline{C}_I)(\neg \overline{K}_i)^2(\overline{E})$.

Corollary 1 (1a) means that if agent i is ignorant of the common knowledge of an event $\overline{E}$ then the event $\overline{E}$ is not common knowledge and the event that $\overline{E}$ is not common knowledge is not common knowledge. Corollary 1 (1b) states that if agent i is ignorant of the common knowledge that $\overline{E}$ is not common knowledge then this chain of the negation of common knowledge continues *ad infinitum*. Corollary 1 (2a) means the following: suppose that agent i is k^2-unaware of an event $\overline{E}$. Then, it is not common knowledge that i knows $\overline{E}$, and it is not common knowledge that i does not know $\overline{E}$. Corollary 1 (2b) studies the implication of agent i's k^∞-unawareness.

5 Conclusion

This paper studied unawareness from the lack of knowledge in a generalized state space model which nests standard non-partitional and generalized partitional state space models. Unawareness can only take two forms: (i) the ignorance of own knowledge and (ii) the ignorance of the knowledge of the lack of knowledge. For either form, an agent is unaware of an event iff she is unaware of the possibility that she knows the event. This paper characterized when unawareness is non-trivial; which properties of unawareness hold and do not hold in a standard or generalized state space model. For example, the agent is unaware of an event iff she is ignorant of being unaware of it. The paper also provided a notion of self-awareness, and showed that an agent may know that she is unaware of something if the objects of knowledge are infinite and if the agent's unawareness fails AU Introspection. The paper also studied non-monotonicity and possible forms of monotonicity of unawareness in knowledgeability. When unawareness is determined by the lack of knowledge, getting more information may cause the agent to become unaware of a new event. One interesting avenue for future research would be to incorporate agents' probabilistic beliefs in the framework of this paper to capture knowledge, probabilistic beliefs, and unawareness in a single coherent framework.

A Appendix

In order to prove Proposition 1, I establish the following lemma.

Lemma A.1. *Fix $\overline{E} \in \mathcal{E}$. Then, $\overline{K}(\overline{E}) \leqslant (\neg\overline{K})^2(\overline{E}) = (\neg\overline{K})^{2n}(\overline{E}) \leqslant (\neg\overline{K})(\neg\overline{E})$. Also, $\overline{K}(\neg\overline{E}) \leqslant (\neg K)^2(\neg\overline{E}) \leqslant (\neg\overline{K})^{2n+1}(\overline{E}) = (\neg\overline{K})^3(\overline{E}) \leqslant (\neg\overline{K})(\overline{E})$.*

Proof of Lemma A.1. I only show the first statement. By Truth Axiom and Monotonicity, $\overline{K}(\overline{E}) \leqslant (\neg\overline{K})^2(\overline{E}) \leqslant (\neg\overline{K})(\neg\overline{E})$. It suffices to show $(\neg\overline{K})^2 = (\neg\overline{K})^4$. Since Truth Axiom implies $\overline{K}(\overline{E}) \leqslant (\neg\overline{K})^2(\overline{E})$, Positive Introspection and Monotonicity imply $\overline{K}(\overline{E}) \leqslant \overline{K}\,\overline{K}(\overline{E}) \leqslant \overline{K}(\neg\overline{K})^2(\overline{E})$. By Monotonicity, $(\neg\overline{K})^2(\overline{E}) \leqslant (\neg\overline{K})^4(\overline{E})$. Conversely, by Truth Axiom, $\overline{K}(\neg\overline{K})^2(\overline{E}) \leqslant (\neg\overline{K})^2(\overline{E})$, i.e., $\overline{K}(\neg\overline{K})(\overline{E}) \leqslant (\neg\overline{K})^3(\overline{E})$. Monotonicity and Positive Introspection imply $(\neg\overline{K})^4(\overline{E}) \leqslant (\neg\overline{K})^2(\overline{E})$. □

Proof of Proposition 1.

1. By Lemma A.1, $\overline{U}^{(\infty)}(\overline{E}) = \overline{U}^{(3)}(\overline{E}) = (\neg\overline{K})^2(\overline{E}) \wedge (\neg\overline{K})^3(\overline{E}) = \overline{U}^{(2)}(\neg\overline{K})(\overline{E})$.

2. First, $\overline{U}^{(2)}(\overline{E}) = (\neg\overline{K})\overline{K}(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{K})(\overline{E}) = \overline{\partial K}(\overline{E})$. Second, $\overline{\partial K}(\overline{E}) = \overline{U}^{(2)}(\overline{E}) = (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{K})(\overline{E}) \leqslant (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{E}) = \overline{\partial}(\overline{E})$. Third, to obtain $\overline{\partial L}^{(2)}\overline{K} = \overline{U}^{(2)} = \overline{\partial K}$, I show $\overline{L}^{(2)}\overline{K} = \overline{K}$:

$$\overline{L}^{(2)}\overline{K}(\overline{E}) = (\neg\overline{K})^2(\overline{E}) \wedge \overline{A}^{(2)}\overline{K}(\overline{E}) = (\neg\overline{K})^2(\overline{E}) \wedge \big(\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{K})(\overline{E})\big) = \overline{K}(\overline{E}).$$

3. First, $\overline{U}^{(\infty)} = \overline{U}^{(2)}(\neg\overline{K}) = \overline{U}^{(2)}(\neg\overline{K})^2 = \overline{\partial}\overline{K}(\neg\overline{K})^2$. Second, I show $\overline{L}^{(\infty)}\overline{K} = \overline{K}(\neg\overline{K})^2$:

$$\begin{aligned}\overline{L}^{(\infty)}\overline{K}(\overline{E}) &= (\neg\overline{K})^2(\overline{E}) \wedge \overline{A}^{(\infty)}\overline{K}(\overline{E}) \\ &= (\neg\overline{K})^2(\overline{E}) \wedge \big(\overline{K}(\neg\overline{K})(\overline{E}) \vee \overline{K}(\neg\overline{K})^2(\overline{E})\big) = \overline{K}(\neg\overline{K})^2(\overline{E}).\end{aligned}$$

Then, I get $\overline{\partial}\overline{L}^{(\infty)}\overline{K}(\overline{E}) = \overline{U}^{(\infty)}(\overline{E})$.

□

The following proposition provides further properties of unawareness. The first statement will be used in establishing Proposition 4 in Section 3.3.

Proposition A.1. *Fix $\overline{E} \in \mathcal{E}$. First, $\overline{U}^{(2)}(\overline{E}) \wedge \overline{U}^{(2)}(\neg\overline{E}) \leqslant \overline{U}^{(\infty)}(\overline{E})$. Second, $\overline{U}^{(2)}(\overline{E}) \vee \overline{U}^{(2)}(\neg\overline{E}) \leqslant \overline{U}^{(2)}\overline{J}(\overline{E})$, where the equality holds if $\overline{K}$ satisfies Finite Conjunction.*

Proof of Proposition A.1. The first assertion follows from Lemma A.1:

$$\begin{aligned}\overline{U}^{(2)}(\overline{E}) \wedge \overline{U}^{(2)}(\neg\overline{E}) &= (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{E}) \wedge (\neg\overline{K})^2(\neg\overline{E}) \\ &\leqslant (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E}) \wedge (\neg\overline{K})^3(\overline{E}) = \overline{U}^{(\infty)}(\overline{E}).\end{aligned}$$

For the second assertion, observing $\overline{KJ} = \overline{J}$, I have

$$\begin{aligned}\overline{U}^{(2)}\overline{J}(\overline{E}) &= (\neg\overline{K})\overline{J}(\overline{E}) \wedge (\neg\overline{K})^2\overline{J}(\overline{E}) = \overline{\partial}(\overline{E}) \wedge (\neg\overline{K})\overline{\partial}(\overline{E}) \\ &= \overline{\partial}(\overline{E}) \wedge (\neg\overline{K})((\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{E})) \\ &\geqslant \overline{\partial}(\overline{E}) \wedge \neg(\overline{K}(\neg\overline{K})(\overline{E}) \wedge \overline{K}(\neg\overline{K})(\neg\overline{E})) \\ &= (\overline{\partial}(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E})) \vee (\overline{\partial}(\overline{E}) \wedge (\neg\overline{K})^2(\neg\overline{E})).\end{aligned}$$

Since $\overline{\partial}(\overline{E}) = \overline{\partial}(\neg\overline{E})$ and since Lemma A.1 implies that

$$\overline{\partial}(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E}) = (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{E}) = \overline{U}^{(2)}(\overline{E}),$$

I obtain $\overline{U}^{(2)}\overline{J}(\overline{E}) \geqslant \overline{U}^{(2)}(\overline{E}) \vee \overline{U}^{(2)}(\neg\overline{E})$. The equality holds when $\overline{K}$ satisfies Finite Conjunction. □

Proof of Proposition 2.

1. (a) If $\overline{\emptyset}^{S(E)} \neq \overline{U}^{(2)}(\overline{E})$ for some $\overline{E} \in \mathcal{E}$, then $\overline{K}(\overline{E}) \neq (\neg\overline{K})^2(\overline{E})$. Thus, $\overline{K}(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$. Conversely, if $\overline{K}(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$ for some $\overline{E} \in \mathcal{E}$, then $\emptyset \neq (\neg K)^2(E) \setminus K(E) = U^{(2)}(E)$. Thus, $\overline{U}^{(2)}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$.

   (b) If $\mathcal{S}^{(2)}$ is non-trivial, then $\overline{U}^{(2)}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$ for some $\overline{E} \in \mathcal{E}$. By (1a), $\overline{K}(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$, i.e., $\mathcal{J} \setminus \mathcal{J}^* \neq \emptyset$. Conversely, if $\mathcal{J} \setminus \mathcal{J}^* \neq \emptyset$, then there is $\overline{K}(\overline{E}) = \overline{E} \in \mathcal{J} \setminus \mathcal{J}^*$, and hence $\overline{U}^{(2)}(\overline{E}) \neq \overline{\emptyset}^{S(E)}$, i.e., $\mathcal{S}^{(2)}$ is non-trivial. The rest follows because $\overline{E} \in \mathcal{J} \setminus \mathcal{J}^*$ iff $\neg\overline{E} \in \mathcal{J}^* \setminus \mathcal{J}$.

2. (a) The assertion follows from part (1a) and $\overline{U}^{(\infty)} = \overline{U}^{(2)}(\neg\overline{K})$.

   (b) By part (1) and $\overline{U}^{(\infty)} = \overline{U}^{(2)}(\neg\overline{K})$, $\mathcal{S}^{(\infty)}$ is non-trivial iff there is $\overline{E} \in \mathcal{E}$ such that $\overline{K}(\neg\overline{K})(\overline{E}) \in \mathcal{J} \setminus \mathcal{J}^*$. Let $\mathcal{S}^{(\infty)}$ be non-trivial, and let $\overline{F} := \overline{K}(\neg\overline{K})(\overline{E}) \in \mathcal{J}\setminus\mathcal{J}^*$. It is enough to show $\overline{K}(\neg\overline{F}) = \overline{K}(\neg\overline{K})^2(\overline{E}) \in \mathcal{J}\setminus\mathcal{J}^*$. Suppose to the contrary that $\overline{K}(\neg\overline{K})^2(\overline{E}) \in \mathcal{J}^*$. Since $\overline{K}(\neg\overline{K})^3(\overline{E}) = (\neg\overline{K})^3(\overline{E})$, it follows that $(\neg\overline{K})^2(\overline{E}) = (\neg\overline{K})^4(\overline{E}) = \overline{K}(\neg\overline{K})^2(\overline{E}) \in \mathcal{J}$. Hence, $\overline{F} = \overline{K}(\neg\overline{K})(\overline{E}) \in \mathcal{J}^*$, a contradiction. Conversely, suppose that there is $\overline{F} \in \mathcal{J} \setminus \mathcal{J}^*$ such that $\overline{K}(\neg\overline{F}) \in \mathcal{J} \setminus \mathcal{J}^*$. Since $\overline{K}(\overline{F}) = \overline{F}$, it follows that $\overline{K}(\neg\overline{K})(\overline{F}) \in \mathcal{J} \setminus \mathcal{J}^*$. Thus, $\mathcal{S}^{(\infty)}$ is non-trivial.

□

To prove (the last statement of) Proposition 3 and Proposition 5, I establish the following intermediate result.

Lemma A.2. *Let $\overline{K} : \mathcal{E} \to \mathcal{E}$ be an operator satisfying Finite Conjunction and Monotonicity. For any $\overline{E}, \overline{F} \in \mathcal{E}$, $\overline{K}(\overline{E} \vee \overline{F}) \vee (\neg\overline{K})(\neg\overline{F}) \leqslant \overline{K}(\overline{E}) \vee (\neg\overline{K})(\neg\overline{F})$.*

Proof of Lemma A.2. By Finite Conjunction and Monotonicity, $\overline{K}(\overline{E}\vee\overline{F})\wedge\overline{K}(\neg\overline{F}) = \overline{K}(\overline{E} \wedge (\neg\overline{F})) \leqslant \overline{K}(\overline{E})$. Now, $\overline{K}(\overline{E} \vee \overline{F}) \vee (\neg\overline{K})(\neg\overline{F}) = (\overline{K}(\overline{E} \vee \overline{F}) \wedge \overline{K}(\neg\overline{F})) \vee (\neg\overline{K})(\neg\overline{F}) \leqslant \overline{K}(\overline{E}) \vee (\neg\overline{K})(\neg\overline{F})$. □

Proof of Proposition 3. Property 6 (A-Introspection). A-Introspection follows because $\overline{A}^{(n)}(\overline{E}) = \bigvee_{r=1}^{n} \overline{K}(\neg\overline{K})^{r-1}(\overline{E})$ is self-evident.

Property 1. First, $\overline{A}^{(n)}(\overline{E}) = \bigvee_{r=1}^{n} \overline{K}(\neg\overline{K})^{r-1}(\overline{E}) \leqslant \overline{K}(\overline{S}^{\uparrow})$. Second, by KU Introspection, $\overline{K}(\overline{S}^{\uparrow}) \geqslant \overline{A}^{(n)}\overline{U}^{(n)}(\overline{E}) \geqslant \overline{K}(\neg\overline{K})\overline{U}^{(n)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$.

Property 2 (Reverse AU Introspection). By Property 1, $\overline{U}^{(n)}\overline{U}^{(n)}(\overline{E}) = (\neg\overline{K})(\overline{S}^{\uparrow}) \leqslant \overline{U}^{(n)}(\overline{E})$ and $\overline{U}^{(n)}\overline{U}^{(n)}\overline{U}^{(n)}(\overline{E}) = \overline{U}^{(n)}\overline{U}^{(n)}(\overline{E})$.

Property 3 (JU Introspection). JU Introspection follows from KU Introspection and A-Introspection: $\overline{\partial U}^{(n)}(\overline{E}) = (\neg\overline{K})\overline{U}^{(n)}(\overline{E}) \wedge (\neg\overline{K})\overline{A}^{(n)}(\overline{E}) = \overline{U}^{(n)}(\overline{E})$.

Property 4 (Weak A-Negative Introspection). Weak A-Negative Introspection follows from the definition of $\overline{A}^{(2)}$.

Property 5 (AK Self-Reflection). AK Self-Reflection follows from $\overline{K} = \overline{KK}$.

Property 7 (Weak AA Self-Reflection). Let $n = 2$. By A-Introspection and KU Introspection, $\overline{U}^{(2)}\overline{A}^{(2)}(\overline{E}) = (\neg\overline{K})\overline{A}^{(2)}(\overline{E}) \wedge (\neg\overline{K})^2\overline{A}^{(2)}(\overline{E}) = \overline{U}^{(2)}(\overline{E}) \wedge (\neg\overline{K})\overline{U}^{(2)}(\overline{E}) = \overline{U}^{(2)}(\overline{E})$. Then, $\overline{A}^{(2)}(\overline{E}) = \overline{A}^{(2)}\overline{A}^{(2)}(\overline{E})$.

Next, let $n = \infty$. By A-Introspection, $\overline{U}^{(\infty)}\overline{A}^{(\infty)}(\overline{E}) \leqslant (\neg\overline{K})\overline{A}^{(\infty)}(\overline{E}) = \overline{U}^{(\infty)}(\overline{E})$. Thus, $\overline{A}^{(\infty)}(\overline{E}) \leqslant \overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E})$. By A-Introspection and KU Introspection, $\overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$. Thus, $\overline{A}^{(\infty)}\overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E}) = \overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E})$.

Property 8. By KU Introspection, $\overline{L}^{(n)}\overline{A}^{(n)}(\overline{E}) = (\neg\overline{K})\overline{U}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}\overline{A}^{(n)}(\overline{E}) = \overline{A}^{(n)}\overline{A}^{(n)}(\overline{E})$.

Property 9 (A-Conjunction). Let $\mathcal{F}$ be a non-empty finite subset of $\mathcal{E}$. I show that $\overline{U}^{(n)}(\bigwedge \mathcal{F}) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}^{(n)}(\overline{E})$. For $n = 2$, by Finite Conjunction and Monotonicity,

$$\overline{U}^{(2)}(\bigwedge \mathcal{F}) = \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})^2(\bigwedge \mathcal{F}) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K})^2(\overline{F})$$

$$= \bigvee_{\overline{E}\in\mathcal{F}} \left((\neg\overline{K})(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K})^2(\overline{F}) \right) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} ((\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})^2(\overline{E})) = \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}^{(2)}(\overline{E}).$$

Note that the same proof works, for example, for any non-empty subset $\mathcal{F}$ of $\mathcal{E}$ if $\mathcal{S}$ satisfies Non-empty Conjunction.

Next, consider $n = \infty$. By Finite Conjunction and Monotonicity,

$$\overline{U}^{(\infty)}(\bigwedge \mathcal{F}) = (\neg\overline{K})^2(\bigwedge \mathcal{F}) \wedge (\neg\overline{K})^3(\bigwedge \mathcal{F}) \leqslant \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K})^2(\overline{F}) \wedge (\neg\overline{K})^2(\bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})(\overline{E})).$$

I show

$$(\neg\overline{K})^2(\bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})(\overline{E})) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})^3(\overline{E}). \tag{A.1}$$

Since

$$\overline{U}^{(\infty)}(\bigwedge \mathcal{F}) \leqslant \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K})^2(\overline{F}) \wedge \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K})^3(\overline{E}) = \bigvee_{\overline{E}\in\mathcal{F}} \left(\bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K})^2(\overline{F}) \wedge (\neg\overline{K})^3(\overline{E}) \right)$$
$$\leqslant \bigvee_{\overline{E}\in\mathcal{F}} ((\neg\overline{K})^2(\overline{E}) \wedge (\neg\overline{K})^3(\overline{E})) = \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}^{(\infty)}(\overline{E}),$$

it suffices to establish Expression (A.1) for $\mathcal{F} = \{\overline{E}_1, \overline{E}_2\}$. Let $\overline{F}_j := (\neg\overline{K})(\overline{E}_j)$, and notice that $(\neg\overline{K})(\neg\overline{F}_j) = \overline{F}_j$ for each j. It follows from Lemma A.2 that

$$\overline{K}(\overline{F}_1 \vee \overline{F}_2) \leqslant \overline{K}(\overline{F}_1 \vee \overline{F}_2) \vee (\neg\overline{K})(\neg\overline{F}_2) \leqslant \overline{K}(\overline{F}_1) \vee (\neg\overline{K})(\neg\overline{F}_2) \leqslant (\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2.$$

Then, $(\neg\overline{K})^2(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K})(\neg((\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2)) = (\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2$. By Lemma A.2, $\overline{K}((\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2) \leqslant \overline{K}(\overline{F}_2) \vee (\neg\overline{K})^2(\overline{F}_1) \leqslant (\neg\overline{K})^2(\overline{F}_1) \vee (\neg\overline{K})^2(\overline{F}_2)$. By Monotonicity and Finite Conjunction, $(\neg\overline{K})^2((\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2) \leqslant (\neg\overline{K})(\neg((\neg\overline{K})^2(\overline{F}_1) \vee (\neg\overline{K})^2(\overline{F}_2))) = (\neg\overline{K})^2(\overline{F}_1) \vee (\neg\overline{K})^2(\overline{F}_2)$. Since $(\neg\overline{K})^2 = (\neg\overline{K})^4$ (Lemma A.1),

$$(\neg\overline{K})^2(\overline{F}_1 \vee \overline{F}_2) = (\neg\overline{K})^4(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K})^2((\neg\overline{K})^2(\overline{F}_1) \vee \overline{F}_2) \leqslant (\neg\overline{K})^2(\overline{F}_1) \vee (\neg\overline{K})^2(\overline{F}_2),$$

which establishes Expression (A.1) as desired. □

Proof of Proposition 4. In the proof, I often denote $\overline{E} = (E, S)$. First, I show that (a) and (b) are equivalent. (a) implies (b) because $\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{E}) = \overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{K})(\overline{E}) \in \mathcal{J}$ for any $\overline{E} \in \mathcal{J}$. Conversely, if (b) holds then $\overline{K}(\overline{E}) \in \mathcal{J}$ implies (a).

Second, I show that (a) is equivalent to Weak Necessitation of $\overline{U}^{(2)}$. Suppose (a). Since $\overline{K}(\neg \overline{K})(\overline{E}) \geqslant \overline{K}(\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{K})(\overline{E}))$, I get $(\neg \overline{K})^2(\overline{E}) \leqslant (\neg \overline{K})(\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{K})(\overline{E})) \leqslant \neg(\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{K})(\overline{E}))$. Since $(\neg \overline{K})(\overline{S}^{\uparrow}) \leqslant (\neg \overline{K})(\overline{E})$, I get $\overline{U}^{(2)}(\overline{E}) \leqslant (\neg \overline{K})(\overline{S}^{\uparrow}) \leqslant \overline{U}^{(2)}(\overline{E})$. Conversely, Weak Necessitation implies $\overline{K}(\overline{S}^{\uparrow}) = \overline{K}(\overline{E}) \vee \overline{K}(\neg \overline{K})(\overline{E})$. Then, (a) follows because $\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg \overline{K})(\overline{E}) = (\overline{K}(\overline{E}) \vee \overline{K}(\neg \overline{K})(\overline{E})) \wedge (\neg \overline{K})(\overline{E}) = \overline{K}(\neg \overline{K})(\overline{E}) \in \mathcal{J}$.

Third, (c) follows from Weak Necessitation of $\overline{U}^{(2)}$ because $\overline{U}^{(2)}(\overline{E}) = (\neg \overline{K})(\overline{S}^{\uparrow}) \leqslant (\neg \overline{K})^3(\overline{E})$. Next, (c) is equivalent to (d). Now, I show that (d) implies Weak Necessitation. By (d) and Proposition 3 (7), $\overline{A}^{(2)} = \overline{A}^{(2)}\overline{A}^{(2)} = \overline{A}^{(\infty)}\overline{A}^{(2)}$. By Propositions 1 (1) and 3, $\overline{A}^{(\infty)}\overline{A}^{(2)}(\overline{E}) = \overline{A}^{(2)}(\neg \overline{K})\overline{A}^{(2)}(\overline{E}) = \overline{A}^{(2)}\overline{U}^{(2)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$. Thus, I obtain $\overline{A}^{(2)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$, which is equivalent to Weak Necessitation.

Fourth, (e) follows from (d) and Proposition 3 (4). Conversely, (e) implies $\overline{U}^{(2)}(\overline{E}) = (\neg \overline{K})(\overline{E}) \wedge (\neg \overline{K})^2(\overline{E}) = (\neg \overline{K})(\overline{E}) \wedge (\overline{K}(\overline{E}) \vee \overline{U}^{(\infty)}(\overline{E})) = \overline{U}^{(\infty)}(\overline{E})$.

Fifth, (f) follows from Weak Necessitation of $\overline{U}^{(2)}$. Conversely, by Proposition A.1, (f) implies $\overline{U}^{(2)}(\overline{E}) = \overline{U}^{(2)}(\overline{E}) \wedge \overline{U}^{(2)}(\neg \overline{E}) \leqslant \overline{U}^{(\infty)}(\overline{E}) \leqslant \overline{U}^{(2)}(\overline{E})$, i.e., (d).

Sixth, (g) follows from Weak Necessitation of $\overline{U}^{(n)}$. Conversely, suppose (g). Reverse AU Introspection and Proposition 3 (1) yield $\overline{U}^{(n)}(\overline{E}) = \overline{U}^{(n)}\overline{U}^{(n)}(\overline{E}) = (\neg \overline{K})(\overline{S}^{\uparrow})$.

Seventh, it follows from A-Introspection (Proposition 3 (6)) that (g) and (h) are equivalent: $\overline{L}^{(n)}\overline{U}^{(n)}(\overline{E}) = (\neg \overline{K})(\overline{A}^{(n)})(\overline{E}) \wedge \overline{A}^{(n)}\overline{U}^{(n)}(\overline{E}) = \overline{U}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}\overline{U}^{(n)}(\overline{E})$.

Eighth, I show that (i) follows from Weak Necessitation of $\overline{U}^{(n)}$. If $(E, S(E)) \leqslant (F, S(F))$, then $\overline{A}^{(n)}(E, S(E)) = \overline{K}(\overline{S(E)}^{\uparrow}) \leqslant \overline{K}(\overline{S(F)}^{\uparrow}) = \overline{A}^{(n)}(F, S(F))$. Conversely, suppose (i). Since $\overline{A}^{(n)}(\overline{\emptyset}^{S}) = \overline{K}(\overline{S}^{\uparrow})$ follows from $\overline{K}(\overline{\emptyset}^{S}) = \overline{\emptyset}^{S}$, I get $\overline{K}(\overline{S}^{\uparrow}) = \overline{A}^{(n)}(\overline{\emptyset}^{S}) \leqslant \overline{A}^{(n)}(E, S) \leqslant \overline{K}(\overline{S}^{\uparrow})$.

Ninth, (j) follows from Weak Necessitation of $\overline{A}^{(\infty)}$. Conversely, since $\overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$, (j) implies that $\overline{A}^{(\infty)}(\overline{E}) = \overline{A}^{(\infty)}\overline{A}^{(\infty)}(\overline{E}) = \overline{K}(\overline{S}^{\uparrow})$.

Tenth, I show that (k) follows from Weak Necessitation. Let $S := \sup(S(E), S(F))$. Then, $\overline{A}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}(\overline{F}) = \overline{K}(\overline{S(E)}^{\uparrow}) \wedge \overline{K}(\overline{S(F)}^{\uparrow}) = \overline{K}(\overline{S}^{\uparrow}) = \overline{A}^{(n)}(\overline{E} \wedge \overline{F})$. Conversely, assume (k). Then, $\overline{A}^{(n)}(\overline{E}) = \overline{A}^{(n)}(\overline{E}) \wedge \overline{A}^{(n)}(\overline{\emptyset}^{S}) = \overline{A}^{(n)}(\overline{\emptyset}^{S}) = \overline{K}(\overline{S}^{\uparrow})$.

□

Proof of Proposition 5. 1. (a) It is enough to prove the first assertion. By AU

Introspection,

$$\overline{\mathbb{U}}^{(n)} = \neg \bigwedge_{\overline{E}\in\mathcal{E}} \overline{A}^{(n)}(\overline{E}) = \neg \bigwedge_{\overline{E}\in\mathcal{E}} \overline{K}(\overline{S(E)}^{\uparrow}) = \neg\overline{K}(\overline{\sup\Sigma}) = \overline{U}^{(n)}(\overline{\sup\Sigma}).$$

(b) Since $\mathcal{E}$ is assumed to be finite, re-label it by $\mathcal{E} = \{\overline{E}_r\}_{r=1}^m$ so that $\overline{\mathbb{U}}^{(n)} = \bigvee_{r=1}^m \overline{U}^{(n)}(\overline{E}_r)$. Let $S_r := S(\overline{E}_r) \in \Sigma$ for each $\overline{E}_r \in \mathcal{E}$.
It follows from Lemma A.2 that $\overline{K}(\overline{E} \vee \overline{F}) \leqslant \overline{K}(\overline{K}(\overline{E}) \vee (\neg\overline{K})(\neg\overline{F}))$ for any $\overline{E}, \overline{F} \in \mathcal{E}$. Substitute $\overline{E} = \overline{U}^{(n)}(\overline{E}_m)$ and $\overline{F} = \bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)$ in the above expression. Since $(\neg K \neg)(\bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)) = \bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)$ follows from Finite Conjunction and Proposition 3 (6), I get

$$\begin{aligned}\overline{K}(\overline{\mathbb{U}}^{(n)}) &\leqslant \overline{K}(\overline{KU}^{(n)}(\overline{E}_m) \vee \bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)) = \overline{K}(\overline{\emptyset}^{S_m} \vee \bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)) \\ &\leqslant \overline{K}(\bigvee_{r=1}^{m-1} \overline{U}^{(n)}(\overline{E}_r)).\end{aligned}$$

By induction, $\overline{K}(\overline{\mathbb{U}}^{(n)}) \leqslant \overline{\emptyset}^{S_1}$. Since $K(\mathbb{U}^{(n)}) = \emptyset$, it follows $\overline{K}(\overline{\mathbb{U}}^{(n)}) = \overline{\emptyset}^{\sup\Sigma}$. Then, $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) = \overline{K}(\overline{\sup\Sigma})$.

(c) I provide a counterexample when $\mathcal{E}$ is not finite in the context of a standard state space (one could embed the following standard state space example into a generalized state space). Let $\Omega = \mathbb{R}$ and $\mathcal{E} = \mathcal{P}(\Omega)$. Note that any infinite complete algebra of sets is uncountable, as it is well known that any infinite σ-algebra is already uncountable. Suppose that the agent's knowledge operator is given by the interior operator on the usual Euclidean topology. Her knowledge satisfies Finite Conjunction and Necessitation as well as Truth Axiom, Monotonicity, and Positive Introspection. Thus, $U^{(n)}$ satisfies Plausibility and KU Introspection. It can be seen that $U^{(n)}$ violates AU Introspection. Now, for any $\omega \in \mathbb{R}$, let $E_\omega = (\omega, +\infty)$. Then, $U^{(2)}(E_\omega) = \partial K E_\omega = \{\omega\}$ and $U^{(\infty)}(E_\omega) = \partial(\neg K)^2 E_\omega = \{\omega\}$. Thus, $K(\bigcup_{\omega\in\Omega} U^{(n)}(E_\omega)) = K(\Omega) = \Omega$. This implies that $K(\overline{U}^{(n)}) = \Omega$.

2. Since $\overline{K}(\overline{\sup\Sigma}) \geqslant \overline{A}(\overline{\mathbb{U}}^{(n)})$, it is enough to show the "if" part. If $\overline{K}(\overline{\mathbb{U}}^{(n)}) = \overline{\emptyset}^{\sup\Sigma}$, then $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) = \overline{K}(\overline{\sup\Sigma}) \neq \overline{\emptyset}^{\sup\Sigma}$. If $\overline{K}(\overline{\mathbb{U}}^{(n)}) \neq \overline{\emptyset}^{\sup\Sigma}$, then $\overline{A}^{(n)}(\overline{\mathbb{U}}^{(n)}) \geqslant \overline{K}(\overline{\mathbb{U}}^{(n)}) \neq \overline{\emptyset}^{\sup\Sigma}$.

□

Proof of Proposition 6. 1. By Proposition 1, substituting $\overline{K}_i(\overline{E})$, $\overline{K}_i(\neg\overline{K}_i)(\overline{E})$, and $\overline{L}_i^{(n)}\overline{K}_i(\overline{E})$ into $\overline{\partial}_j(\cdot) \leqslant \overline{\partial}_i(\cdot)$ yields (1a), (1b), and (1c), respectively.

For (1d), $\overline{K}_j\overline{K}_i = \overline{K}_i$ implies $\overline{U}_j^{(n)}\overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{\partial}_j\overline{U}_i^{(n)}(\overline{E}) = (\neg\overline{K}_j)\overline{U}_i^{(n)}(\overline{E}) \wedge (\neg\overline{K}_j)\overline{A}_i^{(n)}(\overline{E}) \leqslant (\neg\overline{K}_j)\overline{A}_i^{(n)}(\overline{E}) = \overline{U}_i^{(n)}(\overline{E})$.

2. Consider (2a), (2b), and (2c). Recalling Proposition 1, substituting $\overline{K}_j(\overline{E})$, $\overline{K}_j(\neg\overline{K}_j)(\overline{E})$, and $\overline{L}_j^{(n)}\overline{K}_j(\overline{E})$ into $\overline{\partial}_j(\cdot) \leqslant \overline{\partial}_i(\cdot)$ yields the desired results. For (2d), $\overline{\partial}_i\overline{U}_j^{(n)}(\overline{E}) = (\neg\overline{K}_i)\overline{U}_j^{(n)}(\overline{E}) \wedge (\neg\overline{K}_i)\overline{A}_j^{(n)}(\overline{E}) \geqslant (\neg\overline{K}_j)\overline{U}_j^{(n)}(\overline{E}) \wedge (\neg\overline{K}_i)\overline{A}_j^{(n)}(\overline{E}) = (\neg\overline{K}_i)\overline{A}_j^{(n)}(\overline{E}) \geqslant (\neg\overline{K}_j)\overline{A}_j(\overline{E}) = \overline{U}_j^{(n)}(\overline{E})$.

3. By Proposition 1, $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_i\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{K}_j\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_i^{(n)}(\overline{E})$. Then, $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_j\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_j^{(n)}\overline{A}_i^{(n)}(\overline{E})$.

□

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