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A Note on Reduced Strategies and Cognitive Hierarchies in the Extensive and Normal Form*

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Abstract

In a recent paper, Lin & Palfrey (2022, revised 2023) developed a theory of cognitive hierarchies (CH) in sequential games and observed that this solution concept in not reduced-normal-form invariant. In this note I qualify this observation by showing that *the CH model is normal-form invariant*, and that the differences arising from the application of the CH model to the *reduced* normal form depend only on how randomization by level-0 types is modeled. Indeed, while the uniform behavior strategy in the extensive form yields the uniform mixed strategy in the normal form, the latter does not correspond to the uniform randomization in the reduced normal form, because different reduced strategies may correspond to sets of equivalent strategies with different cardinalities. I also note that results in the literature on transformations of sequential games imply that the sequential CH model of Lin & Palfrey is invariant to the interchanging of essentially simultaneous moves, but it is not invariant to coalescing of moves (and, of course, its inverse, sequential agents splitting). Finally, I note that the independence of ex ante beliefs about the level-types of co-players

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is preserved by updated beliefs conditional on information sets in all games with observable deviators.

KEYWORDS: Cognitive hierarchies, sequential games, extensive form, normal form, structurally reduced normal form, coalescing of moves, independence, observable deviators.

This note is prompted by the paper on "Cognitive Hierarchies in Extensive Form Games" by Lin & Palfrey (2022).[1] These authors observe that the Cognitive Hierarchies (CH) model is not reduced normal-form invariant. I qualify this observation by showing that *the CH model is normal-form invariant*, and that the differences arising from the application of the CH model to the *reduced* normal form depend on how randomization by level-0 types is modeled. Indeed, while the uniform behavior strategy in the extensive form yields[2] the uniform mixed strategy in the normal form, the latter does not yield the uniform randomization in the (structurally) reduced normal form, because different reduced strategies may correspond to sets of equivalent strategies with different cardinalities.[3] Specifically, let $\mathbf{r}_i \subseteq S_i$ denote **reduced strategies**, that is, elements of the equivalence partition (quotient space) of the set S_i of pure strategies. Then it may well be the case that $|\mathbf{r}_i'| \neq |\mathbf{r}_i''|$ for different reduced strategies $\mathbf{r}_i'$ and $\mathbf{r}_i''$. Uniform randomization on S_i does not correspond to uniform randomization over reduced strategies, but rather to assigning probability $|\mathbf{r}_i| / |S_i|$ to each reduced strategy $\mathbf{r}_i$. Thus, if the level-0 type of each player i plays each reduced strategy $\mathbf{r}_i$ with probability $|\mathbf{r}_i| / |S_i|$, then applying the CH model to the (structurally) reduced normal form of the given sequential game is equivalent to applying the the CH model to the extensive-form representation of the game. Given known results in the literature on transformations of games represented in extensive form,[4] my observations imply that the sequential CH model is invariant to interchanging essentially simultaneous moves, but it is *not invariant to coalescing of moves* and—of course—its inverse, sequential agents splitting.

[1] The 2023 revision of the paper by Lin & Palfrey takes this note into account. On the cognitive hierarchies model in static games see Camerer et al. (2004) and the other relevant references in Lin & Palfrey (2022).

[2] By means of Kuhn's (1953) transformation of behavior into mixed strategies.

[3] Unless the game tree satisfies a "balancedness" condition. See Remark 5. I use the realization equivalence relation on strategies, which is "structural" because it is independent of payoffs.

[4] See Battigalli et al. (2020) and the relevant references therein.

Representation of Sequential Games Within the class of sequential games *represented* in extensive form,[5] I mostly focus, for the sake of simplicity, on *finite games with perfect information* and *without chance moves*. Since the substantive content of Lin & Palfrey (2022) mostly concerns these games, little is lost by restricting attention to them. Much of the notation is the same as in the Lin & Palfrey (2022, for this class of games). Small (Latin or Greek) letters will typically denote elements of sets represented by the corresponding capital letters. Bold symbols will be used to denote equivalence classes.

I adopt the definition and representation of perfect information games of Chapter 6 of the textbook of Osborne & Rubinstein (1994). The basic primitives of the game

$$\Gamma = \langle I, A, H, P, u \rangle$$

are as follows:

- A *finite* **set of actions** A.
- A *finite* **set of histories** H, that is, finite sequences of actions, including the **empty sequence** $\varnothing$. Thus,

  $$H \subset \{\varnothing\} \cup \left(\bigcup_{k\in\mathbb{N}} A^k \right);$$

  furthermore, H is *closed under the* canonical **prefix-of** relation $\preceq$, which makes it a tree with root $\varnothing$. With this, $A(h) := \{a \in A : (h, a) \in H\}$ is the set of feasible actions given h, and $Z := \{h \in H : A(h) = \emptyset\}$ is the **set of terminal histories**. To avoid trivialities I assume that *there are at least 2 feasible actions at each non terminal history*: $|A(h)| \geq 2$ for every $h \in H \backslash Z$.
- The **player set is** I and the **player function** is $P : H\backslash Z \to I$. With this, $H_i := P^{-1}(i)$ is the set of non-terminal histories where player $i \in I$ is active.[6]

[5] I banned the terms "normal-form game" and "extensive-form game" from my vocabulary, because *the extensive and normal forms are kinds of representations of games, not kinds of games.* Game with simultaneous moves and games with sequential moves are kinds of games.

[6] In Osborne & Rubinstein (1994) and Lin & Palfrey (2022) the player set is instead denoted by N with the convention that $N = \{1, ..., n\}$. Since here such convention (a strict order on the player set) does not play any useful role, I stick to the notation of my (still incomplete) textbook Battigalli et al. (2022). With this, i is an element of I.

- The description of the game is completed by the profile of payoff functions $u = (u_i : Z \to \mathbb{R})_{i \in I}$.
- The analysis also requires the exogenous specification of a probability measure (distribution on level-types) $p_i = (p_{i\ell})_{\ell \in \mathbb{N}_0} \in \Delta(\mathbb{N}_0)$ for each player $i \in I$.

From the aforementioned primitives we *derive*:

- The **set of** (pure) **strategies** $S_i := \times_{h \in H_i} A(h)$ of each player i, and profiles thereof $S := \times_{i \in I} S_i \cong \times_{h \in H \setminus Z} A(h)$, $S_{-i} := \times_{j \neq i} S_j$.
- The **path/outcome function** $O : S \to Z$, which specifies the terminal history induced by each strategy profile.
- The profile of **normal-form** payoff functions $(U_i = u_i \circ O : S \to \mathbb{R})_{i \in I}$.
- The **set of** (structurally, pure) **reduced strategies** (sometimes called "plans of actions") $\mathbf{R}_i := S_i | \approx_i$, where $\approx_i$ is the behavioral/realization[7] equivalence relation

$$s_i' \approx_i s_i'' \iff (\forall s_{-i} \in S_{-i}, O(s_i', s_{-i}) = O(s_i'', s_{-i})),$$

and $\mathbf{R}_i$ is the partition (quotient) of S_i induced by equivalence relation $\approx_i$. I let $\bar{R}_i(s_i)$ denote the set of strategies equivalent to s_i, that is, the reduced strategy $\mathbf{r}_i \in \mathbf{R}_i$ such that $s_i \in \mathbf{r}_i$. I call the partitional map

$$\begin{array}{rccl} \bar{R}_i : & S_i & \to & \mathbf{R}_i \\ & s_i & \mapsto & \{s_i' \in S_i : s_i' \approx_i s_i\} \end{array}$$

"**reduction map**"; note that the right inverse $\bar{R}_i^{-1}$ is the identity map on partition $\mathbf{R}_i$. Also let $\mathbf{R} := \times_{i \in I} \mathbf{R}_i$ and $\mathbf{R}_{-i} := \times_{j \neq i} \mathbf{R}_j$. Since $O(\cdot)$ is constant on each product of cells $\mathbf{r} = \times_{i \in I} \mathbf{r}_i$, it makes sense to write $\mathbf{O} : \mathbf{R} \to Z$, with $\mathbf{O}(\mathbf{r}) = O(s)$ for all $s \in \mathbf{r}$.

- The profile of (structurally) **reduced normal-form** payoff functions $\left(\bar{U}_i : \mathbf{R} \to \mathbb{R}\right)_{i \in I}$ is such that $\bar{U}_i\left(\left(\bar{R}_j(s_j)\right)_{j \in I}\right) = U_i(s)$ for all $s = (s_j)_{j \in I} \in S$ (well posed by definition of $\mathbf{R}$ *via* realization equivalences).

[7] Cf. Rubinstein (1991), Ch. 6.4 in Osborne & Rubinstein (1994), Battigalli et al. (2020), Chapter 9 in Battigalli et al. (2022), and Theorem 1 in Kuhn (1953). I call such strategies "structurally reduced" because the reduction does not depend on payoff fuctions.

- The **set of behavior strategies** $\Sigma_i := \times_{h \in H_i} \Delta(A(h))$ of each player i, with $\Sigma := \times_{i \in I} \Sigma_i$, $\Sigma_{-i} = \times_{j \neq i} \Sigma_j$. The **uniform behavior strategy** of player i is denoted σ_i^0 with

$$\forall h \in H_i, \forall a \in A(h), \sigma_{ih}^0(a) = \frac{1}{|A(h)|}.$$

Randomizations, Uniform Randomization

Remark 1 *For each player $i \in I$, the cardinality of i's strategy set is $|S_i| = \prod_{h \in H_i} |A(h)|$; therefore, the uniform behavior strategy of i yields the uniform mixed strategy under the Kuhn's (1953) transformation that preserves the probabilities of paths.*

Proof. By Kuhn's transformation, behavior strategy $\sigma_i = (\sigma_{ih})_{h \in H_i} \in \Sigma_i$ yields the mixed strategy $\mu_i^{\sigma_i} \in \Delta(S_i)$ such that

$$\forall s_i \in S_i, \mu_i^{\sigma_i}(s_i) = \prod_{h \in H_i} \sigma_{ih}(s_{ih}).$$

Hence, the mixed strategy μ_i^0 obtained from the uniform behavior strategy σ_i^0 satisfies, for every $s_i \in S_i$,

$$\begin{aligned} \mu_i^0(s_i) &= \prod_{h \in H_i} \sigma_{ih}^0(s_{ih}) = \prod_{h \in H_i} \frac{1}{|A(h)|} \\ &= \frac{1}{\prod_{h \in H_i} |A(h)|} = \frac{1}{|S_i|}. \quad \blacksquare \end{aligned}$$

For any $i \in I$ and $s_i \in S_i$, let

$$H_i(s_i) := \{h \in H_i : \exists s_{-i} \in S_{-i}, h \prec O(s_i, s_{-i})\}$$

denote the set of histories where i moves that are allowed (not prevented) by strategy s_i. The following is Theorem 1 in Kuhn (1953):[8]

Lemma 1 *For any player $i \in I$, two strategies are realization equivalent if and only if they allow for the same set of non-terminal histories where i moves and prescribe the same actions at such histories, that is, for all $s_i', s_i'' \in S_i$,*

$$s_i' \approx_i s_i'' \Longleftrightarrow (H_i(s_i') = H_i(s_i'') \wedge (\forall h \in H_i(s_i'), s_{ih}' = s_{ih}'')).$$

[8] Adapted to perfect information games.

Definition 1 *Game Γ has the **one-move** property if no player moves more than once in any path of play, that is, for all $z \in Z$ and $i \in I$, $|\{h \prec z : P(h) = i\}| \leq 1$.*

Remark 2 *Game Γ has the one-move property if and only if reduced and non reduced strategies coincide (that is, if and only if $\mathbf{R}_i$ is the finest partition of S_i for each $i \in I$).*

Proof If Γ has the one-move property, then $H_i(s_i) = H_i$ for every $s_i \in S_i$. Therefore, two strategies of i are not equivalent if and only if they prescribe different actions for at least one $h \in H_i$, that is, if and only if they are different. Thus, $\mathbf{R}_i$ contains only singletons.

Now suppose that Γ does not have the one-move property. Then there are a player $i \in I$ and a pair of histories $h, \bar{h} \in H_i$ such that $h \prec \bar{h}$. Let $\bar{a} \in A(h)$ denote the action such that $(h, \bar{a}) \preceq \bar{h}$ and fix $a^\circ \in A(h) \setminus \{\bar{a}\}$ (a° exists because $|A(h)| \geq 2$ by assumption). Pick two *distinct* actions $a', a'' \in A\left(\bar{h}\right)$ and consider two strategies s_i', s_i'' with the following properties: (1) they select the actions of i specified by history h for all strict prefixes of h where i moves (if any), so that $h \in H_i(s_i') \cap H_i(s_i'')$, (2) they both select a° at h so that $\bar{h} \notin (H_i(s_i') \cup H_i(s_i''))$, (3) $s'_{i\bar{h}} = a'$, $s''_{i\bar{h}} = a''$, and (4) they select the same action at every other history h' where i moves (if any). Then, $H_i(s_i') = H_i(s_i'')$ and s_i' and s_i'' select the same actions at such reachable histories; hence, they are equivalent by Lemma 1: $\bar{R}_i(s_i') = \bar{R}_i(s_i'')$. Yet, $s_i' \neq s_i''$ because $s'_{i\bar{h}} = a' \neq a'' = s''_{i\bar{h}}$. Therefore, the equivalence class $\bar{R}_i(s_i') = \bar{R}_i(s_i'')$ is not a singleton. ■

Given any mixed strategy $\mu_i \in \Delta(S_i)$, we obtain the corresponding image (pushforward) **reduced mixed strategy** $\bar{\mu}_i = \mu_i \circ \bar{R}_i^{-1} \in \Delta(\mathbf{R}_i)$ *via* the reduction map $\bar{R}_i : S_i \to \mathbf{R}_i$, that is, for all $\mathbf{r}_i \in \mathbf{R}_i$,

$$\bar{\mu}_i(\mathbf{r}_i) = \left(\mu_i \circ \bar{R}_i^{-1}\right)(\mathbf{r}_i) = \sum_{s_i \in \mathbf{r}_i} \mu_i(s_i).$$

To ease notation, for any mixed strategy *profile* $\mu = (\mu_i)_{i \in I}$, I write

$$\begin{array}{rccl} \mu \circ \bar{R}^{-1}: & \Delta(S) & \to & \Delta(\mathbf{R}), \\ & \mathbf{r} & \mapsto & \prod_{i \in I} \sum_{s_i \in \mathbf{r}_i} \mu_i(s_i) \end{array}$$

for the image (pushforward) product measure induced by the collective reduction map

$$\begin{array}{rccl} \bar{R}: & S & \to & \mathbf{R} \\ & (s_i)_{i \in I} & \mapsto & \left(\bar{R}_i(s_i)\right)_{i \in I}. \end{array}$$

Remark 3 *For the purposes of expected-payoff calculations, the only measures that matter are the probability measures on reduced strategies induced by each mixed strategy, that is, for every* $\mu = \left(\mu_j\right)_{j\in I} \in \times_{j\in I}\Delta\left(S_j\right)$ *and* $i \in I$,

$$\mathbb{E}_{\mu}\left(U_i\right) = \sum_{s\in S} u_i\left(O\left(s\right)\right)\prod_{j\in I}\mu_j\left(s_j\right) = \sum_{\mathbf{r}\in\mathbf{R}} u_i\left(\mathbf{O}\left(\mathbf{r}\right)\right)\sum_{s\in\bar{R}^{-1}(\mathbf{r})}\prod_{j\in I}\mu_j\left(s_j\right) = \mathbb{E}_{\mu\circ\bar{R}^{-1}}\left(\bar{U}_i\right).$$

Remark 4 *For each player* $i \in I$, *the mixed reduced strategy induced both by the uniform behavior strategy* σ_i^0 *and by the uniform (non-reduced) mixed strategy* μ_i^0 *is* $\mu_i^0 \circ \bar{R}_i^{-1}$ *with* $\left(\mu_i^0 \circ \bar{R}_i^{-1}\right)\left(\mathbf{r}_i\right) = \left|\mathbf{r}_i\right| / \left|S_i\right|$; $\mu_i^0 \circ \bar{R}_i^{-1}$ *is uniform in every one-move game, but there are games where* $\mu_i^0 \circ \bar{R}_i^{-1}$ *is not uniform.*

The last claim of the remark is illustrated by Centipede-like games (those where, for each h of height 2 or more, $A\left(h\right)$ contains a terminating action T and a continuation (pass) action P. See Figure A, where $\left|H_1\right| = \left|H_2\right| = n \geq 2$: the set of strategies has cardinality $\left|S_i\right| = 2^{\left|H_i\right|}$; the set of reduced strategies has cardinality $\left|\mathbf{R}_i\right| = \left|H_i\right|+1$ (player i can either take at the k^{th} opportunity, with $k \in \{1, ..., \left|H_i\right|\}$, or always pass), the cardinality of reduced strategy $\mathbf{T}_{i,k} \subseteq S_i$ (taking at the k^{th} opportunity) is twice the cardinality of reduced strategy $\mathbf{T}_{i,k+1}$: $\left|\mathbf{T}_{i,k}\right| = 2\left|\mathbf{T}_{i,k+1}\right|$.[9]

$$\begin{array}{ccccccccc}
1 & \xrightarrow{P} & 2 & \xrightarrow{P} & 1 & \xrightarrow{P} & \ldots\ 2 & \xrightarrow{P} & \binom{1+2nc}{2nc} \\
T\downarrow & & T\downarrow & & T\downarrow & & T\downarrow & & \\
\binom{1}{0} & & \binom{c}{1+c} & & \binom{1+2c}{2c} & & \binom{(2n-1)c}{1+(2n-1)c} & &
\end{array}$$

Figure A

Remark 5 *The following "balancedness" property is a necessary and sufficient condition for the uniformity of* $\mu_i^0 \circ \bar{R}_i^{-1}$: *for each* $h \in H_i$ *and each pair of actions* $a', a'' \in A\left(h\right)$, *the sets of continuation strategies of* i *in the subgames with roots* $\left(h, a'\right)$ *and* $\left(h, a''\right)$ *have the same cardinalities, that is,*

$$\prod_{\bar{h}\in H_i\cap\{h':(h,a')\preceq h'\}}\left|A\left(\bar{h}\right)\right| = \prod_{\hat{h}\in H_i\cap\{h'':(h,a'')\preceq h''\}}\left|A\left(\hat{h}\right)\right|,$$

where the product is 1 *by convention if the intersection is empty.*[10]

[9] Cf. Figure 6 in Lin & Palfrey (2022) (Figure 5 Lin & Palfrey 2023).

[10] The interpretation of this convention is that, if there are no further moves of i, his only continuation strategy is to "wait."

The game tree of Figure B illustrates. The balancedness property is not satisfied: let $h = \varnothing$ (empty sequence), in the subgame with root $(h, a') = (a')$ player 1 has (by convention) one continuation strategy, in the subgame with root $(h, a'') = (a'')$ she has two continuation strategies.

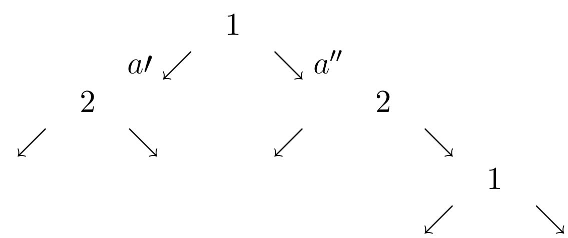

Figure B

Given that each level-type k of player j uses behavior strategy σ_{jk} and under uncertainty about the level-type k of player j, the conditional predictive probabilities[11] of j's actions assigned by *player* i *of level* $\ell + 1$ are obtained from the behavior strategy mixture (Selten, 1975) $\widetilde{\sigma}_j^\ell$ with *ex ante* subjective weights p_{jk}^ℓ ($p_{j\cdot}^\ell$ is the normalized truncation of p_j with support $\{0, ..., \ell\}$). First, for each $\bar{h} \in H_j$ and level k, obtain the updated probability of k conditional on $\bar{h}$ by Bayes rule, just looking at player j:

$$\nu_j^\ell \left(k|\bar{h}\right) = \frac{p_{jk}^\ell \prod_{h \in H_i \cap \left\{h' : h' \prec \bar{h}\right\}} \sigma_{jk,h}\left(\alpha\left(h, \bar{h}\right)\right)}{\sum_{k'=0}^{\ell} p_{jk'} \prod_{h \in H_i \cap \left\{h' : h' \prec \bar{h}\right\}} \sigma_{jk',h}\left(\alpha\left(h, \bar{h}\right)\right)},$$

where $\alpha\left(h, \bar{h}\right)$ is the action $\bar{a}$ such that $(h, \bar{a}) \preceq \bar{h}$. Next, for each $a \in A\left(\bar{h}\right)$, let

$$\widetilde{\sigma}_{j\bar{h}}^\ell (a) = \sum_{k=0}^{\ell} \nu_j^\ell \left(k|\bar{h}\right) \sigma_{jk,\bar{h}} (a).$$

The profile of behavior strategy mixtures describing the predictive probabilities assigned by player i of level $\ell + 1$ to the co-players' actions is denoted $\widetilde{\sigma}_{-i}^\ell = \left(\widetilde{\sigma}_j^\ell\right)_{j \neq i}$.

Starting from the product measure $p_{-i}^\ell = \times_{j \neq i} p_j^\ell$, the updated beliefs of player i of level-type $\ell + 1$ on the levels/types of the co-players conditional

[11] In compliance with the language of Bayesian statistics, a **predictive probability** is the probability of an observable event, possibly conditional on another observable event: unlike levels/types, histories and actions are observable.

on $\bar{h}$ is the product measure $\times_{j\neq i}\nu_j^{\ell}\left(\cdot|\bar{h}\right)$ (see, e.g., Lin & Palfrey 2022, and the more general argument provided below about games with observable deviators).

Best replies Using Kuhn's transformations and the assumption that every pure strategy profile of the co-players has strictly positive probability due to the presence of a positive fraction of level-0 types for each co-player (role) $j \neq i$, we can equivalently express i's conjectures about co-players as products of (1) totally mixed strategies $\mu_j \in \Delta^{\circ}\left(S_j\right)$, (2) totally mixed reduced strategies $\bar{\mu}_j \in \Delta^{\circ}\left(\mathbf{R}_i\right)$, or (3) totally randomized (predictive) behavior strategies $\hat{\sigma}_j \in \times_{h\in H_j}\Delta^{\circ}\left(A\left(h\right)\right)$.

For all $h \in H_i$ and $\sigma_i \in \times_{h'\in H_i}\Delta\left(A\left(h'\right)\right)$, let $\sigma_i^{\succeq h} \in \times_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}}\Delta\left(A\left(h'\right)\right)$ denote the restriction of σ_i to the subgame with root h; symbol $s_i^{\succeq h} \in \times_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}}A\left(h'\right)$ has the analogous meaning for pure strategies. With this,

$$\text{supp}\left(\sigma_i^{\succeq h}\right) := \left\{s_i^{\succeq h} : \forall h' \in H_i \cap \left\{\bar{h} : \bar{h} \succeq h\right\}, \sigma_{ih'}^{\succeq h}\left(s_{ih'}^{\succeq h}\right) > 0\right\}$$

denotes the support of $\sigma_i^{\succeq h}$, that is, the support of the h-subgame mixed strategy obtained from $\sigma_i^{\succeq h}$ by means of Kuhn's transformation:

$$s_i^{\succeq h} \in \text{supp}\left(\sigma_i^{\succeq h}\right) \Longleftrightarrow \prod_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}} \sigma_{ih'}\left(s_{ih'}\right) > 0.$$

We define randomized best replies by assuming, in the spirit of the Cognitive Hierarchies literature, that ties at the top are broken by randomizing uniformly on top actions.

Definition 2 *The* ***sequential best reply*** *of i to (predictive) conjecture $\hat{\sigma}_{-i} \in \times_{j\neq i}\left(\times_{h\in H_j}\Delta\left(A\left(h\right)\right)\right)$ is the (possibly degenerate) behavior strategy $\sigma_i = \mathrm{BR}_i\left(\hat{\sigma}_{-i}\right)$ that maximizes expected payoff given $\hat{\sigma}_{-i}$ in every subgame and such that each local randomization is uniform on its support, that is, for every $h \in H_i$,*

$$\begin{aligned} supp\left(\sigma_i^{\succeq h}\right) &= \arg\max_{s_i^{\succeq h}} \sum_{z\succeq h} P\left(z|h; s_i^{\succeq h}, \hat{\sigma}_{-i}\right) u_i\left(z\right), \\ \forall a &\in supp\left(\sigma_{ih}\right), \ \sigma_{ih}\left(a\right) = \frac{1}{\left|supp\left(\sigma_{ih}\right)\right|}. \end{aligned}$$

Remark 6 *By the One-Deviation Principle, σ_i is the sequential best reply of i to conjecture $\hat{\sigma}_{-i}$—that is, $\sigma_i = \mathrm{BR}_i(\hat{\sigma}_{-i})$—if and only if, for every $h \in H_i$,*

$$\begin{aligned} supp(\sigma_{ih}) &= \arg\max_{a\in A(h)} \sum_{z\succeq(h,a)} P(z|(h,a);\sigma_i,\hat{\sigma}_{-i})\, u_i(z), \\ \forall a &\in supp(\sigma_{ih}),\ \sigma_{ih}(a) = \frac{1}{|supp(\sigma_{ih})|}, \end{aligned}$$

where $P(z|(h,a);\sigma_i,\hat{\sigma}_{-i})$ is the probability of z conditional on (h,a) when behavior complies with σ_i and $\hat{\sigma}_{-i}$ in the subgame with root (h,a).

Let $H_i(\mu_i) := \bigcup_{s_i\in\mathrm{supp}(\mu_i)} H_i(s_i)$ denote the the set of non terminal histories allowed (not precluded) by mixed strategy μ_i. For any behavior strategy σ_i that is realization equivalent to μ_i, write $H_i(\sigma_i) = H_i(\mu_i)$.

Definition 3 *The **weakly sequential best reply** of i to $\hat{\sigma}_{-i} \in \times_{j\neq i}\left(\times_{h\in H_j}\Delta(A(h))\right)$ is the (possibly degenerate) behavior strategy $\bar{\sigma}_i = \overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})$ such that, for every $h \in H_i(\sigma_i)$,*

$$\begin{aligned} supp\left(\bar{\sigma}_i^{\succeq h}\right) &= \arg\max_{s_i^{\succeq h}} \sum_{z\succeq h} P\left(z|h; s_i^{\succeq h},\hat{\sigma}_{-i}\right) u_i(z), \\ \forall a &\in supp(\bar{\sigma}_{ih}),\ \bar{\sigma}_{ih}(a) = \frac{1}{|supp(\bar{\sigma}_{ih})|}, \end{aligned}$$

and furthermore

$$\forall h \in H_i \backslash H_i(\bar{\sigma}_i), \forall a \in A(h),\ \bar{\sigma}_{ih}(a) = \frac{1}{|A(h)|}.$$

Note that the specification of $\bar{\sigma}_i$ outside $H_i(\bar{\sigma}_i)$ is immaterial, but the uniform distribution is in the spirit of the CH model.

Remark 7 *Weak sequential best replies are invariant to realization equivalences: for all players $i \in I$, conjectures $\hat{\sigma}_{-i}$, and strategies $s_i \in supp\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$, $\bar{R}_i(s_i) \subseteq supp\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$.*

Remark 8 *Sequential best replies and weakly sequential best replies coincide and yield the same expected payoffs on realizable histories: for all $i \in I$ and $\hat{\sigma}_{-i} \in \times_{j\neq i}\left(\times_{h\in H_j}\Delta(A(h))\right)$,*

$$H_i(\mathrm{BR}_i(\hat{\sigma}_{-i})) = H_i\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right),$$

$$\forall h \in H_i\left(\mathrm{BR}_i\left(\hat{\sigma}_{-i}\right)\right),\ \mathrm{BR}_{ih}\left(\hat{\sigma}_{-i}\right) = \overline{\mathrm{BR}}_{ih}\left(\hat{\sigma}_{-i}\right),$$

and

$$\sum_{z \succeq h} P\left(z|h; \mathrm{BR}_i^{\succeq h}\left(\hat{\sigma}_{-i}\right), \hat{\sigma}_{-i}\right) u_i(z) = \sum_{z \succeq h} P\left(z|h; \overline{\mathrm{BR}}^{\succeq h}\left(\hat{\sigma}_{-i}\right), \hat{\sigma}_{-i}\right) u_i(z).$$

Best-Reply Equivalence Between Normal- and Extensive-Form Representations

Definition 4 *The* ***ex ante best reply*** *of i to $\mu_{-i} \in \Delta(S_{-i})$ is the (possibly degenerate) mixed strategy μ_i^* such that*

$$\begin{aligned}\mathrm{supp}\left(\mu_i^*\right) &= \arg\max_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} U_i\left(s_i, s_{-i}\right) \mu_{-i}\left(s_{-i}\right),\\ \forall s_i &\in \mathrm{supp}\left(\mu_i^*\right),\ \mu_i\left(s_i\right) = \frac{1}{\left|\mathrm{supp}\left(\mu_i^*\right)\right|}.\end{aligned}$$

In what follows, for every behavioral strategy profile $\sigma_{-i} \in \Sigma_{-i}$ of the co-players, let $\mu_{-i}^{\sigma_{-i}} \in \Delta(S_{-i})$ denote any realization equivalent profile of mixed strategies, such as the one obtained by means of Kuhn's transformation, i.e., the product measure $\mu_{-i}^{\sigma_{-i}} \in \Delta(S_{-i})$ such that

$$\forall s_{-i} \in S_{-i},\ \mu_{-i}^{\sigma_{-i}}\left(s_{-i}\right) = \prod_{j \neq i} \prod_{h \in H_j} \sigma_{jh}\left(s_{jh}\right).$$

Lemma 2 *For all strictly positive conjectures, ex ante best replies coincide with weakly sequential best replies: specifically, for all $i \in I$ and $\hat{\sigma}_{-i} \in \Sigma_{-i} \cap \mathbb{R}_{++}^{H_{-i}}$, the ex ante best reply to $\mu_{-i}^{\hat{\sigma}_{-i}}$ is the Kuhn's transformation of $\overline{\mathrm{BR}}_i\left(\hat{\sigma}_{-i}\right)$.*

Proof It is well known that if every strategy profile of the co-players is deemed possible ex ante, then ex ante expected payoff maximization is equivalent to expected payoff maximization conditional on each history allowed by the optimizing strategy. As for the probabilities assigned by the mixed best reply, observe that—since all the actions in the support of the sequential best reply $\mathrm{BR}_i\left(\hat{\sigma}_{-i}\right)$ realization equivalent to weak sequential best reply $\overline{\mathrm{BR}}_i\left(\hat{\sigma}_{-i}\right)$ yield the same, maximal conditional expected payoff and

$\mathrm{BR}_{ih}(\hat{\sigma}_{-i}) = \overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})$ for all $h \in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i})) = H_i\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$—then a kind of exchangeability property holds:

$$\begin{aligned}\mathrm{supp}(\mu_i^*) &= \left(\times_{h\in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))}\mathrm{supp}(\mathrm{BR}_{ih}(\hat{\sigma}_{-i}))\right) \times \left(\times_{h'\in H_i\setminus H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))} A(h')\right) \\ &= \mathrm{supp}\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right),\end{aligned}$$

where μ_i^* is the ex ante best reply. Therefore, for every $s_i \in$supp(μ_i^*),

$$\begin{aligned}\mu_i^*(s_i) &= \frac{1}{|\mathrm{supp}(\mu_i^*)|} \\ &= \frac{1}{\prod_{h\in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))}|\mathrm{supp}(\mathrm{BR}_{ih}(\hat{\sigma}_{-i}))| \cdot \prod_{h'\in H_i\setminus H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))}|A(h')|} \\ &= \frac{1}{\prod_{h\in H_i}\left|\mathrm{supp}\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)\right|} \\ &= \prod_{h\in H_i}\frac{1}{\left|\mathrm{supp}\left(\overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})\right)\right|} = \prod_{h\in H_i}\overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})(s_{ih}).\ \heartsuit\end{aligned}$$

■

Recall that p_j^ℓ denotes the ℓ-truncation of p_j, that is, for every level-type $k \in \{0, ..., \ell\}$, $p_{jk}^\ell = p_{jk}\left(\sum_{\kappa=0}^{\ell} p_{j\kappa}\right)^{-1}$ and $p_{jm}^\ell = 0$ for $\ell < m$. With this, p_j^ℓ is the initial belief of player i of level-type $\ell+1$ about the level-type of player j. Recall that $\widetilde{\sigma}_{-i}^\ell$ is the profile of behavior strategy mixtures representing the predictive probabilities assigned by player i of level-type $\ell+1$ to the co-players' actions. Similarly, in a game with simultaneous moves (such as the normal form of the given sequential game), we let μ_j^k denote the mixed strategy of level-type k of player j, so that the conjecture of player i of level-type $\ell+1$ about the co-players' strategies is $\widetilde{\mu}_{-i}^\ell = \times_{j\neq i}\left(\sum_{k=0}^{\ell} p_{jk}^\ell \mu_j^k\right)$.

Proposition 1 *Consider the cognitive-hierarchies models applied to the normal-form and extensive-form representations of a finite game (with perfect information). For every player $i \in I$ and every level $\ell \geq 0$, the level-$(\ell+1)$ mixed best reply $\mu_i^{\ell+1}$ to conjecture $\widetilde{\mu}_{-i}^\ell = \times_{j\neq i}\left(\sum_{k=0}^{\ell} p_{jk}^\ell \mu_j^k\right)$ in the normal form is the Kuhn's transformation of the weakly sequential best reply $\bar{\sigma}_i^{\ell+1} = \overline{\mathrm{BR}}_i\left(\widetilde{\sigma}_{-i}^\ell\right)$ to behavior strategy mixture $\widetilde{\sigma}_{-i}^\ell$ in the extensive form, which is realization equivalent to the sequential best reply $\sigma_i^{\ell+1} = \mathrm{BR}_i\left(\widetilde{\sigma}_{-i}^\ell\right)$.*

Sketch of proof The proof is by induction on ℓ. The basis step $\ell = 0$ follows from Remark 1 and Lemma 2, because $\widetilde{\mu}^0_{-i} = \times_{j\neq i}\mu^0_j$, where μ^0_j is the uniform (hence, strictly positive) probability measure induced by the uniform behavior strategy σ^0_j. For $\ell > 0$, suppose by way of induction that the result holds for each $k \in \{0, ..., \ell\}$ and fix any $i \in I$. One can show that the strictly positive conjecture $\widetilde{\sigma}^\ell_{-i}$ is realization equivalent to $\widetilde{\mu}^\ell_{-i}$. Thus, Lemma 2 yields the result. ♡

Corollary 1 *Consider the* modified *cognitive-hierarchies model applied to the (structurally) reduced strategic form where the level-*0 *type of each player i strictly randomize with the (possibly non-uniform) reduced mixed strategy $\bar{\mu}^0_i = \mu^0_i \circ \bar{R}^{-1}_i \in \Delta(\mathbf{R}_i)$ obtained from the uniform mixed strategy $\mu^0_i \in \Delta(S_i)$. For every player $i \in I$ and every level $\ell \geq 0$, if the reduced mixed strategy of level $\ell + 1$ of i is pure (degenerate, $\bar{\mu}^{\ell+1}_i = \delta_{\mathbf{r}^{\ell+1}_i}$), then the corresponding pure reduced strategy $\mathbf{r}^{\ell+1}_i$ satisfies $\mathbf{r}^{\ell+1}_i = supp\left(\mu^{\ell+1}_i\right)$, where $\mu^{\ell+1}_i \in \Delta(S_i)$ is the mixed strategy of level $\ell + 1$ of i in the normal form.*

Games with Imperfect Information The foregoing analysis extends seamlessly to games with **observable actions**, where players may choose simultaneously at some stage and previous moves are perfectly observed (see Battigalli et al. 2023). As for games with *imperfectly* observable actions, the main complication is due to the presence of information sets. The main change in this case is that the conditional belief about co-players level-types at an information set in games with more than two players need not be a product measure (see Example 3 in Figure 3 of Lin & Palfrey 2022). It is, however, a product measure in all games with **observable deviators**,[12] that is, games where, for every player i and information set $\mathbf{h}_i \in \mathbf{H}_i$, the set

$$S(\mathbf{h}_i) := \{s \in S : \exists h \in \mathbf{h}_i, h \prec O(s)\}$$

of pure strategy profiles inducing a path through $\mathbf{h}_i$ is a Cartesian product of its projections, $S(\mathbf{h}_i) = \times_{j\in I} S_j(\mathbf{h}_i)$, where

$$S_j(\mathbf{h}_i) := \mathrm{proj}_{S_j} S(\mathbf{h}_i) := \{s_j \in S_j : \exists s_{-j} \in S_{-j}, (s_j, s_{-j}) \in S(\mathbf{h}_i)\}.$$

[12] On observable deviators see Fudenberg & Levine (1993) and Battigalli (1996, 1997). The notation used here for information sets is the same as Battigalli et al. (2020) and it hints at the fact that, under the assumption that players have perfect memory (an assumption about their cognitive traits), information sets correspond to personal histories of own actions and messages (signals) received concerning previous play (see Battigalli & Generoso 2021).

Remark 9 *Suppose that the information structure satisfies the* observable deviators *property, that is,* $S(\mathbf{h}_i) = \times_{j\in I} S_j(\mathbf{h}_i)$ *for all players* $i \in I$ *and information sets* $\mathbf{h}_i \in \mathbf{H}_i$. *Then players' updated beliefs about co-players' level-types conditional of their information sets are product measures.*

Sketch of proof Following the hint in footnote 8 of Battigalli (1996), model i's uncertainty about co-players $j \neq i$ as distributional strategies $\delta_j \in \Delta(\Theta_j \times S_j)$, where $\Theta_j \cong \mathbb{N}_0$ is the set of level-types of player j. (Of course, the ex ante belief on $\times_{j\neq i}\Theta_j$ of level-type $k+1$ of player i is the product of the normalized truncations on $\{0, ..., k\}$.) The initial belief about co-players is the product measure $\delta_{-i} = \times_{j\neq i}\delta_j$. Observable deviators implies that the updated probability of profile $(\theta_{-i}, s_{-i}) = (\theta_j, s_j)_{j\neq i}$ conditional on $\mathbf{h}_i \in \mathbf{H}_i$ is

$$\begin{aligned}
\delta_{-i}(\theta_{-i}, s_{-i}|\Theta_{-i} \times S_{-i}(\mathbf{h}_i)) &= \frac{\delta_{-i}(\theta_{-i}, s_{-i})}{\delta_{-i}(\Theta_{-i} \times S_{-i}(\mathbf{h}_i))} \\
&= \frac{\delta_{-i}(\theta_{-i}, s_{-i})}{\delta_{-i}(\times_{j\neq i}\Theta_j \times S_j(\mathbf{h}_i))} \\
&= \frac{\prod_{j\neq i}\delta_j(\theta_j, s_j)}{\prod_{j\neq i}\delta_j(\Theta_j \times S_j(\mathbf{h}_i))} \\
&= \prod_{j\neq i}\frac{\delta_j(\theta_j, s_j)}{\delta_j(\Theta_j \times S_j(\mathbf{h}_i))} \\
&= \prod_{j\neq i}\delta_j(\theta_j, s_j|\Theta_j \times S_j(\mathbf{h}_i)),
\end{aligned}$$

where the denominators are strictly positive because there is a strictly positive fraction of level-0 types, who play every action with strictly positive probability. The conditional probability of each profiles of co-players level-types $\theta_{-i} = (\theta_j)_{j\neq i}$ is the (product) marginal of $\delta_{-i}(\cdot|\Theta_{-i} \times S_{-i}(\mathbf{h}_i))$:

$$\nu_i(\theta_{-i}|\mathbf{h}_i) = \prod_{j\neq i}\delta_j(\{\theta_j\} \times S_j(\mathbf{h}_i)|\Theta_j \times S_j(\mathbf{h}_i)). \heartsuit$$

The main results stated for perfect information games also hold for all sequential games (assuming perfect recall): essentially, in Remark 1 one has to replace, for each player $i \in I$ histories $h \in H_i$ with information sets $\mathbf{h}_i \in \mathbf{H}_i$ (corresponding to personal histories of i, see Battigalli & Generoso

2021), the cardinality of the strategy set S_i is $|S_i| = \times_{\mathbf{h}_i \in \mathbf{H}_i} |A(\mathbf{h}_i)|$ and the counting argument implies that the uniform behavior strategy σ_i^0 yields the uniform mixed strategy μ_i^0 by means of Kuhn's transformation. Results on sequential, weakly sequential and ex ante best replies extend seamlessly to all sequential games as long as players have perfect recall (which makes conditional expected utility maximization dynamically consistent). The extension of Proposition 1 to sequential games with imperfect information follows.

With this, known results on transformations of extensive form structures yield the following:

Remark 10 *The cognitive hierarchies model is invariant to interchanging of essentially simultaneous moves, but it is not invariant to coalescing sequential moves by the same player and its inverse, sequential agents splitting.*

To see why this is true, consider that Battigalli et al. (2020) prove that two extensive form structures have the "same" map $\mathbf{O} : \mathbf{S} \to Z$ from structurally reduced strategy profiles to induced terminal histories (up to isomorphisms) if and only if they one can transform one into the other by means of a sequence of interchanging and coalescing/splitting transformations. One can also show that two extensive form structures have the same map $O : S \to Z$ (up to isomorphisms) if and only if one can transform one into the other by means of a sequence in interchanging transformations (see Bonanno 1992). On the one hand, the latter results and the extension of Proposition 1 to imperfect information games implies that the cognitive hierarchies model is invariant to interchanging essentially simultaneous moves. On the other hand, the result by Battigalli et al. (2020) and Remarks 2 and 5 imply that the cognitive hierarchies model is not invariant to sequential agent splitting, that can make a one-move (or balanced) game not one-move (or unbalanced). The following common-interests game illustrates.

$$\begin{array}{ccccccc} 1 & \xrightarrow{C} & 2 & \xrightarrow{c} & 2 & \xrightarrow{a} & \binom{7}{7} \\ D\downarrow & & d\downarrow & & b\downarrow & & \\ \binom{2}{2} & & \binom{0}{0} & & \binom{0}{0} & & \end{array}$$

Figure C'

In the game depicted in Figure C', the best reply by player 1 to the uniform behavior strategy of player 2 is D, because C yields $\frac{7}{4} < 2$ in expectation, because the sequence of actions (c, a) by player 2, that yields 7 utils has probability $\frac{1}{4}$.

$$\begin{array}{ccccc} 1 & \xrightarrow{C} & 2 & \xrightarrow{ca} & \binom{7}{7} \\ D\downarrow & & d\downarrow & \searrow cb & \\ \binom{2}{2} & & \binom{0}{0} & & \binom{0}{0} \end{array}$$

Figure C”

By coalescing the sequential moves of player 2, we obtain the game in Figure C”, which has the *same reduced normal form* as the game in Figure C’. Now uniform randomization assigns probability $\frac{1}{3}$ to the action/strategy ca of player 2 that yields 7 utils and the best reply of player 1 is C, which yields $\frac{7}{4} > 2$ in expectation.

Discussion The cognitive hierarchies model is mostly used to organize data of experimental games. Sequential games are often played in experiments with the strategy method by making subjects irreversibly choose among *reduced* strategies. Thus, the question is whether subjects who are presented with a sequential game and then have to choose between reduced strategies are better modeled by assuming that they think of uniform randomization as equalizing the probabilities of possible actions at any given node of the sequential game, or equalizing the probabilities of reduced strategies. Palfrey & Lin (2023) report interesting evidence supporting the latter hypothesis.

Be as it may, we have to recognize that the normal form $\mathcal{N}(\Gamma)$ of a sequential game Γ and the reduced normal form $\mathcal{RN}(\Gamma)$—*interpreted as games where player irreversibly and covertly choose strategies (reduced strategies) in advance*—are different from each other (except when Γ is a one-move game), and that they very different from the sequential game Γ. Whether players should behave “in the same way” in Γ, $\mathcal{N}(\Gamma)$, and $\mathcal{RN}(\Gamma)$—or whether behavior should be invariant to some specific transformations of the game—cannot but depend on the adopted theory of strategic interaction and the corresponding solution concept. It is well known that some solution concepts like Nash equilibrium and iterated admissibility are essentially reduced-normal-form invariant,[13] while others like trembling hand perfect equilibrium, sequential equilibrium, initial rationalizability and strong rationalizability

[13] It is less well known that also selfconfirming equilibrium is essentially reduced-normal form invariant. See Battigalli et al (2019).

are not reduced normal form invariant.[14] Similarly, some solution concepts are invariant to transformations like interchanging essentially simultaneous moves and coalescing/sequential agents splitting: these transformations do not change the structurally reduced normal form (see Battigalli et al.2020); thus, all the reduced-normal-form invariant solution concepts are necessarily invariant to these transformations, but there other solution concepts, like initial and strong rationalizability, are invariant as well. We proved that the cognitive hierarchies model with uniform randomization is normal-form invariant, although it is not reduced-normal-form invariant. Therefore, the model is invariant to interchanging essentially simultaneous moves, but not invariant to coalescing/sequential agent splitting. Are these lacks of invariances mere "representation effects"? My position is that, even if different games can be obtained from each other by some transformations preserving some basic structures, *they remain different and should not be presumed to be played in the same way unless one explicitly spells out and adopts a theory entailing this.* Some solution concepts have clear and explicit foundations in theories of strategic reasoning, or learning, or adaptive play. If we like those theories, we must accept the equivalences and differences they entail.

References

[1] Battigalli, P. (1996): "Strategic Independence and Perfect Bayesian Equilibria," *Journal of Economic Theory*, 70, 201-234.

[2] Battigalli, P. (1997): "Games with Observable Deviators," in *Decisions, Games and Markets*, ed. by P. Battigalli, A. Montesano and F. Panunzi. Dordrecht: Kluwer Academic Publishers, 57-96.

[3] Battigalli, P. and N. Generoso, (2021): "Information Flows and Memory in Games," IGIER W.P. 678, Bocconi University.

[4] Battigalli, P., E. Catonini, G. Lanzani, and M. Marinacci (2019): "Ambiguity Attitudes and Self-Confirming Equilibrium in Sequential Games," *Games and Economic Behavior*, 115, 1-29.

[14] On initial (also called "weak") rationalizability and strong rationalizability see, e.g., the textbook of Battigalli et al (2023) and the relevant references therein.

[5] Battigalli, P., P. Leonetti, and F. Maccheroni (2020): "Behavioral Equivalence of Extensive Game Structures," *Games and Economic Behavior*, 121, 533-547.

[6] Battigalli, P., E. Catonini, and N. De Vito (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.

[7] Bonanno, G. (1992): "Set-Theoretic Equivalence of Extensive-Form Games. *International Journal of Game Theory*, 20, 429–447.

[8] Camerer, C. F., T.-H. Ho, and J.-K. Chong (2004): "A Cognitive Hierarchy Model of Games," *Quarterly Journal of Economics*, 119, 861–898.

[9] Fudenberg, D., and D.K. Levine (1993): "Self-Confirming Equilibrium," *Econometrica*, 61, 523-545.

[10] Kuhn, H.W. (1953): "Extensive Games and the Problem of Information," in *Contributions to the Theory of Games II*, ed. by H.W. Kuhn and A.W. Tucker. Princeton: Princeton University Press, 193-216.

[11] Lin, P-H, and T. Palfrey (2022): "Cognitive Hierarchies in Extensive Form Games," W.P. 1460. California Institute of Technology.

[12] Lin, P-H, and T. Palfrey (2023): "Cognitive Hierarchies in Extensive Form Games," revision of W.P. 1460. California Institute of Technology.

[13] Osborne, M., and A. Rubinstein (1994): *A Course in Game Theory*. Cambridge MA: MIT Press.

[14] Rubinstein, A. (1991): "Comments on the Interpretation of Game Theory," *Econometrica*, 59, 909-904.

[15] Selten, R. (1975): "Re-examination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25-55.